Stochastics II | **Universität Ulm**
WS 2017/2018 | Prof. Dr. Evgeny Spodarev
December 5, 2017 | Dr. Vitalii Makogin

Exercise sheet 7 (total – 16 points) till December 13, 2017

Exercise 7-1 (2 points)

Let $\{X(t), t \in \mathbb{R}\}$ be a centered Gaussian process with covariance C_X. Find mean and covariance functions for the process $\{Y(t) = X^2(t), t \in \mathbb{R}\}$.

Exercise 7-2 (2 points)

Let W be the Wiener process. Find the characteristic function of $aW(t) + bW(s)$, where $a, b, t, s > 0$.

Exercise 7-3 (4 points)

The fractional Brownian motion with Hurst index $H \in (0,1)$ is the centered Gaussian process $\{B^H(t), t \in \mathbb{R}\}$ with covariance $C_{B^H}(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), t, s \in \mathbb{R}$. Prove that the fractional Brownian motion (in particular, the Wiener process) is a process with stationary increments, that is, the distribution of $\{X(s) := B^H(t+s) - B^H(t), s \geq 0\}$ does not depend on t.

Exercise 7-4 (4 points)

The Brownian bridge is the centered Gaussian process $\{B(t), t \in [0,1]\}$ with covariance function $C_B(t,s) = t \wedge s - st, s, t \in [0,1]$.

1. Let $\{B(t), t \in [0,1]\}$ be the Brownian bridge. Prove that

$$\{X(t) = (1+t)B\left(1 - \frac{1}{1+t}\right), t \in \mathbb{R}_+\}$$

   is a Wiener process.

2. Let $\{W(t), t \in [0,1]\}$ be the Wiener process. Prove that the process

$$\{X(t) = W(t) - tW(1), t \in [0,1]\}$$

   is a Brownian bridge.

Exercise 7-5 (4 points)

Let $\{W(t), t \geq 0\}$ be the two-dimensional Wiener process, that is $W(t) = (W_1(t), W_2(t)), t \geq 0$, where W_1 and W_2 are two independent Wiener processes. Let $B(0,r) = \{x \in \mathbb{R}^2 : \|x\| \leq r\}$, $r > 0$ and $[a,b] = [a_1,b_1] \times [a_2,b_2], 0 \leq a_1 < b_1, 0 \leq a_2 < b_2$. Find

1. $\mathbf{P}(W(t) \in B(0,r))$.
2. $\mathbf{P}(W(t) \in [a,b])$.