Exercise sheet 6 (total – 19 points) till December 6, 2017

Exercise 6-1 (2 points)

Find the renewal function $H(t), t \geq 0$ of a renewal process $\{N(t), t \geq 0\}$ if the interarrival times have gamma distribution $\Gamma(\lambda, 2)$.

Exercise 6-2 (2 points)

Customers in a store are conventionally divided into m categories. Probability that a customer is of the k-th category is equal to p_k, and distribution function of his purchase equals F_k in this case. Assume that a cash desk breaks down with probability α while serving each customer. Find the Laplace transform of money that a cash desk obtains until the first breakdown.

Exercise 6-3 (3 points)

Assume that $\zeta, \eta_1, \eta_2, \ldots$ are independent random variables, ζ takes values in $\mathbb{Z}_+$, $\mathbf{P}(\zeta = k) = p_k$, and random variables $\{\eta_k, k \in \mathbb{N}\}$ are i.i.d. and non-negative. Let $\hat{l}_\eta(s)$ be the Laplace transform of η_1. Find the Laplace transform of $\sum_{k=1}^{\zeta} \eta_k$.

Exercise 6-4 (6 points)

A computer network router has accepted n independent packets, and processes them in consecutive order. Assume that each packet has type A with probability p and type B with probability $q = 1 - p$, and processing times have distribution functions F_A and F_B, respectively.
(a) Find the Laplace transform of a time span necessary for all packets' service.
Particularly, find answers in the cases when
(b) F_A and F_B have exponential distribution with parameters α and β, respectively,
(c) F_A and F_B are distribution functions of constant random variables equal to α and β, respectively.

Exercise 6-5 (6 points)

Service time of a spare part has an exponential distribution with a parameter α. A processing of a new spare part starts immediately after the previous spare part was completely processed. However, if some detail is processing for more than a period of time β, then the machine overheats and stops. The service of the first spare part starts at the moment $t = 0$. Let ξ be the time instant when the machine stops.
(a) Prove that its distribution function F_ξ satisfies the following equation:

$$F_\xi(x) = e^{-\alpha\beta}\mathbb{I}\{x \geq \beta\} + \int_0^x F_\xi(x-y)\alpha e^{-\alpha y}\mathbb{I}\{y \in [0, \beta)\}dy.$$

(b) Find $\mathbf{E}\xi$, $\mathbf{Var}\xi$.
Hint: Write down an equation for $\bar{F}_\xi(x) = 1 - F_\xi(x)$. And then integrate the left and right hand sides to get $\mathbf{E}\xi$.