Exercise sheet 8 (total – 20 points) till December 20, 2017

Exercise 8-1 (6 points)
Let \( W = \{W(t), \ t \geq 0\} \) be a Wiener process. Define the process of the maximum as \( M = \{M_t = \max_{s \in [0,t]} W(s), t \geq 0\} \). Show:

1. (2 points) The probability density of \( M_t \) is given by
   \[
   f_{M_t}(x) = \sqrt{\frac{2}{\pi t}} \exp \left( -\frac{x^2}{2t} \right) \mathbb{1} \{x \geq 0\}.
   \]

2. (2 points) The expectation and variance of \( M_t \) are given via
   \[
   \mathbb{E}M_t = \sqrt{\frac{2t}{\pi}}, \ \text{Var}M_t = t(1 - \frac{2}{\pi}).
   \]

3. (2 points) Let \( \tau(x) := \min\{s \geq 0, W(s) = x\} \) be the first time when \( W \) attains the value \( x \). Prove that \( \tau(x) \) has Lévy distribution with density
   \[
   f_\tau(x)(y) = \frac{x}{\sqrt{2\pi y^2}} \exp \left( -\frac{x^2}{2y} \right) \mathbb{1} \{y \geq 0\}.
   \]
   Show that \( \mathbb{E} \tau(x) = \infty \).

Exercise 8-2 (3 points)
Let \( X = \{X(t) := \int_0^t W(s)ds, t \geq 0\} \), where \( W \) is the Wiener process. Find the distribution of random variable \( X(t) \) for \( t > 0 \).

Hint: Recall that a limit of Gaussian random variables is also Gaussian.

Exercise 8-3 (11 points)

1. (6 points) Write a program in R which simulates the trajectory of the Wiener process on \([0,T]\)
   - (a) by using an approximation with Schauder functions and input parameters \( t, T \) and \( m \), where \( t \) is a finite dimensional vector of locations in \([0,T]\) and \( m \) is the cut-off parameter of the series expansion;
   - (a) by using the independence and the distribution of the increments of \( W \), with input parameter \( t \) defined as in (a);
   - (c) by using Donsker’s invariance principle: for every \( n \in \mathbb{N} \) we define \( \{\tilde{W}^n(t), \ t \in [0,1]\} \) by
     \[
     \tilde{W}^n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} + (nt - \lfloor nt \rfloor)\frac{Z_{\lceil nt \rceil}}{\sqrt{n}},
     \]
     where \( S_i = Z_1 + \ldots + Z_i, \ i \geq 1, \ S_0 = 0, \) where \( Z_1, Z_2, \ldots \) are i.i.d. r.v.’s with \( \mathbb{E}Z_i = 0, \text{Var}Z_i = 1 \). Experiment with different distributions of \( Z_i, i \geq 1 \) (at least three).

2. (1 point) Simulate 500 trajectories of a Wiener process on \([0,5]\) in cases (a)-(c). Take \( m = 10 \) in (a) and \( t = (t_0, \ldots, t_{1000}) \) in (a)-(b), where \( t_0 = 0 \) and \( t_k = kT/1000, k = 1, \ldots, 1000 \). Take \( n = 1000 \) in (c). Plot one trajectory for each case.

3. (2 points) For each simulated trajectory \( \tilde{W} \) compute the approximation \( \tilde{M} = \max_{t=0,\ldots,1000} \tilde{W}(t) \) of random variable \( M \). Compare the empirical distribution of \( \tilde{M} \) with the distribution of \( M \) from Exercise 8-1 using Kolmogorov-Smirnov test and Kolmogorov’s distance as a measure.

4. (2 points) For each simulated trajectory \( \tilde{W} \) compute the approximation \( \tilde{X} = \frac{1}{\sqrt{n}} \sum_{i=0}^{1000} \tilde{W}(t) \) of random variable \( X \). Compare the empirical distribution of \( \tilde{X} \) with the distribution of \( X \) from Exercise 8-2 using Kolmogorov-Smirnov test and Kolmogorov’s distance as a measure.