Stochastics II | **Universität Ulm**
SoSe 2016 | Prof. Dr. Evgeny Spodarev
December 7, 2016 | Dr. Vitalii Makogin

Exercise sheet 8 (total – 19 points) till December 14, 2016

Exercise 5-1 (2 points)

Let W be the Wiener process. Find the characteristic function for $W(2) + 2W(1)$.

Exercise 5-2 (4 points)

Let W be the Wiener process. Find:

1. $\mathbf{E}(W(t))^m, m \in \mathbb{N}$,
2. $\mathbf{E}\exp(2W(1) + W(2))$,
3. $\mathbf{E}\cos(2W(1) + W(2))$,
4. $\mathbf{E}[W(2) > 1 | W(1) < 2]$.

Exercise 5-3 (3 points)

Let $X = \{X(t) := \int_0^t W(s)ds, t \geq 0\}$, where W is the Wiener process. Find the distribution of random variable $X(T)$ for $T > 0$.

Hint: Recall that a limit of Gaussian random variables is also Gaussian.

Exercise 5-4 (10 points)

(6 points) Write a program in R which simulates the trajectory of the Wiener process on $[0, T]$

(a) by using approximation with Schauder functions and input parameters t, T and m, where t is a finite dimensional vector of locations in $[0, T]$ and m is the cutt-off parameter of the series expansion;

(a) by using the independence and the distribution of the increments of W, with input parameter t defined as in (a);

(c) by using ideas for the proof of Theorem 3.3.1: Let $\{W(t), t \in [0, 1]\}$ be the Wiener process and $Z_1, Z_2, \ldots$ a sequence of independent random variables with $\mathbf{P}(Z_i = 1) = \mathbf{P}(Z_i = -1) = \frac{1}{2}$ for all $i \in \mathbb{Z}_+$. For every $n \in \mathbb{N}$ we define $\{\tilde{W}^n(t),\ t \in [0, 1]\}$ by $\tilde{W}^n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + (nt - \lfloor nt \rfloor)\frac{Z_{\lfloor nt \rfloor + 1}}{\sqrt{n}}$, where $S_i = Z_1 + \ldots + Z_i$, $i \geq 1$, $S_0 = 0$.

(1 point) Simulate 500 trajectories of a Wiener process on $[0, 5]$ in cases (a)-(c). Take $m = 10$ in (a) and $t = (t_0, ..., t_{1000})$ in (b), where $t_0 = 0$ and $t_k = kT/1000, k = 1, ..., 1000$. Take $n = 1000$ in (c).

(1 point) Plot one trajectory for all cases.

(3 points) For each simulated trajectory $\tilde{W}$ compute the approximation $\tilde{X}(5) = \frac{1}{1000}\sum_{i=0}^{1000} \tilde{W}(t_i)$ of random variable $X(5)$. Compare the empirical distribution of $\tilde{X}(5)$ with the distribution of $X(5)$ from Exercise 5-3 using Kolmogorov-Smirnov test and Kolmogorov's distance as a measure. Which method of simulation is better?