Stochastics II **Universität Ulm**
WS 2017/2018 Prof. Dr. Evgeny Spodarev
December 13, 2017 Dr. Vitalii Makogin

Exercise sheet 8 (total – 20 points) till December 20, 2017

Exercise 8-1 (6 points)

Let $W = \{W(t),\ t \geq 0\}$ be a Wiener process. Define the process of the maximum as $M = \{M_t = \max_{s\in[0,t]} W(s), t \geq 0\}$. Show:

1. (2 points) The probability density of M_t is given by
$$f_{M_t}(x) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathbb{1}\{x \geq 0\}.$$
2. (2 points) The expectation and variance of M_t are given via $\mathbf{E}M_t = \sqrt{\frac{2t}{\pi}}$, $\mathbf{Var}M_t = t(1 - \frac{2}{\pi})$.
3. (2 points) Let $\tau(x) := \min\{s \geq 0, W(s) = x\}$ be the first time when W attains the value x. Prove that $\tau(x)$ has Lévy distribution with density
$$f_{\tau(x)}(y) = \frac{x}{\sqrt{2\pi y^3}} \exp\left(-\frac{x^2}{2y}\right) \mathbb{1}\{y \geq 0\}.$$
Show that $\mathbf{E}\tau(x) = \infty$.

Exercise 8-2 (3 points)

Let $X = \{X(t) := \int_0^t W(s)ds, t \geq 0\}$, where W is the Wiener process. Find the distribution of random variable $X(t)$ for $t > 0$.

Hint: Recall that a limit of Gaussian random variables is also Gaussian.

Exercise 8-3 (11 points)

1. (6 points) Write a program in R which simulates the trajectory of the Wiener process on $[0, T]$
   (a) by using an approximation with Schauder functions and input parameters t, T and m, where t is a finite dimensional vector of locations in $[0, T]$ and m is the cut-off parameter of the series expansion;
   (a) by using the independence and the distribution of the increments of W, with input parameter t defined as in (a);
   (c) by using Donsker's invariance principle: for every $n \in \mathbb{N}$ we define $\{\tilde{W}^n(t),\ t \in [0, 1]\}$ by $\tilde{W}^n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + (nt - \lfloor nt \rfloor)\frac{Z_{\lfloor nt \rfloor + 1}}{\sqrt{n}}$, where $S_i = Z_1 + \ldots + Z_i$, $i \geq 1$, $S_0 = 0$, where $Z_1, Z_2, \ldots$ are i.i.d. r.v.'s with $\mathbf{E}Z_i = 0, \mathbf{Var}Z_i = 1$. Experiment with different distributions of $Z_i, i \geq 1$ (at least three).
2. (1 point) Simulate 500 trajectories of a Wiener process on $[0, 5]$ in cases (a)-(c). Take $m = 10$ in (a) and $t = (t_0, ..., t_{1000})$ in (a)-(b), where $t_0 = 0$ and $t_k = kT/1000, k = 1, ..., 1000$. Take $n = 1000$ in (c). Plot one trajectory for each case.
3. (2 points) For each simulated trajectory $\tilde{W}$ compute the approximation $\tilde{M}_5 = \max_{i=0,...,1000} \tilde{W}(t_i)$ of random variable M_5. Compare the empirical distribution of $\tilde{M}_5$ with the distribution of M_5 from Exercise 8-1 using Kolmogorov-Smirnov test and Kolmogorov's distance as a measure.
4. (2 points) For each simulated trajectory $\tilde{W}$ compute the approximation $\tilde{X}(5) = \frac{1}{1000} \sum_{i=0}^{1000} \tilde{W}(t_i)$ of random variable $X(5)$. Compare the empirical distribution of $\tilde{X}(5)$ with the distribution of $X(5)$ from Exercise 8-2 using Kolmogorov-Smirnov test and Kolmogorov's distance as a measure.