Stochastics II
WS 2017/2018
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Exercise sheet 14 (total – 18 points) till February 14, 2018

Exercise 14-1 (3 points)

Let $X_n, n \geq 0$ be a stationary sequence, and $g : \mathbb{R}^\infty \to \mathbb{R}$ be a measurable function. Prove that the random sequence $Y_n := g(X_{n+1}, X_{n+2}, \ldots), n \geq 0$ is stationary as well. Prove that if $\{X_n, n \geq 0\}$ is ergodic then the sequence $\{Y_n, n \geq 0\}$ is ergodic, too.

Exercise 14-2 (3 points)

Let $X = \{X_n, n \geq 1\}$ be a stationary random sequence with $\mu = \mathbf{E}X_n$ such that $\mathbf{Cov}(X_0, X_n) \to 0$, as $n \to \infty$. Show that

$$\frac{1}{n}\sum_{j=1}^{n} X_j \xrightarrow{L_2} \mu, n \to \infty.$$

Exercise 14-3 (4 points)

Let $\{N_t, t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Consider the process $X_t := \xi(-1)^{N_t}$, $t \geq 0$, where ξ is a random variable independent of N with $\mathbf{P}(\xi = -1) = \mathbf{P}(\xi = 1) = 1/2$.

1. (2 points) Compute the mean value and the covariance function of the process X. Show that the random sequence $\{X_n, n \geq 0\}$ is stationary in wide sense.
2. (2 points) Find the spectral density of the covariance function of the random sequence $\{X_n, n \geq 0\}$.

Exercise 14-4 (5 points)

Let $\{W(t), t \in \mathbb{R}_+\}$ be a Wiener process. Define the family of random variables $Z((a, b]) := W(b) - W(a)$ on the semiring $\mathcal{K} = \{(a, b], -\infty < a < b < \infty\}$.

1. (1 point) Show that Z is an orthogonally scattered random measure on $\mathcal{K}$.
2. (2 points) Let $I(f)$ be the stochastic integral of $f \in L^2(\mathbb{R})$ with respect to Z. Show that $I(f)$ is a Gaussian random variable. Find $\mathbf{E}I(f)$ and $\mathbf{E}[I(f)^2]$.
3. (2 points) Prove that $I(f)$ is a Gaussian random variable for any orthogonally scattered Gaussian random measure Z.

Exercise 14-5 (3 points)

Let Z be the orthogonal random measure from Exercise 15-4.

1. (1 point) Find its structure measure μ.
2. (1 points) Find
$$\mathbf{E}\left|\int_0^\pi \sin t \; dZ(t)\right|^2.$$
3. (1 points) Find
$$\mathbf{E}\left(\int_0^1 t \; dZ(t)\overline{\int_0^1 (2+t^2) \; dZ(t)}\right).$$