Stochastics II | **Universität Ulm**

WS 2017/2018 | Prof. Dr. Evgeny Spodarev

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Exercise sheet 13 (total – 18 points) till February 7, 2018

Exercise 13-1 (2 points)

Derive Corollary 6.1.2 from Corollary 6.1.1 in the Lecture notes. Namely, prove that if $\sum_{n=1}^{\infty} a_n^2 < \infty$, where $\{a_n\}_{n\in\mathbb{N}}$ is a deterministic sequence, and $\{\delta_n\}$ is a sequence of i.i.d. random variables with $\mathbf{E}\delta_n = 0$, $\mathbf{Var}\delta_n = \sigma^2 < \infty$, $n \in \mathbb{N}$, then the sequence $\sum_{n=1}^{\infty} a_n\delta_n$ converges a.s.

Exercise 13-2 (2 points)

Prove Corollary 6.2.1 in the Lecture notes: Let $X \geq 0$ be a random variable, $A = \{\omega \in \Omega : X(\omega) > 0\}$. Then it holds for almost all $\omega \in A$ that $\sum_{n=0}^{\infty} X(T^n(\omega)) = +\infty$, where T is a measure preserving map.

Exercise 13-3 (2 points)

Let $T : \Omega \to \Omega$ be a measure preserving map. Let B be any set with $T^{-1}(B) \subset B$ and let $C = \cap_{n\geq 0} T^{-n}(B)$. Show that $T^{-1}(C) = C$.

Exercise 13-4 (4 points)

Let $T : \Omega \to \Omega$ be a measure preserving map.

1. (2 points) Show that the set of all invariant events w.r.t. T is a σ-algebra J.
2. (2 points) Show that the set of all almost invariant events w.r.t. T is a σ-algebra J^*.

Exercise 13-5 (2 points)

Let a stationary sequence $X_n, n \geq 0$ be generated by a random variable X_0 and a measure preserving map T. Assume that X is m-dependent, that is, families of random variables $\{X_k, k \leq n\}$ and $\{X_j, j \geq n+m\}$ are independent for any n. Prove that T is ergodic.

Exercise 13-6 (6 points)

Let $\Omega = \mathbb{R}^2$ and P be a normal distribution in $\mathbb{R}^2$ with zero mean and identity matrix of covariances. Assume that transformation $T : \Omega \to \Omega$ acts in polar coordinates as $T((r,\varphi)) = (r, 2\varphi \ (mod\ 2\pi))$, $r \geq 0, 0 \leq \varphi < 2\pi$.

1. (2 point) Prove that T preserves the measure P.
2. (4 points) Find the limit

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{k=0}^{n-1} f(T^k(x)) \right), \ x \in \mathbb{R}^2$$

for $f_1 = x_1^2$, $f_2(x) = x_1, x_2$.

Hint: At first, prove this fact for the functions of the form $f(r,\varphi) = \sum_{k=0}^{m} c_k \mathbb{I}\{\varphi \in [\alpha_k, \beta_k]\}\mathbb{I}\{r \in [x_k, y_k]\}$, and then pass to a limit.