Stochastics II
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Exercise sheet 1 (total – 20 points) till October 25, 2017

Exercise 1-1 (2 points)

Let ξ be a random variable with distribution function F. Prove that $X(t), t \in \mathbb{R}$ is a stochastic process, if

1. $X(t) = \max(\xi, t^2), t \in \mathbb{R}$.
2. $X(t) = \max(e^{\xi t} - K, 0), t \in \mathbb{R}, K > 0$.

Draw the sample paths of the process X. Find one-dimensional distributions of the process X.

Exercise 1-2 (4 points)

Let $\xi_1, \ldots, \xi_n$ be i.i.d. r.v.'s with distribution function F, and $X(t) = \frac{1}{n}\#\{k|\xi_k \leq t\} = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}\{\xi_k \leq t\}, t \in \mathbb{R}$. Draw the sample paths of the process X. Find all m-dimensional distributions of the process X[1], $n \geq 1, m \geq 1$.

Exercise 1-3 (4 points)

Prove the following result (based on Kolmogorov's theorem).

Proposition 1. *The family of measures $\mathbf{P}_{t_1,\ldots,t_d}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), (t_1, \ldots, t_d) \in T^d, d \geq 1$, satisfies the conditions of symmetry and consistency iff for all $d \geq 2, (s_1, \ldots, s_d) \in \mathbb{R}^d$ and $(t_1, \ldots, t_d) \in T^d$ it holds $\varphi_{\mathbf{P}_{t_1,\ldots,t_d}}((s_1, \ldots, s_d)) = \varphi_{\mathbf{P}_{t_{i_1},\ldots,t_{i_d}}}((s_{i_1}, \ldots, s_{i_d}))$ for any permutation $(1, \ldots, d) \to (i_1, \ldots, i_d)$, and $\varphi_{\mathbf{P}_{t_1,\ldots,t_{d-1}}}((s_1, \ldots, s_{d-1})) = \varphi_{\mathbf{P}_{t_1,\ldots,t_d}}((s_1, \ldots, s_{d-1}, 0))$.*

Exercise 1-4 (4 points)

1. (1 point) Let $Z = (\xi_1, \ldots, \xi_n)^T$ be a random vector with independent $N(0,1)$-distributed components. Let V be a $n \times n$ matrix, $\bar{\mu} = (\mu_1, \ldots, \mu_n)^T$. Calculate the characteristic function of $n-$dimensional Gaussian random vector $Y = AZ + \bar{\mu}$. Find its mean vector and covariance matrix $\Sigma = (\sigma_{i,j})_{i,j}^n$.
2. (2 point) Show the existence of a random function with finite dimensional multivariate Gaussian distributions and specify spaces $(S_{t_1,\ldots,t_n}, \mathcal{B}_{t_1,\ldots,t_n})$.
3. (1 point) Find the finite dimensional distributions of Gaussian white noise.

Exercise 1-5 (3 points)

Two devices start to operate at the instant of time $t = 0$. They operate independently of each other for random periods of time and after that they shut down. The operating time of the $i-$th device has a distribution function $F_i, i = 1, 2$. Let $X(t)$ be the number of operating devices at the instant t. Find one- and two-dimensional distributions of the process $\{X(t), t \in \mathbb{R}_+\}$.

Exercise 1-6 (3 points)

Let G be a set of functions $g : \mathbb{R}^d \to \mathbb{R}$ which are non-decreasing in each coordinate. Show that $G \in \mathcal{B}_T$ $(\mathcal{S}_t = \mathbb{R}, t \in T = \mathbb{R}^d)$. Let $\mathbf{X} = \{X_t = \xi(1 + f(t)\eta), t \in \mathbb{R}^d\}$, where $f : \mathbb{R}^d \to \mathbb{R}$ is a non-decreasing function in each coordinate, and ξ, η are independent $U[-1,2]-$distributed random variables. Denote by $\mathbf{P}_\mathbf{X}$ probability measure (distribution, law) of $\mathbf{X}$ on $(\mathcal{S}_T, \mathcal{B}_T)$. Find $\mathbf{P}_\mathbf{X}(G)$.

[1] note that $X = \hat{F}_n$ is the empirical distribution function based on the sample $(\xi_1, \ldots, \xi_n)$