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Exercise sheet 9 (total – 19 points) till January 10, 2018

Exercise 9-1 (3 points)

Let $W = \{W(t),\ t \in [0,1]\}$ be a Wiener process and $L = \operatorname{argmax}_{t\in[0,1]} W(t)$. Show that

$$P(L \le x) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad x \in [0,1].$$

Hint: Use the fact that $\max_{r\in[0,t]} W(r) \overset{d}{=} |W(t)|$.

Exercise 9-2 (4 points)

Let $\{X(t), t \ge 0\}$ Let W and N be the independent Wiener process and Poisson process with intensity λ, respectively. Prove that the process $\{X(t) = \sigma W(N(t)), t \ge 0\}$, $\sigma > 0$ has the same finite-dimensional distributions as a compound Poisson process $\{Y(t), t \ge 0\}$ with Lévy measure

$$\nu(dx) = \frac{\lambda}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right), x \in \mathbb{R}.$$

Exercise 9-3 (4 points)

For the following distribution functions prove that they are infinitely divisible and find parameters (a, b, ν) in the Lévy-Khintchin representations of characteristic functions.

1. (2 point) Cauchy distribution.
2. (2 point) Gamma distribution.

Exercise 9-4 (3 points)

Show that the distribution of a bounded random variable Z is infinitely divisible if and only if Z is constant.

Exercise 9-5 (4 points)

Show that the function $\varphi : \mathbb{R} \to \mathbb{C}$ given by

$$\varphi(z) = \exp(\psi(z)), \qquad \psi(z) = 2 \sum_{k=-\infty}^{\infty} (\cos(2^k z) - 1)$$

is the characteristic function of an infinitely divisible random variable.

Hint: Use the Lévy-Khintchine representation with Lévy measure $\nu(\pm 2^k) = 2^{-k}, k \in \mathbb{Z}$.