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Exercise sheet 9 (total – 18 points) till December 21, 2016

Exercise 9-1 (2 points)

Let W be the Wiener process. Find mean and covariance function of W^2. Determine the marginal distributions of W^2.

Exercise 9-2 (6 points)

Let $W = \{W(t),\ t \in [0,1]\}$ be a Wiener process. Define the process of the maximum as $M = \{M(t) = \max_{s\in[0,t]} W(s), t \geq 0\}$. Show:

1. (2 points) The probability density of $M(t)$ is given by

$$f_{M(t)}(x) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathbb{1}\{x \geq 0\}.$$

Hint: Use the fact that $\mathbf{P}(M(t) > x) = 2\mathbf{P}(W(t) > x)$.

2. (2 points) The expectation and variance of $M(t)$ are given via

$$\mathbf{E}M(t) = \sqrt{\frac{2t}{\pi}}, \qquad \mathbf{Var}M(t) = t(1 - \frac{2}{\pi}).$$

3. (2 points) Let $\tau(x) := \min\{s \in \mathbb{R}, W(s) = x\}$ be the first time when W attains the value x. Prove that $\tau(x)$ has Lévy distribution with density

$$f_{\tau(x)}(y) = \frac{x}{\sqrt{2\pi y^3}} \exp\left(-\frac{x^2}{2y}\right) \mathbb{1}\{y \geq 0\}.$$

Show that $\mathbf{E}\tau(x) = \infty$.

Exercise 9-3 (4 points)

Let $W = \{W(t),\ t \in [0,1]\}$ be a Wiener process and $L = \operatorname{argmax}_{t\in[0,1]} W(t)$. Show that

$$P(L \leq x) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad x \in [0,1].$$

Hint: Use the fact that $\max_{r\in[0,t]} W(r) \overset{d}{=} |W(t)|$.

Exercise 9-4 (3 points)

Let W and N be the independent Wiener process and Poisson process with intensity λ , respectively. Find the mean and covariance of the process $X(t) = W(N(t)), t \geq 0$. Is X a process with independent increments?

Exercise 9-5 (3 points)

Suppose that W and $\bar{W}$ are two independent Wiener processes and let $\rho \in [-1,1]$ be a constant. Is the process $X(t) := \rho W(t) + \sqrt{1-\rho^2}\bar{W}(t)$ a Wiener process?