# Exercise sheet 12 (total – 18 points) till January 31, 2018

## Exercise 12-1 (2 points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a discrete martingale and $\tau$ a discrete stopping time w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Show that$\{X_{\min\{\tau,n\}}\}_{n\in\mathbb{N}}$ is also a martingale w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.

## Exercise 12-2 (4 points)

(*Doob decomposition*) Let a random sequence $\{X_n, n \geq 0\}$ be a submartingale w.r.t. the filtration $\{\mathcal{F}_n, n \geq 0\}$. Prove that there exists a martingale $\{M_n, n \geq 0\}$ and non-decreasing integrable random sequence $\{A_n, n \geq 0\}$ such that $A_0 = 0$, $A_n$ is $\mathcal{F}_{n-1}$-measurable for each $n \geq 1$, and

$$X_n = M_n + A_n, \forall n \geq 0.$$

## Exercise 12-3 (3 points)

Let the stochastic process $X = \{X(t), t \geq 0\}$ be adapted and càdlàg. Show that

$$\mathbf{P}\left(\sup_{0\leq v\leq t} X(v) > x\right) \leq \frac{\mathbf{E}X(t)^2}{x^2 + \mathbf{E}X(t)^2}$$

holds for arbitrary $x > 0$ and $t \geq 0$, if $X$ is a submartingale with $\mathbf{E}X(t) = 0$ and $\mathbf{E}X(t)^2 < \infty$.

## Exercise 12-4 (3 points)

Let $X = \{X(n), n \in \mathbb{N}\}$ be a martingale. Show that the sequence of random variables $X(\tau \wedge 1), X(\tau \wedge 2), \ldots$ is uniformly integrable for every finite stopping time $\tau$, if $\mathbf{E}|X(\tau)| < \infty$ and $\mathbf{E}(|X(n)|\mathbb{I}_{\{\tau>n\}}) \to 0$ for $n \to \infty$.

## Exercise 12-5 (6 points)

1. (3 points) Prove that if $\{X_n, n \geq 0\}$ is a non-negative supermartingale, then $\exists \lim_{n\to\infty} X_n = X$ a.s. and $\mathbf{E}X \leq \mathbf{E}X_0$.
   Hint: Use the upcrossing inequality: if $\{Y_n, n \geq 0\}$ is a submartingale then $(b-a)\mathbf{E}U_n \leq \mathbf{E}(Y_n - a)_+ - \mathbf{E}(Y_0 - a)_+$, where $U_n$ is the number of upcrossings of interval $(a, b)$ by $Y_m, m \geq 0$ completed by time $n$.

2. (3 points) Let random sequences $\{X_n, n \geq 0\}$ and $\{Y_n, n \geq 0\}$ be a.s. non-negative, integrable and adapted to the filtration $\mathcal{F}_n, n \geq 0$.

   Suppose $\mathbf{E}(X_{n+1}|\mathcal{F}_n) \leq (1 + Y_n)X_n$, $\forall n \geq 0$ with $\sum_{n=0}^{\infty} Y_n < \infty$ a.s. Prove that $X_n$ converges a.s. to a finite limit.
   Hint: Consider $Z_n = X_n / \prod_{i=1}^{n-1}(1 + Y_i), n \geq 1$.