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Exercise sheet 4 (total – 19 points) till November 22, 2017

Exercise 4-1 (2 points)

Prove that the Wiener process has a modification with Hölder-continuous sample paths.

Exercise 4-2 (4 points)

Prove that there does not exist a process with independent increments $\{X(t), t \geq 0\}$ such that

1. (2 point) $X(0)$ has a non-atomic distribution, but the distribution of $X(1)$ has an atom.
2. (2 points) The distribution of $X(0)$ is absolutely continuous but the distribution of $X(1)$ is not.

Hint: Let μ be a probability measure on $\mathcal{B}(\mathbb{R})$, and let φ be its characteristic function. Then

$$\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t)dt.$$

If $\lim_{t\to\pm\infty} |\varphi(t)| = 0$, then μ has no atoms.

Exercise 4-3 (4 points)

Prove the following statements.

1. (2 point) The Wiener process possesses independent increments.
2. (2 points) The Poisson process with intensity λ has independent increments.

Hint: Find the characteristic function of increments and verify the formula (1.7.1) from the lecture notes.

Exercise 4-4 (2 points)

Let $Z_k, k \in \mathbb{N}$ be independent identically distributed random variables with mean 0 and finite variance σ^2, and let $X_n = \sum_{k=1}^{n} Z_k$. Let τ be a $\mathbb{N}$-valued random variable which is independent of the future w.r.t. $Z_k, k \in \mathbb{N}$ and $\mathbf{E}\tau < \infty$. Show that $\mathbf{Var}(X_\tau) = \sigma^2 \mathbf{E}\tau$

Exercise 4-5 (4 points)

Let $\{N(t), t \geq 0\}$ be renewal process with interarrival times $T_k, k \in \mathbb{N}$. Prove that $\mathbf{E}e^{\theta N(t)} < \infty$ for some strictly positive θ whenever $\mathbf{E}T_1 > 0$.

Hint: Consider the renewal process with interarrival times $\tilde{T}_k = \varepsilon \mathbb{I}\{T_k \geq \varepsilon\}$ for some suitable $\varepsilon > 0$.

Exercise 4-6 (3 points)

Let the time between the renewals of a renewal process N be $U[0,1]$-distributed. Find the mean and variance of $N(t), 0 \leq t \leq 1$.