Long range dependence of heavy tailed random functions

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Abstract

We introduce a definition of long range dependence of random processes and fields on an index space $T \subseteq \mathbb{R}^d$ in terms of integrability of the covariance of indicators that a random function exceeds any given level. This definition is particularly designed to cover the case of random functions with infinite variance. We show the value of this new definition and its connection to limit theorems on some examples including subordinated Gaussian as well as random volatility fields and time series.

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1 Introduction

Let $X = \{X_t, t \in T\}$ be a stationary random field on an index subset $T$ of $\mathbb{R}^d, d \geq 1$, defined on an abstract probability space $(\Omega, \mathcal{F}, P)$. If $X_0$ is square integrable the property of long range dependence can be defined as

$$\int_T |C(t)| \, dt = +\infty$$

(1)

where $C(t) = \text{Cov}(X_0, X_t), t \in T$. There are also other definitions e.g. in terms of spectral density of $X$ being unbounded at zero, growth comparison of partial sums (Allan sample variance), the order of the variance of sums going to infinity, etc., see the modern reviews in [13], [4], [37] for processes and [23] for random fields. All these approaches are not equivalent to each other.

More importantly, there is no unified approach to define long memory property if $X$ is heavy tailed, that is with infinite variance. Many authors
use the phenomenon of phase transition in certain parameters of the field (such as stability index, Hurst index, heaviness of the tails, etc.) regarding their different limiting behaviour. To give just few examples, we mention [41] for the subordinated heavy-tailed Gaussian time series whereas [36], [34] and [33] consider the extreme value behaviour of partial maxima of stable random processes and fields and a connection with their ergodic properties. Papers [10, 29] analyze different measures of dependence (such as \( \alpha \)-spectral covariance) for linear random fields with infinite variance lying in the domain of attraction of a stable law. Those are used to define various types of memory and prove corresponding limit theorems for partial sums. The drawback of all these approaches is that they are often statistically not tractable and tailored for a particular class of random functions.

The goal of our paper is to give a simple uniform view into long range dependence which applies to any stationary (light or heavy tailed) random field \( X \); see Definition 3.1. The right statistic to study appears to be the volume of excursion sets of the field. As explained in Section 3.1, we link our definition of long memory to limit theorems for level sets (the full picture is presented later in Section 4). In Section 3.2 we show that all rapidly mixing random fields are short range dependent in the sense of new definition. No moment assumptions are needed there. In Section 3.3, the sufficient conditions for a subordinated Gaussian (possibly heavy-tailed) random field to be short or long range dependent are given. In the next section, the same is done for stochastic volatility random fields of the form \( X_t = G(Y_t)Z_t \). Different sources of long range dependence are described.

Section 4 explains how the new definition is linked to the limiting behaviour of integrals \( \int_{W_n} g(X_t)dt \) as \( n \to \infty \). First, in case of \( g(x) = x \) we indicate in Sections 4.1.1 and 4.1.2 that our definition of long range dependence and non-standard behaviour in limit theorems for the empirical mean do not coincide. This is not surprising, since the definition is supposed to capture behaviour in limit theorems for excursion sets. This is illustrated in Section 4.1.3 and Section 4.2, where we have to develop limiting theory for integral functionals of random volatility models.

This includes the case of limit theorems for the volume of level sets of \( X \). For better readability, proofs of the most of results are moved to Appendix.

2 Preliminaries

Recall that \( T \) is a subset of \( \mathbb{R}^d \). Let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), and let \( \nu_d(\cdot) \) be the \( d \)-dimensional Lebesgue measure. Let \( \| \cdot \| \) be a norm in the Euclidean space \( \mathbb{R}^d \). For two functions \( f, g : \mathbb{R} \to \mathbb{R} \) we write \( f(x) \sim g(x), x \to \infty \).
$a$ if $\lim_{x \to a} f(x)/g(x) = 1$ where $g(x) \neq 0$ for all $x \in \mathbb{R}$. Let $\langle f, g \rangle = \int f(x)g(x)\,dx$ be the inner product in the space $L^2(\mathbb{R})$ of square integrable functions. Additionally, we shall make use of the inner product $\langle f, g \rangle_\varphi = \int f(x)g(x)\varphi(x)\,dx$ in the space $L^2_\varphi(\mathbb{R})$ of functions which are square integrable with the weight $\varphi$, where $\varphi$ is the standard normal density. For a finite measure $\mu$ on $\mathbb{R}$, let $\text{supp}(\mu)$ be its support, i.e., the largest measurable subset of positive $\mu$-measure in $\mathbb{R}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We say that $X = \{X_t, t \in T\}$ is a white noise if it consists of i.i.d. random variables $X_t$. Let $F_X(x) = P(X_0 \leq x)$ and $\bar{F}_X(x) = 1 - F_X(x)$ be the marginal cumulative distribution probability function or the tail distribution function of $X$, respectively. Let $F_{X,Y}(x,y) = P(X \leq x,Y \leq y)$, $x,y \in \mathbb{R}$ be the bivariate distribution function of a random vector $(X,Y)$. Later on we make use of the known formula
\begin{equation}
\text{Cov}(X,Y) = \mathbb{E}(\text{Cov}(X,Y|A)) + \text{Cov}(\mathbb{E}(X|A), \mathbb{E}(Y|A))
\end{equation}
for any $\sigma$–algebra $A \subset \mathcal{F}$.

A random function $X = \{X_t, t \in T\}$ is called associated (A) if
\[ \text{Cov}(f(X_I), g(X_J)) \geq 0 \]
for any finite subset $I \subset T$ and for any bounded coordinatewise non–decreasing Borel functions $f, g : \mathbb{R}^{|I|} \to \mathbb{R}$ where $X_I = \{X_t, t \in I\}$. $X$ is called positively (PA) or negatively (NA) associated if $\text{Cov}(f(X_I), g(X_J)) \geq 0$ ($\leq 0$, resp.) for all finite disjoint subsets $I, J \subset T$, and for any bounded coordinatewise non–decreasing Borel functions $f : \mathbb{R}^{|I|} \to \mathbb{R}$, $g : \mathbb{R}^{|J|} \to \mathbb{R}$, see e.g. [6].

We use the notation $B \sim S_\alpha(\sigma, 1, 0)$ for a random variable $B$ which is $\alpha$-stable and totally skewed with scale parameter $\sigma > 0$, cf. [38].

## 3 Long range dependence

Consider a real–valued stationary random field $X$. Introduce
\[ \text{Cov}_X(t, u, v) = \text{Cov}(1(X_0 > u), 1(X_t > v)), \quad t \in T, \ x, v \in \mathbb{R}. \]
It is always defined as the indicators involved are bounded functions.

**Definition 3.1.** A random field $X$ is called short range dependent (s.r.d.) if for any finite measure $\mu$ on $\mathbb{R}$
\[ \sigma^2_{\mu,X} := \int \int T_{\mathbb{R}^2} |\text{Cov}_X(t, u, v)| \mu(du) \mu(dv) \, dt < +\infty. \]
X is long range dependent (l.r.d.) if there exists a finite measure \( \mu \) on \( \mathbb{R} \) such that \( \sigma_{\mu,X}^2 = +\infty \). For discrete parameter random fields (say, if \( T \subseteq \mathbb{Z}^d \)), the \( \int_T dt \) in the above lines should be replaced by \( \sum_{t \in T} \).

3.1 Motivation and explanation

Assume that \( X \) is wide sense stationary with covariance function \( C(t) = \text{Cov}(X_0, X_t) \), \( t \in T \), and moreover,

\[
\text{Cov}_X(t, u, v) \geq 0 \quad \text{or} \quad \leq 0 \quad \text{for all} \quad t \in T, \ u, v \in \mathbb{R}.
\]  

(3)

Examples of \( X \) with this property are all PA or NA- random functions. Applying [24, Lemma 2], we have (the equality is originally attributed to Hoeffding (1940))

\[
C(t) = \int_{\mathbb{R}^2} \text{Cov}_X(t, u, v) \, du \, dv.
\]  

(4)

Then, \( X \) is long range dependent if

\[
\int_T |C(t)| \, dt = \int_T \int_{\mathbb{R}^2} |\text{Cov}_X(t, u, v)| \, du \, dv \, dt = +\infty
\]

which agrees with the classical definition.

However, Definition 3.1 suggests to integrate \( |\text{Cov}_X(t, u, v)| \) with respect to a finite measure \( \mu \times \mu \) instead of Lebesgue measure \( du \, dv \). This has to do with the asymptotic behavior of volumes of excursions of \( X \) above levels \( u, v \). Recall the functional central limit theorem for normed volumes of excursion sets of \( X \) at level \( u \) proven in [28] (see also [42, Theorem 9, p. 234] for a generalization of this result to fields without a finite second moment).

Namely, for a large class of weakly dependent stationary random fields \( X \in \mathcal{A} \) on \( \mathbb{R}^d \), the function \( \int \text{Cov}_X(t, u, v) \, dt, \ u, v \in \mathbb{R} \) is the covariance function of the centered Gaussian process which appears as a limit of

\[
\frac{\nu_d \left( \{ t \in [0, n]^d : X_t > u \} \right) - n^d \bar{F}_X(u)}{n^{d/2}}, \quad u \in \mathbb{R}, \quad n \to \infty
\]  

(5)

in Skorokhod topology \( \mathcal{D}(\mathbb{R}) \). By the continuous mapping theorem, it holds

\[
\int_{\mathbb{R}} \nu_d \left( \{ t \in [0, n]^d : X_t > u \} \right) \mu(du) - n^d \int_{\mathbb{R}} \bar{F}_X(u) \mu(du) \to N(0, \sigma_{\mu,X}^2) \quad (6)
\]

as \( n \to \infty \) for any finite measure \( \mu \) with \( \sigma_{\mu,X}^2 \) as in Definition 3.1. So \( X \) is s.r.d. if the asymptotic covariance \( \sigma_{\mu,X}^2 \) in the central limit theorem (6) is
finite for any finite integration measure \( \mu \) prescribing the choice of levels \( u \). On the contrary,

\[
\sigma^2_{\mu,X} = +\infty \tag{7}
\]

for \( \mu = \delta_{\{u_0\}} \) means no central limit theorem (CLT) for the excursion volume of \( X \) at level \( u_0 \). If the measure \( \mu \) in (7) is discrete concentrated at a finite number of levels \( u_i, i = 1, \ldots, m \), this means no multivariate CLT for the excursion volumes at these levels. Finally, relation (7) for diffuse finite measures \( \mu \) yields no functional CLT for normed volumes (5) of level sets of \( X \) at levels \( u \in \text{supp}(\mu) \). In these three cases, we say that \( X \) is l.r.d.

In terms of potential theory, the value \( \sigma^2_{\mu,X} \) in Definition 3.1 is the energy of measure \( \mu \) with symmetric kernel \( K(u,v) = \int T |\text{Cov}_X(t,u,v)| dt \), cf. [22, p. 77 ff.].

3.2 Checking the short or long range dependence

Denote by \( P_{\mu}(\cdot) = \mu(\cdot)/\mu(\mathbb{R}) \) the probability measure associated with the finite measure \( \mu \) on \( \mathbb{R} \). Let \( U, V \) be two independent random variables with distribution \( P_{\mu} \). Then the variance \( \sigma^2_{\mu,X} \) from Definition 3.1 rewrites

\[
\frac{\sigma^2_{\mu,X}}{\mu^2(\mathbb{R})} = \int_T \mathbb{E}|\text{Cov}_X(t,U,V)| dt = \int_T \mathbb{E}|F_{X_{u_0},X_{u_0}}(U,V) - F_X(U)F_X(V)| dt.
\]

This relation may be sometimes useful to check the s.r.d. of \( X \) showing the finiteness of \( \sigma^2_{\mu,X} \) for any i.i.d. random variables \( U \) and \( V \). In particular, it may be used to check the s.r.d. statistically when we plug in the empirical uni- and bivariate distribution functions into this formula as estimates of \( F_{X_{u_0},X_{u_0}} \) and \( F_X \) and then use procedures similar to the inference of the extreme value coefficient.

By stationarity of \( X \), it holds \( \text{Cov}_X(t,u,v) = \text{Cov}_X(-t,u,v) \) for any \( t, -t \in T, u, v \in \mathbb{R} \). Hence, for \( T = \mathbb{R} \) it is enough to check that

\[
\int_0^\infty |\text{Cov}_X(t,u_0,u_0)| dt = +\infty
\]

for some \( u_0 \in \mathbb{R} \). For \( T = \mathbb{Z} \) it is sufficient to consider \( \sum_{t=1}^\infty |\text{Cov}_X(t,u_0,u_0)| = +\infty \).

In certain cases, \( \text{Cov}_X(t,u,v) \) can be computed explicitly, for instance, if \( X \) is a centered stationary unit variance Gaussian random field with covariance function \( C(t) \). Then we have

\[
\text{Cov}_X(t,u,v) = \frac{1}{2\pi} \int_0^{C(t)} \frac{1}{\sqrt{1-r^2}} \exp \left\{ -\frac{u^2 - 2ruv + v^2}{2(1-r^2)} \right\} dr, \tag{8}
\]
see [7, Lemma 2].

**Link between short-range dependence and mixing.** Let \( \mathcal{U}, \mathcal{V} \) be two sub-\( \sigma \)-algebras of \( \mathcal{F} \). Introduce the \( z \)-mixing coefficient \( z(\mathcal{U}, \mathcal{V}) \) (where \( z \in \{\alpha, \beta, \phi, \psi, \rho\} \)) as in [12, p.3]. For instance, it is given for \( z = \alpha \) by

\[
\alpha(\mathcal{U}, \mathcal{V}) = \sup \{ |P(U \cap V) - P(U)P(V)| : U \in \mathcal{U}, V \in \mathcal{V} \}.
\]

Let \( X = \{X_t, t \in T\} \) be a random function. Let \( X_C = \{X_t, t \in C\}, C \subset T, \) and \( \sigma_{X_C} \) be the \( \sigma \)-algebra generated by \( X_C \). If \( |C| \) is the cardinality of a finite set \( C \) then the \( z \)-mixing coefficient of \( X \) is given by

\[
z_X(k, u, v) = \sup\{z(\sigma_{X_A}, \sigma_{X_B}) : d(A, B) \geq k, |A| \leq u, |B| \leq v\},
\]

where \( u, v \in \mathbb{N} \) and \( d(A, B) \) is the Hausdorff distance between finite subsets \( A \) and \( B \) generated by the metric on \( \mathbb{R}^d \). The interrelations between different mixing coefficients \( z_X, z \in \{\alpha, \beta, \phi, \psi, \rho\} \) are given e.g. in [12, p.4, Proposition 1].

We state the result that links mixing properties and short-range dependence. The field \( X \) may be non–Gaussian and have infinite variance.

**Theorem 3.2.** Let \( X = \{X_t, t \in T\} \) be a stationary random field with \( z \)-mixing rate satisfying \( \int_T z_X(\|t\|, 1, 1) \, dt < +\infty \) where \( z \in \{\alpha, \beta, \phi, \psi, \rho\} \). Then \( X \) is short range dependent with

\[
\int_T \int_{\mathbb{R}^2} |\operatorname{Cov}(X(t, u, v)| \, du \, dv \, dt \leq 8 \int_T z_X(\|t\|, 1, 1) \, dt \cdot \mu^2(\mathbb{R}) < +\infty.
\]

**Proof.** Without loss of generality, prove the result for \( \alpha \)-mixing \( X \). Introduce random variables \( \xi(u) = 1(X_0 > u) \), \( \eta(v) = 1(X_t > v) \), where \( t \in T \), \( u, v \in \mathbb{R} \). Then, by the covariance inequality in [12, p. 9, Theorem 3] connecting the covariance of random variables with their mixing rates we have

\[
\int_T \int_{\mathbb{R}^2} |\operatorname{Cov}(X(t, u, v)| \mu(du)\mu(dv) dt = \int_T \int_{\mathbb{R}^2} |\operatorname{Cov}(\xi(u), \eta(v))| \mu(du)\mu(dv) dt
\leq 8 \int_T \alpha(\sigma_{X_0}, \sigma_{X_t}) dt \int_{\mathbb{R}^2} \|\xi(u)\|_{\infty} \|\eta(v)\|_{\infty} \mu(du)\mu(dv)
\leq 8 \int_T \alpha_X(\|t\|, 1, 1) dt \cdot \mu^2(\mathbb{R}) < +\infty
\]

where \( ||Y||_{\infty} = \operatorname{Ess-sup}(Y) \), \( 0 \leq \xi(u), \eta(v) \leq 1 \) a.s. for all real \( u, v \). \( \Box \)
To illustrate the above theorem, we let $Y = \{Y_t, t \in \mathbb{N}\}$ to be a stationary a.s. non-negative $\psi$–mixing random sequence with univariate cumulative distribution function $F_Y$ and $\int_{\mathbb{R}^d} \psi_Y(||t||, 1, 1) dt < +\infty$. Examples of $\psi$–mixing random sequences can be found e.g. in [12, Example 4, p.19] (see also references therein), [16, Theorem 2.2], [31, Proof of Claim 2.5], [5], [39, p. 54-55]. Let $F_Z^{-1}$ be the quantile function of a random variable $Z$ with $E|G|^2 = +\infty$. Set $G(x) = F_Z^{-1}(F_X(x))$, $x \geq 0$, then $X_t = G(Y_t)$, $t \in \mathbb{N}$ is $\psi$–mixing as well. Moreover, it is s.r.d. by the last theorem and has infinite variance because of $X_0 \overset{d}{=} Z$.

**Remark 3.3.** For a Gaussian $\phi$–mixing random function $X$, the statement of Theorem 3.2 is trivial, since such $X$ is $m$–dependent [17, Theorem 17.3.2], and the integral $\int_0^\infty |\text{Cov}_X(t, u, v)| dt$ in Definition 3.1 is bounded by $2m$ for any $u, v \in \mathbb{R}$.

### 3.3 Subordinated Gaussian random functions

Recall that $\varphi(x)$ is the density of the standard normal law. We use the notation $\Phi(x)$ for its c.d.f. Introduce the Hermite polynomials $H_n$ of degree $n$, $n \in \mathbb{N}_0$ by

$$H_n(x) = (-1)^n \varphi^{-1}(x) \varphi^{(n)}(x)$$

where $\varphi^{(n)}$ is the $n$-th derivative of $\varphi$. Clearly, it holds

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \ldots$$

For even orders $n$, Hermite polynomials are even functions, whereas for odd $n$ they are odd functions. It is well known that Hermite polynomials form an orthogonal basis in $L^2_\varphi(\mathbb{R})$. For any function $f \in L^2_\varphi(\mathbb{R})$ with $\langle f, 1 \rangle_\varphi = 0$ let $\text{rank}(f) = \min\{n \in \mathbb{N} : \langle f, H_n \rangle_\varphi \neq 0\}$ be the Hermite rank of $f$.

Furthermore, the Hermite rank can also be defined for functions with infinite variance, as long as $E|G^{1+\theta}(Y)| < \infty$ for some $\theta \in (0, 1)$; see [41] or [4, Section 4.3.5].

Let $Y = \{Y_t, t \in T\}$ be a stationary centered Gaussian real-valued random function with $\text{Var} Y_t = 1$ and $\rho(t) = \text{Cov}(Y_0, Y_t)$, $t \in T$. The subordinated Gaussian random function $X$ is defined by $X_t = G(Y_t)$, $t \in T$, where $G : \mathbb{R} \to \text{Im}(G) \subseteq \mathbb{R}$ is a measurable function. First assume that $X$ is square integrable. Let $C(t) = \text{Cov}(X_0, X_t)$, $t \in T$.

The following lemma is proven in [35, Lemma 10.2]:

7
Lemma 3.4. Let $Z_1, Z_2$ be standard normal random variables with $\rho = \text{cov}(Z_1, Z_2)$, and let $F$, $G$ be functions satisfying $\mathbb{E}F^2(Z_1), \mathbb{E}G^2(Z_1) < +\infty$. Then

$$\text{Cov}(F(Z_1), G(Z_2)) = \sum_{k=1}^{\infty} \frac{\langle F, H_k \rangle \varphi \langle G, H_k \rangle \varphi}{k!} \rho^k.$$ 

Assuming $\rho(t) \geq 0$ for all $t \in T$ and applying this lemma to our subordinated process $X = G(Y)$ we get that it is s.r.d. if

$$\int_T |C(t)| \, dt = \sum_{k=1}^{\infty} \frac{\langle G, H_k \rangle^2}{k!} \int_T \rho^k(t) \, dt < +\infty. \quad (9)$$

We shall see that an analogous result holds also if $X$ has no finite second moment. Introduce the condition

$$(\rho) \ |\rho(t)| < 1 \text{ for all } t \neq 0 \text{ if } T \text{ is countable and for } \nu_d-\text{almost every } t \in T \text{ if } T \text{ is uncountable.}$$

The following result gives the conditions for s.r.d of a subordinated Gaussian random field. Its proof is given in Appendix.

Theorem 3.5. Let $Y$ be a Gaussian random function introduced above. Let $X$ be a subordinated Gaussian random function defined by $X_t = G(Y_t)$, $t \in T$, where $G$ is a right-continuous strictly monotone (increasing or decreasing) function. Assume that the condition $(\rho)$ holds. Let

$$b_k(\mu) = \left( \int_{\text{Im}(G)} H_k(G^{-}(u)) \varphi(G^{-}(u)) \, \mu(du) \right)^2 \quad (10)$$

where $G^{-}$ is the generalized inverse of $G$ if $G$ is increasing or of $-G$ if $G$ is decreasing. Then $X$ is s.r.d. if

$$\sum_{k=1}^{\infty} \frac{b_{k-1}(\mu)}{k!} \int_T |\rho(t)|\rho(t)^{k-1} \, dt < +\infty \quad (11)$$

for any finite measure $\mu$ on $\mathbb{R}$.

Corollary 3.6. Assume that the conditions of Theorem 3.5 hold.

1. Let $\mu(dx) = f(x) \, dx$ for an $f \in L^1(\mathbb{R})$, $f(x) \geq 0$ for all $x \in \mathbb{R}$. If $G \in C^1(\mathbb{R})$ and $\text{Im}(G) = \mathbb{R}$ then $b_k(\mu) = \langle G''f(G), H_k \rangle^2$, $k \in \mathbb{N}$. In this case, all coefficients $b_k(\mu)$ are finite if for some $\theta \in (0, 1)$ it holds $E[|G'(Y_0)f(G(Y_0))|^{1+\theta}] < +\infty$. If $G'f(G)$ is an even function then $b_k(\mu) = 0$ for all natural odd $k$. 
2. If $X_t = G(|Y_t|)$, $t \in T$, then the s.r.d. condition (11) modifies to

$$
\sum_{k=1}^{\infty} \frac{b_{2k-1}(\mu)}{(2k)!} \int_T \rho(t)^{2k} \, dt < +\infty. \quad (12)
$$

**Remark 3.7.** Based on Theorem 3.5 and Corollary 3.6, l.r.d. conditions can also be formulated, e.g.,

1. $X = G(Y)$ is l.r.d. if $\exists u_0 \in \mathbb{R} : b_k(\delta_{\{u_0\}}) < +\infty$ for all $k$ and series (11) diverges to $+\infty$.

2. If the initial process $Y$ is s.r.d. then all powers of $\rho$ are integrable on $T$ and the long memory of $X = G(|Y|)$ can only come from function $G$. This can happen e.g. if its Fourier coefficients $b_k(\mu)$ decrease to zero slowly enough. Conversely, if $Y$ is l.r.d., $0 < b_{2k-1}(\mu) < +\infty$ for all $k \in \mathbb{N}$ and some finite measure $\mu$, there exists $k \in \mathbb{N}$ s.t. $\int_T \rho^{2k}(t) \, dt = +\infty$ then $X$ is l.r.d.

Let us illustrate the last point of Remark 3.7 by an example.

**Example 3.8.** Let $G(x) = e^{x^2/(2\alpha)}$, $\alpha > 0$, $T = \mathbb{R}^d$. Then it is easy to see that

$$
P(|X_0| > x) = L(x)x^{-\alpha},
$$

where $L(x) = \sqrt{2/(\pi \log x)}$. For $\alpha \in (1, 2]$, it holds $\mathbb{E} X_0 < \infty$, $\mathbb{E} X_0^2 = +\infty$.

To compute $b_{2k-1}(\mu)$, we notice that

$$
\sqrt{b_{2k-1}(\mu)} = \frac{1}{\sqrt{2\pi}} \int_1^\infty u^{-\alpha} H_{2k-1}(\sqrt{2\alpha \log u}) \, \mu(du), \quad k \in \mathbb{N}.
$$

Using the upper bound $|H_{2k-1}(x)| \leq xe^{x^2/(2k-1)!!}/4$, $x \geq 0$ from [1, p. 787] one can show that

$$
b_{2k-1}(\mu) \leq \frac{\alpha}{16\pi} [(2k-1)!!]^2 \left( \int_1^\infty u^{-\alpha/2} \sqrt{\log u} \, \mu(du) \right)^2
\leq \frac{\alpha}{4\pi} \mu^2([1, +\infty)) [(2k-1)!!]^2 < +\infty
$$

for all $k \in \mathbb{N}$. Now by Stirling’s formula [3, Theorem 1.4.2], we get

$$
\frac{[(2k-1)!!]^2}{(2k)!} \sim \frac{C_3}{\sqrt{k}}, \quad k \to +\infty
$$

(13)
for $C_3 > 0$, so
\[
\frac{b_{2k-1}(\mu)}{(2k)!} = O\left(\frac{1}{\sqrt{k}}\right), \quad k \to +\infty.
\] (14)

Assume that $\rho(t) \sim \|t\|^{-\eta}$ as $\|t\| \to +\infty$, $\eta > 0$. Then $X = e^{Y^2/(2\alpha)}$, $\alpha > 0$ is

- l.r.d. if $\eta \in (0, d/2]$ since then $\int_{\mathbb{R}^d} \rho^2(t) \, dt = +\infty$,
- s.r.d. if $\eta > d/2$ since then we have
\[
\int_{\mathbb{R}^d} \rho^{2k}(t) \, dt = O(k^{-1}) \quad \text{as} \quad k \to +\infty,
\]

and the series (12) behaves as $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} < +\infty$.

Here the source of long memory of $X$ is the l.r.d. field $Y$. If $\alpha > 2$ the variance of $X_0$ is finite, and our results agree with Definition (1) by relation (9) if we notice that $\operatorname{rank}(G) = 2$.

Note that for $\eta \in (d/2, d)$ the Gaussian field $Y$ is l.r.d. but the subordinated field $X = e^{Y^2/(2\alpha)}$ is s.r.d.

### 3.4 Stochastic volatility models

We present a way of constructing random fields with long memory by introducing a random volatility $G(Y_t)$ (being a deterministic function of a random scaling field $Y = \{Y_t, t \in T\}$) of a random field $Z = \{Z_t, t \in T\}$ where $Y$ and $Z$ are independent. An overview of random volatility models and their applications in finance can be found in e.g. [40] and [2, Part II]. For each $t \in T$, $X_t = G(Y_t)Z_t$ is a scale mixture of $G(Y_t)$ and $Z_t$, see [43, Chapter VI, p. 345].

For a finite measure $\mu$, introduce the functional
\[
D(G(Y), Z_0) = \int_T \int_{\mathbb{R}^2} \operatorname{Cov}\left(\tilde{F}_Z(u/G(Y_0)), \tilde{F}_Z(v/G(Y_t))\right) \mu(du)\mu(dv) \, dt.
\]

**Lemma 3.9.** Let a random field $X = \{X_t, t \in T\}$ be given by $X_t = G(Y_t)Z_t$ where $Y = \{Y_t, t \in T\}$ and $Z = \{Z_t, t \in T\}$ are independent stationary random fields, $Z$ has property (3), $G : \mathbb{R} \to \mathbb{R}_+$ and $P(G(Y_t) = 0) = 0$ for
all \( t \in T \). Then

\[
\int \int_{T \times R^2} \Cov_X(t, u, v) \, \mu(du) \mu(dv) \, dt = D(G(Y), Z_0) + \int \int_{T \times R^2} \mathbb{E}[\Cov_Z(t, u/G(Y_0), v/G(Y_t))] \, \mu(du) \, \mu(dv) \, dt. \tag{15}
\]

**Proof.** Without loss of generality, assume \( G \geq 0 \). Apply relation (2) to \( \mathbf{1}(X_0 > u), \mathbf{1}(X_t > v) \) to get

\[
\Cov_X(t, u, v) = \mathbb{E}[\Cov(\mathbf{1}(Z_0 > u/G(Y_0)), \mathbf{1}(Z_t > u/G(Y_t))|Y)] + \Cov(\bar{F}_Z(u/G(Y_0)), \bar{F}_Z(v/G(Y_t))) := I_1(u, v, t) + I_2(u, v, t).
\]

Use the independence of \( Y \) and \( Z \) together with Tonelli theorem to obtain

\[
\int \int_{T \times R^2} I_1(u, v, t) \, \mu(du) \, \mu(dv) \, dt = \int \int_{T \times R^2} \mathbb{E}[\Cov_Z(t, u/G(Y_0), v/G(Y_t))] \, \mu(du) \, \mu(dv) \, dt,
\]

and hence

\[
\int \int_{T \times R^2} \Cov_X(t, u, v) \, \mu(du) \, \mu(dv) \, dt = D(G(Y), Z_0) + \int \int_{T \times R^2} \mathbb{E}[\Cov_Z(t, u/G(Y_0), v/G(Y_t))] \, \mu(du) \, \mu(dv) \, dt.
\]

\( \square \)

Let us illustrate the use of Lemma 3.9. In the first of the corollaries, the dependence of \( Z \) is intensified by scaling it with values of structureless \( G(Y_t) \).

**Corollary 3.10.** Let \( X_t = G(Y_t)Z_t, t \in T \) be a random volatility field, where \( Y = \{Y_t, t \in T\} \) and \( Z = \{Z_t, t \in T\} \) are independent stationary random fields, \( Z \) has property (3), \( G: \mathbb{R} \to \mathbb{R}_+ \) and \( P(G(Y_t) = 0) = 0 \) for all \( t \in T \). Assume that \( Y \) is a white noise, and

\[
\sup_{u,v \in \text{Supp}(\mu)} \int_{T} |\mathbb{E}[\Cov_Z(t, u/G(Y_0), v/G(Y_t))]| \, dt < +\infty \tag{16}
\]

for any finite measure \( \mu \). Then \( X \) is s.r.d.
Proof. Using the notation of the proof of Theorem 3.9, \( Y \) being a white noise means \( I_2(u, v, t) = 0 \) for any \( t \neq 0, u, v \in \mathbb{R} \). Then

\[
\int_T \int_{\mathbb{R}^2} \text{Cov}_X(t, u, v) \mu(du) \mu(dv) \, dt
\]

\[
= \int_T \int_{\mathbb{R}^2} |\mathbb{E} [\text{Cov}_Z(t, u/Y_0, v/Y_1)]| \, dt \mu(du) \mu(dv) < +\infty
\]
since \( \mu \) is finite. \( \square \)

Corollary 3.11. Let the random field \( X \) be given by \( X_t = AZ_t, t \in T \), \( |T| = +\infty \) where \( A > 0 \) a.s., \( A \) and \( Z \) are independent and \( Z \in \mathcal{PA} \) is stationary. Then \( X \) is l.r.d. if there exists \( u_0 \in \mathbb{R} : \bar{F}_Z(u_0/A) \neq \text{const} \) a.s.

The above corollary evidently holds true if e.g. \( Z_0 \sim \text{Exp}(\lambda), A \sim \text{Frechet}(1) \) for any \( \lambda > 0 \). It also clearly applies to a subgaussian random field \( X \) where \( A = \sqrt{\mathcal{B}}, B \sim S_{\alpha/2}(\cos \pi \alpha/4, 1, 0), \alpha \in (0, 2) \), and \( Z \) is a centered stationary Gaussian random field with covariance function \( C(t) \geq 0 \) for all \( t \in T \) and a non–degenerate tail \( \bar{F}_Z \). Here \( Z \) does not need to be heavy–tailed.

The following corollary describes the situation where light-tailed \( Y \) is responsible for the l.r.d. of \( X \), while \( Z \) – for heavy tails.

Corollary 3.12. For the random field \( X = \{X_t, t \in T\} \) given by \( X_t = Y_tZ_t, t \in T \), assume that random fields \( Y = \{Y_t, t \in T\} \) and \( Z = \{Z_t, t \in T\} \) are stationary and independent. Assume that \( Z_0 \) has a regularly varying tail, that is, \( P(Z_0 > x) \sim L(x)/x^\alpha \) as \( x \to +\infty \) for some \( \alpha > 0 \) where the function \( L \) is slowly varying at \( +\infty \). For \( Y_0 > 0 \) a.s. assume that \( \mathbb{E} Y_0^\delta < \infty \) and \( \mathbb{E} (Y_0^\delta Y_t^\delta) < \infty \) for some \( \delta > \alpha \) and all \( t \in T \). Let \( Y, Z \in \mathcal{PA}(\mathcal{NA}) \). Then \( X \) is l.r.d. if \( Y^\alpha = \{Y_t^\alpha, t \in T\} \) is l.r.d.

Now we scale a l.r.d. (possibly heavy–tailed) random field \( Z \) by a random volatility \( G(Y) \) being a subordinated Gaussian random field.

Lemma 3.13. Let \( X_t = G(Y_t)Z_t \) be a random field as in Lemma 3.9. Assume additionally that \( Y \) is a centered Gaussian random field with unit variance and covariance function \( \rho(t) \geq 0 \) satisfying condition \( (\rho) \). Then

\[
D(G(Y), Z_0) = \sum_{k=1}^{\infty} \frac{\left( \int_{\mathbb{R}} \langle \bar{F}_Z(u/G(\cdot)), H_k(\cdot) \rangle \mu(du) \right)^2}{k!} \int_T \rho^k(t) \, dt.
\]
The following example illustrates our definition of l.r.d. in the context of a popular long memory stochastic volatility model that is used in econometrics to model log-returns of stocks, see [4, p.70ff] and references therein.

Example 3.14. Assume that \( X = \{X_t, t \in \mathbb{Z}\} \) has a form \( X_t = e^{Y_t^2/4} Z_t \), where \( Z_t \) is a sequence of i.i.d. random variables, while \( Y_t \) is a centered stationary Gaussian long memory sequence with unit variance and covariance function \( \rho \) satisfying condition (\( \rho \)). Both sequences \( Z_t \) and \( Y_t \) are assumed to be independent from each other. From Example 3.8 we know that \( e^{Y_0^2/4} \) is regularly varying with index \( \alpha = 2 \). By Breiman’s lemma the tail distribution function of \( |X_0| \) is also regularly varying with index \( \alpha = 2 \) and hence \( X_0 \) has infinite variance. Choose \( \mu = \delta_{u_0} \) for some \( u_0 \in \mathbb{R} \). Lemmata 3.9 and 3.13 yield

\[
\sum_{t=1}^{\infty} \int \text{Cov}_X(t,u,v) \mu(du) \mu(dv) = \sum_{k=1}^{\infty} \frac{\langle \tilde{F}_Z(u_0/G), H_k \rangle^2}{k!} \sum_{t=1}^{\infty} \rho^k(t),
\]

where \( G(x) = e^{x^2/4} \). Since \( \tilde{F}_Z(u_0/G) \) is symmetric, monotone nondecreasing and bounded we get \( \langle \tilde{F}_Z(u_0/G), H_k \rangle = 0 \) for all odd \( k \), and it is finite for all even \( k \in \mathbb{N} \). Moreover, by Lemma 4.1, 2) we have rank \( (\tilde{F}_Z(u_0/G)) = 2 \). It is clear then that \( X \) is l.r.d. if \( \sum_{t=1}^{\infty} \rho^2(t) = +\infty \). In particular, if \( \rho(t) \sim |t|^{-\eta} \) as \( |t| \to \infty \), then l.r.d. occurs if \( \eta \in (0, 1/2] \).

4 Limit theorems

In this section, we investigate connections between Definition 3.1 and limit theorems for random volatility and subordinated Gaussian random fields. We focus on the volume of their excursion sets.

We start with a technical lemma which will play the major role later on.

Lemma 4.1. Let \( Y, Z \) be independent random variables such that \( Y \sim N(0,1) \). For any monotone right-continuous non-constant function \( G : \mathbb{R} \to \mathbb{R}_\pm \) with \( \nu_1 (\{x \in \mathbb{R} : G(x) = 0\}) = 0 \), consider the functions \( \tilde{G}(y) = G(|y|) \) and

\[
\zeta_{G,Z,u}(y) = \mathbb{E}[1\{G(y)Z > u\}] - P(G(Y)Z > u), \quad y \in \mathbb{R}
\]

for a fixed \( u > 0 \) if \( G \geq 0 \) and \( u < 0 \) if \( G \leq 0 \). Then the following holds:

1. Let \( G : \mathbb{R} \to \mathbb{R}_\pm \) be as above such that \( \mathbb{E}|G(Y)|^{1+\theta} < +\infty \) for some \( \theta \in (0,1] \). Then rank \( (G) = \text{rank} (\zeta_{G,1,u}) = \text{rank} (\zeta_{G,Z,u}) = 1 \).
2. Let $G : \mathbb{R}_+ \to \mathbb{R}_\pm$ be as above such that $E |\tilde{G}(Y)|^{1+\theta} < +\infty$ for some $\theta \in (0, 1]$, $G^-(u) \neq 0$. Then $\text{rank}(\tilde{G}) = \text{rank}(\zeta_{\tilde{G},1,u}) = \text{rank}(\zeta_{\tilde{G},Z,u}) = 2$.

Remark 4.2.  
1. If $Z \equiv 1$ the assertion of Lemma 4.1, 1) holds under milder assumptions on $G$ and $u$. Thus, let $G : \mathbb{R} \to \mathbb{R}$ be a monotone right–continuous non–constant function such that $E |G(Y)|^{1+\theta} < +\infty$ for some $\theta \in (0, 1]$. Then for any $u \in \mathbb{R} \text{ rank}(G) = \text{rank}(\zeta_{G,1,u}) = 1$.

2. The assumption of nonnegative or nonpositive $G$ is essential to the statement $\text{rank}(\zeta_{G,Z,u}) = 1$ of Lemma 4.1, 1) since for $G(y) = y$ and symmetric $Z$ we have $E[Y1\{YZ > u\}] = 0$, so the Hermite rank of $\zeta_{G,Z,u}$ is greater than 1. Similarly, one can construct examples of functions $G$ with $\text{rank}(\zeta_{G,Z,u}) > 2$ for some $u \in \mathbb{R}$ if the assumptions of Lemma 4.1, 2) do not hold. For instance, $G^-(u) = 0$ means that $\text{rank}(\zeta_{G,Z,u}) \geq 4$.

3. If $G$ is nonnegative or nonpositive and $u = 0$ then it is easily seen that $\zeta_{G,Z,0} \equiv 0$ and, formally speaking, its Hermite rank is infinite.

4.1 Link between l.r.d. and limit theorems

How does Definition (3.1) connect to limit theorems for random functions? In order to answer to this question, we have to specify the statistic whose limiting behaviour we consider.

From now on, we assume the random function $X$ to be measurable. It will be shown in this section that the limiting behaviour of the usual empirical mean $\int_{W_n} X_t \, dt/\nu_d(W_n)$ as $n \to +\infty$ cannot be directly related to this definition. Instead, the volume of excursions of $X$ over a finite fixed level $u$ becomes a more appropriate statistic to look at.

In what follows, $L$ will indicate a slowly varying function at infinity, that can be different at each of its occurrences.

4.1.1 Empirical mean: finite variance case

If the homogeneous random field $X$ is not long range dependent then one should expect that

$$n^{-d/2} \int_{W_n} (X_t - E[X_0]) \, dt$$

converges to a normal limit as $n \to \infty$, where $W_n = n \cdot W$ and $W \subset \mathbb{R}^d$ is a convex body of positive volume containing the origin in its interior. This has been done in the literature under some additional weak dependence.
assumptions, like $\alpha$-mixing (see [8, 15, 18, 20]) or quasi-association [19]. If $X = G(Y)$ is a subordinated Gaussian isotropic random field, $\rho(t)$ is the correlation function of $Y$ and the Hermite rank of $G$ is $q$ then one requires $\rho \in L^q(\mathbb{R}^d)$, see [26, Theorem 1].

If the random field $X$ is long range dependent then one should expect either a non-central limit theorem or a central limit theorem with normalization different from $n^{-d/2}$. To illustrate this, let us start with well-understood time series ($d = 1, T = \mathbb{Z}$). If $\{X_t, t \in \mathbb{Z}\}$ has a form $X_t = G(Y_t)$, where $Y$ is a long memory Gaussian process (i.e. the Gaussian process with non-summable covariances) and $G$ is a function of Hermite rank 1 such that $\mathbb{E}[G(Y_0)] = 0$ and $\text{Var}(G(Y_0)) < \infty$, then the covariances $\text{Cov}(G(Y_0), G(Y_t))$ are not summable, $\text{Var}(S_n) = \text{Var}(\sum_{t=1}^{n} X_t)$ grows at rate faster than $n$ and under additional technical assumptions $S_n$ converges to a normal limit with a normalization greater than $\sqrt{n}$. If however $G$ has the Hermite rank greater than 1, then we have three possibilities:

- the covariances $\text{Cov}(G(Y_0), G(Y_t))$ are summable and the usual central limit theorem holds;

- the covariances are not summable, $\text{Var}(S_n)$ grows faster than $\sqrt{n}$, and the properly normalized $S_n$ converges to a Hermite-Rosenblatt process. These classical results are attributed to Dobrushin, Major and Taqqu; see [44, 11] and [4, Section 4.2] for a review;

- The covariances are not summable, $\sum_{t \in \mathbb{Z}} \text{Cov}(G(Y_0), G(Y_t)) = 0$, and $\text{Var}(S_n)$ grows slower than $n$. Then the limit of normalized $S_n$ is non-Gaussian as well, see [14, Theorem 9].

There is still no general theory for the limit theorems of long range dependent square integrable random fields. For a review on (both isotropic and anisotropic) random fields indexed by $T = \mathbb{Z}^d$, see the paper [23]. Recent papers [21, 10, 30] give more (non)central limit theorems for (functionals of) l.r.d. linear random fields on $\mathbb{Z}^d$ with finite or infinite variance. They also focus on the question how the shape of sampling domains $W_n$ influences the limit.

Let us focus on subordinated stationary isotropic Gaussian random fields as considered in [18, 25, 26]. Let $X = \{X_t, t \in \mathbb{R}^d\}$ where $X_t = G(Y_t)$ and $Y = \{Y_t, t \in \mathbb{R}^d\}$ is a stationary isotropic l.r.d. centered Gaussian random field with covariance function $\rho(t) = \|t\|^{-\eta}L(\|t\|)$, $\eta \in (0, d/q)$. Here $\mathbb{E}G^2(Y_0) < +\infty$ and $q$ is the Hermite rank of $G$. Under some technical assumptions on the spectral density $f(\lambda)$ of $Y$ (cf. [26, Assumption 2]) it
holds
\[ n^{q/2 - d} L^{-q/2}(n) \int_{W_n} G(Y_t) \, dt \xrightarrow{d} R, \quad n \to +\infty, \]
where
\[ R = (\gamma(d, \eta))^{q/2} \int_{\mathbb{R}^{d q}} \int_{W_n} e^{i(\lambda_1 + \ldots + \lambda_q, u)} du \frac{B(d\lambda_1) \cdots B(d\lambda_q)}{(|\lambda_1| \cdots |\lambda_q|)^{(d-\eta)/2}}, \]
\[ \gamma(d, \eta) = \frac{\Gamma((d - \eta)/2)}{2^{\eta} \pi^{d/2} \Gamma(\eta/2)}, \]
and \( \int_{\mathbb{R}^{d q}} \) is the multiple Wiener–Ito integral with respect to a complex Gaussian white noise measure \( B \) (with structural measure being the spectral measure of \( Y \), cf. [18, Section 2.9]). It is easy to see that in case \( q = 1 \) the distribution of \( R \) is Gaussian. However, the normalization \( n^{q/2 - d} L^{-1/2}(n) \) differs from the CLT–common normalizing factor \( n^{-d/2} \) which agrees with the fact that \( X \) is l.r.d. in the sense of the usual definition (1). For \( q \geq 2 \), one gets a \( q \)-Rosenblatt–type distribution for \( R \), see [45, 27] and references therein for its properties in the case \( q = 2 \).

### 4.1.2 Empirical mean: infinite variance case

Of course, in case of the infinite variance, we cannot link the behaviour of the empirical mean to the usual definition of long range dependence. Furthermore, we would like to show that Definition 3.1 does not describe the behavior of integrals or partial sums of the field \( X \) if \( X \) has infinite variance. For that, we use the framework of time series \( X = \{X_t, t \in \mathbb{Z}\} \) where many more models have been widely explored, as compared to (continuous-time) random fields.

Clearly, if the random variables \( X_t \) are i.i.d. with infinite variance then they are not long range dependent in our sense. In particular, assume that \( X_t \) are regularly varying with index \( \alpha \in (1, 2) \), that is, \( P(|X_0| > x) \sim x^{-\alpha} L(x) \) as \( x \to +\infty \), where \( L : \mathbb{R}_+ \to \mathbb{R}_+ \) is a slowly varying at infinity function. Then \( S_n = \sum_{t=1}^{n} (X_t - E[X_t]) \) converges to a stable limit with normalization \( L^{-1}(n)n^{-1/\alpha} \). This behavior is usually attributed to s.r.d.

Consider (similarly as in Section 3.3) a subordinated time series \( X_t = G(|Y_t|), t \in \mathbb{Z}, \) where \( \{Y_t, t \in \mathbb{Z}\} \) is a centered Gaussian long memory linear time series with nondecreasing covariance function \( \rho(t) = \text{Cov}(Y_0, Y_t) \sim |t|^{-\eta} L(t), t \to +\infty, \eta \in (0, 1), \) and such that \( P(|X_0| > x) \sim x^{-\alpha} L(x), \) \( \alpha \in (0, 2) \). It is further assumed that \( G \) has Hermite rank \( q \). By Corollary
3.6, 2) \( X \) is short range dependent in the sense of Definition 3.1 whenever for any finite measure \( \mu \) on \( \mathbb{R} \)

\[
\sum_{k=1}^{\infty} \frac{b_{2k-1}(\mu)}{(2k)!} \sum_{t=1}^{\infty} \rho^{2k}(t) < +\infty.
\] (20)

We note that

\[
\sum_{t=1}^{\infty} \rho^{2k}(t) \leq C_0 \int_{1}^{\infty} \frac{L^{2k}(t)}{t^{2k\eta}} \, dt \leq \int_{1}^{\infty} \frac{C_1 \, dt}{t^{2k(\eta-\delta)}}, \quad k \in \mathbb{N},
\] (21)

where \( \delta > 0 \) is arbitrary and \( C_0, C_1 > 0 \) are some constants. It holds since \( L(t) \leq C_2 t^{\delta} \) for \( t \geq t_0 \) where \( t_0 > 0 \) is large enough and \( C_2 = C_2(\delta, t_0) = (1 + \delta)L(t_0)/t_0^{\delta} \leq 1 \) for large \( t_0 \), cf. [32, Proposition 2.6]. The right-hand side of (21) is finite and equal to \( O(1/k) \) whenever \( \eta \in (1/2, 1) \) since \( \delta > 0 \) can be chosen arbitrarily small. The series in (21) diverges if \( \eta \in (0, 1/2) \). If \( \eta = 1/2 \) the summability of the series in (21) depends on the particular form of the slowly varying function \( L \) and will not be discussed here.

Thus, for \( \eta \in (1/2, 1) \) \( X \) is s.r.d. whenever

\[
\sum_{k=1}^{\infty} \frac{b_{2k-1}(\mu)}{(2k)!k} < +\infty
\] (22)

for any finite measure \( \mu \).

Now we have to consider a special example of function \( G \) in order to get more explicit results for the s.r.d. case. As in Example 3.8, set \( G(x) = e^{x^2/(2\alpha)} \), \( \alpha \in (0, 2] \). By relation (14), condition (22) is satisfied for \( \eta \in (1/2, 1) \) since \( \delta > 0 \) can be chosen arbitrarily small. The case \( \alpha \in (0, 1) \) the limit is always stable. The case \( \alpha = 1 \) is more delicate. Summarizing, the limit can be of either "weakly dependent-type" (with the normalization \( L^{-1}(n) n^{-1/\alpha} \)) or "long memory-type". For convenience, we give the comparison of the long or short memory behaviour of the above subordinated Gaussian time series \( X \) according to Definition 3.1 and [41] in Table 1. There, some discrepancies are seen, that is Definition 3.1 does not agree with the asymptotic behaviour of \( S_n \).

The discussion of this section yields that our definition of l.r.d. does not capture the (non-standard) behaviour of partial sums. This is not surprising, since it is supposed to capture behaviour in limit theorems for excursion sets.
<table>
<thead>
<tr>
<th>Memory</th>
<th>Definition (3.1)</th>
<th>Paper [41]</th>
<th>Limit of norm. $S_n$</th>
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<tbody>
<tr>
<td>s.r.d.</td>
<td>$\frac{1}{2} &lt; \eta &lt; 1$</td>
<td>$1 - \frac{1}{\alpha} &lt; \eta &lt; 1$</td>
<td>$\alpha$–stable</td>
</tr>
<tr>
<td>l.r.d.</td>
<td>$0 &lt; \eta &lt; \frac{1}{2}$</td>
<td>$0 &lt; \eta &lt; 1 - \frac{1}{\alpha}$</td>
<td>Rosenblatt</td>
</tr>
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Table 1: Short or long memory of $X_t = e^{Y_t^2/(2\alpha)}$ in the infinite variance case $\alpha \in (1, 2)$ in dependence of the long memory parameter $\eta$ of $Y$.

### 4.1.3 Volume of level sets: subordinated Gaussian case

Consider the limiting behavior of the volume of level sets of (infinite variance) subordinated Gaussian random field $X = \{X_t, t \in \mathbb{R}^d\}$. Let $X_t = G(Y_t)$ where $Y = \{Y_t, t \in \mathbb{R}^d\}$ is a centered stationary isotropic Gaussian unit variance random field with covariance function $\rho$ as at the end of Section 4.1.1, that is $\rho(t) = \|t\|^{-\eta}L(\|t\|)$. $Y$ is l.r.d. in the sense of Definition (1).

Assume $G : \mathbb{R} \to \mathbb{R}$ to be a monotone right–continuous function such that $\mathbb{E}(G(Y)|^{1+\theta} < +\infty$ with $\theta \in (0, 1)$. Let the variance of $X_0$ be infinite. For any $u \in \mathbb{R}$ introduce the function $g_u(x) = \zeta_{G,1,u}(x)$, where $\zeta_{G,1,u}$ is given in (18). By Remark 4.2, 1), the Hermite ranks of $G$ and $g_u$ are equal to one.

By Section 4.1.1, if $\eta \in (0, d)$ then

$$\int_{W_n} g_u(Y_t) dt \over n^{d-\eta/2}L^{1/2}(n) = \int_{W_n} 1 (G(Y_t) > u) dt - \nu_d(W_n)P(G(Y_0) > u) \overset{d}{\to} R$$

as $n \to +\infty$ where $R$ is given in (19). The normalization in this limit theorem is not of CLT-type $n^{-d/2}$ which should be attributed to the l.r.d. case. Let us compare this behavior with Definition 3.1. As an example, we consider

$$G(x) = \text{sgn}(x) \left(e^{x^2/\beta^2} - 1\right), \quad x \in \mathbb{R}$$

for some $\beta > \sqrt{2(1+\theta)}$. Note that it is possible that the variance of $X = G(Y)$ is infinite. Set $\mu = \delta_{0}$. By Remark 3.7, 1) we get $b_k(\mu) = H_k^2(0)/(2\pi) < +\infty$ for any $k \geq 0$, $b_0 > 0$, $b_1 = 0$, etc. By the choice $\rho(t) = \|t\|^{-\eta}L(\|t\|)$, $\eta \in (0, d)$ we get that $\int_{\mathbb{R}^d} |\rho(t)| dt = +\infty$, and the series (11) diverges. Then $X$ is l.r.d. in the sense of Definition 3.1 for $\eta \in (0, d)$ which is in accordance with the above limit theorem.

### 4.2 Limit theorems for the integrals of functionals of l.r.d. random volatility fields

In order to illustrate how our definition of l.r.d. matches limit theorems for volumes of level sets for random volatility models, unlike as in the subordinated Gaussian case, a general asymptotic theory has to be developed.
Let $X$ be a random volatility field of the form $X_t = G(Y_t)Z_t$, $t \in T$, $T = \mathbb{R}^d$, where

- $\{G(Y_t), t \in T\}$ is a subordinated Gaussian random field,
- $\{Z_t, t \in T\}$ is a white noise,
- the random fields $Y$ and $Z$ are independent.

Our goal is to prove limit theorems for $\int_{W_n} g(X_t) dt$ as $n \to \infty$, where $W_n = n \cdot W$, $W$ is chosen as in Section 4.1, and $g$ is a real valued function such that

$$
\mathbb{E}[g(X_0)] = 0, \quad \mathbb{E}[g^2(X_0)] > 0.
$$

Introduce the function

$$
\xi(y) = \mathbb{E}[g(G(y)Z_0)].
$$

It follows from (23) that for $\nu_1$–almost every $y \in \mathbb{R}$

$$
\xi(y) < \infty.
$$

By (23) we also have $\mathbb{E}[\xi(Y_0)] = 0$. Let

$$
J(m) = \langle \xi, H_m \rangle = \mathbb{E}[H_m(Y_0) g(G(Y_0)Z_0)]
$$

be the $m$th Hermite coefficient of $\xi$. We recall that a sufficient condition for the finiteness of $J(m)$ is

$$
\mathbb{E}[|g(X_0)|^{1+\theta}] = \mathbb{E}[|\xi(Y_0)|^{1+\theta}] = \mathbb{E}

\left[\mathbb{E}[g(G(Y_0)Z_0) \mid \mathcal{Y}]^1+\theta\right] < \infty
$$

for some $\theta \in (0, 1)$. Let rank $\langle \xi \rangle = q$. Furthermore, set

$$
m(y,Z_t) = g(G(y)Z_t) - \mathbb{E}[g(G(y)Z_t)] = g(G(y)Z_t) - \xi(y),
$$

which is almost everywhere finite by (24), and $\chi(y) = \mathbb{E}[m^2(y,Z_0)]$. We also assume

$$
\mathbb{E}[\chi^3(Y_0)] < \infty.
$$

Note that under (26), using Lyapunov inequality on a space of finite measure and the stationarity of $Y_t$, we have for any compact set $I$ that

$$
\mathbb{E}

\left[\left(\int_I \chi(Y_t)dt \right)^3\right] < \infty.
$$
Theorem 4.3. Assume that random field \( X_t = G(Y_t)Z_t, \ t \in \mathbb{R}^d \), be as above, where additionally

- \( Y \) is a homogeneous isotropic centered Gaussian random field with the covariance function \( \rho(t) = \mathbb{E}[Y_0Y_t] = ||t||^{-\eta}L(||t||), \ \eta \in (0, d/q) \) and \( L \) is slowly varying at infinity,

- \( Y \) has a spectral density \( f(\lambda) \) which is continuous for all \( \lambda \neq 0 \) and decreasing in a neighborhood of 0.

Assume that (23), (25) with \( \theta = 1 \), (26) hold.

1. If \( \xi(y) \equiv 0 \) then
\[
n^{-d/2} \int_{W_n} g(X_t) \, dt \xrightarrow{d} N(0, \sigma^2), \quad n \to +\infty, \tag{27}
\]
where \( \sigma^2 = \mathbb{E}[g^2(X_0)] \nu_d(W) > 0 \).

2. If \( \xi(y) \not\equiv 0 \) then
\[
n^{m/2-d-\eta/2} L(n) \int_{W_n} g(X_t) \, dt \xrightarrow{d} R, \quad n \to +\infty, \tag{28}
\]
where the random variable \( R \) is given in (19).

4.3 Examples

Example 4.4. Assume that \( g(y) = y, \ \mathbb{E}[G^2(Y_0)] < \infty \) and \( \mathbb{E}[Z_0] = 0 \). Then \( \xi(y) = G(y)\mathbb{E}[Z_0] \equiv 0 \) and (27) always holds. In this case, there is no contribution from the long memory of the random field \( Y_t \).

Example 4.5. Assume that \( g(y) = y - \mathbb{E}[G(Y_0)Z_0], \ \mathbb{E}[Z_0] \not\equiv 0 \). Then \( \xi(y) = \mathbb{E}[Z_0] \{G(y) - \mathbb{E}[G(Y_0)]\} \). Condition (26) is satisfied if \( \mathbb{E}[Z_0^3] < +\infty, \mathbb{E}[G^4(Y_0)] < +\infty \). In this case \( \xi(y) \not\equiv 0 \), and (28) always holds.

Example 4.6. Assume that \( g(y) = g_u(y) = 1\{y > u\} - P(G(Y_0)Z_0 > u) \) where \( G \) is nonnegative or nonpositive \( \nu_1 \)-a.e. Then
\[
\xi(y) = \mathbb{E}[1\{G(y)Z_0 > u\}] - P(G(Y_0)Z_0 > u) \not\equiv 0
\]
if \( u \not\equiv 0 \), so case (28) applies. If \( u = 0 \) then \( \xi(y) \equiv 0 \) (compare Remark 4.2, 3)), so case (27) holds true.
Example 4.7. Let the random volatility field \( X_t = G(|Y_t|)Z_t \), \( t \in \mathbb{R}^d \) be as in Lemma 3.13 where \( \{Z_t\} \) is a heavy–tailed white noise, \( \mathbb{E}Z_0^2 = +\infty \). Let \( G(x) \geq 0 \) be as in Lemma 4.1, 2) and \( \rho(t) \sim \|t\|^{-\eta} \) as \( \|t\| \to +\infty \) be nonnegative. Similarly to Example 3.14, an analogue of relation (17) holds true: for \( \mu = \delta_{u_0} \), \( u_0 > 0 \) we have

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \text{Cov}_X(t,u,v)\mu(du)\mu(dv)dt = \sum_{k=1}^{\infty} \frac{\langle \tilde{F}_Z(u_0/\tilde{G}), H_k \rangle^2}{k!} \int_{\mathbb{R}^d} \rho^k(t) dt,
\]

where \( \tilde{G}(y) = G(|y|) \), \( y \in \mathbb{R} \). Since rank \( (\tilde{F}_Z(u_0/\tilde{G})) = 2 \), \( X \) is l.r.d. in the sense of Definition 3.1 if \( \int_{\mathbb{R}^d} \rho^2(t) dt = +\infty \), that is, if \( \eta < d/2 \).

Consider function \( \xi \) from Example 4.6 with \( u = u_0 > 0 \) and \( \tilde{G} \) instead of \( G \). By Lemma 4.1, 2) rank \( (\xi) = 2 \). By Theorem 4.3 and Example 4.6, the asymptotic behavior of the volume of the level sets of \( X \) at niveau \( u_0 \) is l.r.d. if \( \eta \in (0,d/2) \) which is in agreement with our definition.

5 Summary and outlook

We proposed a new definition of long memory for stationary random functions \( X \) indexed by any set \( T \subset \mathbb{R}^d \) which works also for heavy tailed \( X \). We could show that this definition fits well the asymptotic behavior of the volume of the excursion set of \( X \) at a level \( u \in \mathbb{R} \) in a unboundedly growing observation window \( W_n \). This connection to non–central limit theorems was proven for a class of random volatility fields with a subordinated l.r.d. Gaussian volatility.

6 Appendix: Proofs

We start with the following technical result. If \( Y \) is itself a stationary centered Gaussian random field with \( \text{Var} \ Y_0 = 1 \) and covariance function \( C(t) \) we have

\[
\text{Cov}_Y(t,u,v) = \frac{1}{2\pi} \int_0^{\text{C}(t)} \frac{1}{\sqrt{1-r^2}} \exp \left\{ \frac{-u^2 - 2ruv + v^2}{2(1-r^2)} \right\} dr,
\]

see [7, Lemma 2].

**Proof of Theorem 3.5.** Consider representation (29). Since the density \( f_{U,V} \) of a bivariate normal distribution with zero mean, unit variances and correlation coefficient \( \mp r \) equals

\[
\frac{1}{2\pi\sqrt{1-r^2}} \exp \left( -\frac{x^2 \pm 2rxy + y^2}{2(1-r^2)} \right) \geq 0
\]
then it is easy to see that

\[ |\text{Cov}_Y(t, x, y)| = \frac{1}{2\pi} \int_0^{\frac{|\rho(t)|}{\sqrt{1-r^2}}} \exp \left( -\frac{x^2 - 2\text{sign}(\rho(t)) r x y + y^2}{2(1-r^2)} \right) \, dr. \]

Since \( G \) is strictly monotone, by properties of the generalized inverse of \( G \) we have

\[
\int_T^\infty \int_{-\infty}^{\infty} \int_{T(\text{Im}(G))^2} |\text{Cov}_X(t, u, v)| \mu(du) \mu(dv) \, dt = \\
\int_T^\infty \int_{-\infty}^{\infty} |\text{Cov}_Y(t, G^{-}(u), G^{-}(v))| \mu(du) \mu(dv) \, dt = \\
\int_T^\infty \int_{0}^{\frac{|\rho(t)|}{\sqrt{1-r^2}}} \exp \left( \frac{(G^{-}(u))^2 - 2\text{sign}(\rho(t)) r G^{-}(u) G^{-}(v) + (G^{-}(v))^2}{2(1-r^2)} \right) \, dr \mu(du) \mu(dv) \, dt.
\]

By [9, Formula (21.12.5)] for the density \( f_{U,V} \) with correlation coefficient \( \text{sign}(\rho(t)) r \in (-1, 1) \) it holds

\[
f_{U,V}(x, y) = \sum_{k=0}^{\infty} \frac{\Phi^{(k+1)}(x) \Phi^{(k+1)}(y)}{k!} (\text{sign}(\rho(t)) r)^k, \quad x, y \in \mathbb{R}. \quad (30)
\]

By condition \( \nu_d(\{t \in T : (|\rho(t)| = 1)\}) = 0 \), the above series converges uniformly for \( r \in (-1; 1) \), so integration over \( r \in [0; |\rho(t)|] \) and summation with respect to \( k \) can be interchanged. Then the above triple integral reads

\[
\int_T \int_{\text{Im}(G)^2} \int_0^{\frac{|\rho(t)|}{\sqrt{1-r^2}}} \Phi^{(k+1)}(G^{-}(u)) \Phi^{(k+1)}(G^{-}(v)) \frac{(\text{sign}(\rho(t)) r)^k}{k!} \, dr \mu(du) \mu(dv) \, dt \\
= \int_T \int_{\text{Im}(G)^2} \varphi(G^{-}(u)) \varphi(G^{-}(v)) \sum_{k=0}^{\infty} \frac{H_k(G^{-}(u)) H_k(G^{-}(v))}{k!} (\text{sign}(\rho(t))^k \\
\quad \times \frac{|\rho(t)|^{k+1}}{k+1} \, dr \mu(du) \mu(dv) \, dt \\
= \int_T \int_{\text{Im}(G)^2} |\rho(t)| \varphi(G^{-}(u)) \varphi(G^{-}(v)) \sum_{k=0}^{\infty} \frac{H_k(G^{-}(u)) H_k(G^{-}(v))}{(k+1)k!} (\text{sign}(\rho(t))^k r)^k \, dr \mu(du) \mu(dv) \, dt.
\]
Abel’s uniform convergence test allows us to interchange the sum and the integral over $Im(G)^2$. Since $b_k \geq 0$ we get

\[
\int_T \sum_{k=0}^{\infty} \int_{Im(G)^2} \varphi(G^{-}(u))\varphi(G^{-}(v)) \frac{H_k(G^{-}(u))H_k(G^{-}(v))}{(k + 1)!} |\rho(t)| |\rho(t)^k dt |\mu(du)\mu(dv)dt \\
= \int_T \sum_{k=0}^{\infty} \frac{1}{(k + 1)!} \left( \int_{Im(G)} \varphi(G^{-}(u))H_k(G^{-}(u))\mu(du) \right)^2 |\rho(t)| |\rho(t)^k dt \\
= \int_T \sum_{k=0}^{\infty} \frac{b_k(\mu)}{(k + 1)!} |\rho(t)| |\rho(t)^k dt = \sum_{k=1}^{\infty} \frac{b_k(\mu)}{k!} \int_T |\rho(t)| |\rho(t)^{k-1} dt,
\]

where the integral over $T$ and the sum are interchangeable by Tonelli’s theorem subdividing $T$ into parts $T^+ = \{ t \in T : \rho(t) \geq 0 \}$ and $T^- = \{ t \in T : \rho(t) < 0 \}$. Then $X = G(Y)$ has short memory if

\[
\sum_{k=1}^{\infty} \frac{b_k(\mu)}{k!} \int_T |\rho(t)| |\rho(t)^{k-1} dt < +\infty
\]

for any finite measure $\mu$ on $\mathbb{R}$.

\[\square\]

\textbf{Proof of Corollary 3.6.} 1. It follows from relation (10) using the change of variables $u = G(x)$ and by [4, Lemma 4.21].

2. W.l.o.g. assume $G$ to be an increasing function. Since the probability density of the centered uni- and bivariate Gaussian distribution is invariant under transformation $x \mapsto -x, y \mapsto -y$ we get

\[
\text{Cov}_X(t, u, v) = P(|Y_0| > G^{-}(u), |Y_t| > G^{-}(v)) \\
- P(|Y_0| > G^{-}(u))P(|Y_t| > G^{-}(v)) \\
= 2 \left( P(Y_0 > G^{-}(u), Y_t > G^{-}(v)) \right) - P(Y_0 > G^{-}(u))P(Y_t > G^{-}(v)) \\
+ P(Y_0 > G^{-}(u), Y_t < -G^{-}(v)) - P(Y_0 > G^{-}(u))P(Y_t < -G^{-}(v)) \right).
\]

Denote $Z = -Y_t$, $x = G^{-}(u)$, $y = G^{-}(v)$. It holds

\[
P(Y_0 > x, Y_t > y) - P(Y_0 > x)P(Y_t > y) = \text{Cov}(1(Y_0 \geq x), 1(Y_t \geq y)),
\]

\[
P(Y_0 > x, Y_t < -y) - P(Y_0 > x)P(Y_t < -y) = \text{Cov}(1(Y_0 > x), 1(Z > y)).
\]
Since \( \text{Cov}(Y_0, Z) = -\rho(t) \) and \( xy = G^-(u)G^-(v) \geq 0 \) we have by formula (29) that

\[
|\text{Cov}_X(t, u, v)| = \frac{2}{2\pi} \left| \int_0^\rho(t) \frac{1}{\sqrt{1-r^2}} \exp \left( -\frac{x^2 - 2rxy + y^2}{2(1-r^2)} \right) dr \right| \\
+ \left. \int_0^{-\rho(t)} \frac{1}{\sqrt{1-r^2}} \exp \left( -\frac{x^2 - 2rxy + y^2}{2(1-r^2)} \right) dr \right| \\
= \frac{|\rho(t)|}{\pi \sqrt{1-r^2}} \left( \exp \left( -\frac{x^2 - 2rxy + y^2}{2(1-r^2)} \right) - \exp \left( -\frac{x^2 + 2rxy + y^2}{2(1-r^2)} \right) \right) \frac{dr}{\pi \sqrt{1-r^2}}.
\]

Similarly to the proof of Theorem 3.5, we use representation (30) to write

\[
\int_T^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\text{Cov}_X(t, u, v)| \mu(du) \mu(dv) dt \\
= 2 \int_T \int_{\text{Im}(G)} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \int_0^{2\kappa} \phi(x) |\rho(t)|^{k+1} \mu(du) \mu(dv) dt \\
= \int_T \sum_{k=1}^{\infty} \frac{2}{(2k)!} \left( \int_{\text{Im}(G)} H_{2k-1}(G^-(u)) \phi(G^-(u)) \mu(du) \right)^2 |\rho(t)|^{2k} dt \\
= 4 \sum_{k=1}^{\infty} \frac{b_{2k-1}(\mu)}{(2k)!} \int_T \rho(t)^{2k} dt.
\]

Proof of Corollary 3.11. Choose \( \mu = \delta_{(u_0)} \), \( u_0 \in \mathbb{R} \) and write

\[
\int_T \int_{\mathbb{R}^2} \text{Cov}_X(t, u, v) \mu(du) \mu(dv) dt = \int_T \text{Cov} \left( \tilde{F}_Z(u_0/A), \tilde{F}_Z(u_0/A) \right) dt \\
+ \int_T \mathbb{E} \left[ \text{Cov}_Z(t, u_0/A, u_0/A) \right] dt \geq \int_T \text{Var} \left( \tilde{F}_Z(u_0/A) \right) dt = +\infty
\]

since \( Z \in \text{PA} \), \( \tilde{F}_Z(u_0/A) \) is non-degenerate and bounded. \( \square \)
Proof of Corollary 3.12. Without loss of generality assume \( Z, Y \in \mathcal{P}A \). Then \( Y^\alpha \in \mathcal{P}A \), too, and the second term in (15) is nonnegative. Denote \( A_{u,v}(t) = \text{Cov}(\bar{F}_Z(u/Y_0), \bar{F}_Z(v/Y_t)) \), \( u, v \in \mathbb{R}_+, t \in T \).

Since \( Y \in \mathcal{P}A \) and the function \( \bar{F}_Z(u/\cdot) \) is bounded and nondecreasing for \( u > 0 \) we get \( A_{u,v}(t) \geq 0 \) for all \( u,v \in \mathbb{R}_+, t \in T \). Using the regular variation of the tail of \( Z_0 \), the independence of \( Y \) and \( Z \) and Potter bound [32, Proposition 2.6] one can easily show that under the above assumptions on the integrability of \( Y \) it holds

\[
A_{u,v}(t) \sim \bar{F}_Z(u)\bar{F}_Z(v)\text{Cov}(Y_0^\alpha, Y_t^\alpha), \quad u,v \to +\infty,
\]

for any \( t \in T \). Then for sufficiently large \( N > 0 \) there exists \( u_0 > N \) such that for the Dirac measure \( \mu = \delta_{\{u_0\}} \) and some \( \varepsilon \in (0,1) \) we have

\[
\int_T \int_{\mathbb{R}^2} \text{Cov}_X(t,u,v)\mu(du)\mu(dv)dt \geq \int_T A_{u_0,u_0}(t)dt \geq \varepsilon \bar{F}_Z^2(u_0)\int_T \text{Cov}(Y_0^\alpha, Y_t^\alpha) dt
\]

which is infinite if \( Y^\alpha \) is l.r.d. Thus, \( X = YZ \) is l.r.d. if \( Y^\alpha \) is l.r.d. \( \square \)

Proof of Lemma 3.13. Without loss of generality, assume \( G \) to be nonnegative. By Lemma 3.4, Fubini and Tonelli theorems for \( G_u(y) = \bar{F}_Z(u/G(y)) \) we get

\[
D(G(Y), Z_0) = \int_T \int_{\mathbb{R}^2} \text{Cov}(G_u(Y_0), G_v(Y_t)) \mu(du)\mu(dv) dt
\]

\[
= \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}} \langle G_u, H_k \rangle_{\varphi} \mu(du) \right)^2 \int_T \rho^k(t) dt.
\]

The change of order of the sum and integrals is justified by Weierstrass uniform convergence test since for almost all \( t \in T \)

\[
\sum_{k=1}^{\infty} \frac{|\langle G_u, H_k \rangle_{\varphi} |^2}{k!} \rho^k(t) \leq \sum_{k=1}^{\infty} \frac{(1, |H_k|)_{\varphi}^2}{k!} \rho^k(t) \leq \sum_{k=1}^{\infty} \rho^k(t) < \infty
\]

due to \( (1, |H_k|)_{\varphi} \leq \sqrt{k!} \) by Cauchy–Schwarz inequality and due to condition (\( \rho \)). \( \square \)

Proof of Lemma 4.1. 1. If \( G : \mathbb{R} \to \mathbb{R} \) is monotone then \( \text{rank}(G) = 1 \) due to

\[
\langle G, H_1 \rangle_{\varphi} = \mathbb{E}[YG(Y)] = \int_0^\infty (G(y) - G(-y)) y\varphi(y)dy \neq 0. \quad (31)
\]
What is the Hermite rank of \( \zeta_{G,Z,u} \)? First consider \( Z \equiv 1 \). Since the Hermite rank of \( y \mapsto 1 \{ y > u \} - \bar{F}_Y(u) \) is one we can write
\[
\langle \zeta_{G,1,u}, H_1 \rangle \varphi = \mathbb{E}[Y 1 \{ G(Y) > u \}] = \mathbb{E}[Y 1 \{ Y > G^-(u) \}] \neq 0,
\]
where \( \mathbb{E}Y = 0 \) and \( G \) is non-decreasing w.l.o.g. Hence, rank \( (\zeta_{G,1,u}) = 1 \) for any \( u \in \mathbb{R} \). Now let \( G : \mathbb{R} \to \mathbb{R}_+ \) and \( Z \) be arbitrary. W.l.o.g. assume \( G \) to be nonnegative. Then
\[
\langle \zeta_{G,Z,u}, H_1 \rangle \varphi = \int_{\mathbb{R}} \bar{F}_Z(u/G(y)) y \varphi(y) dy \neq 0,
\]
since for any \( u \neq 0 \) the function \( y \mapsto \bar{F}_Z(u/G(y)) \) is monotone, and we can use the reasoning (31). For nonpositive \( G \) replace \( \bar{F}_Z \) above by \( F_Z \).

2. W.l.o.g. assume that \( G \) is nonnegative and nondecreasing. Prove that rank \( (\widetilde{G}) = 2 \).

Clearly, since \( y \mapsto G(|y|) \) is even, we have \( \mathbb{E}[YG(|Y|)] = 0 \). Now,
\[
\mathbb{E}[H_2(Y)G(|Y|)] = 2 \int_{0}^{\infty} G(y)(y^2 - 1) \varphi(y) dy.
\]
We note that
\[
\int_{0}^{\infty} (y^2 - 1) \varphi(y) dy = 0
\]
and hence
\[
\int_{0}^{1} (y^2 - 1) \varphi(y) dy = - \int_{1}^{\infty} (y^2 - 1) \varphi(y) dy.
\]
Also, by the mean value theorem, due to monotonicity of non-constant \( G \), there exists \( y_0 \in [0,1) \) such that
\[
\int_{0}^{1} G(y)(y^2 - 1) \varphi(y) dy = G(y_0) \int_{0}^{1} (y^2 - 1) \varphi(y) dy.
\]
Therefore,
\[
\int_{0}^{\infty} G(y)(y^2 - 1) \varphi(y) dy \\
\geq G(y_0) \int_{0}^{1} (y^2 - 1) \varphi(y) dy + G(1) \int_{1}^{\infty} (y^2 - 1) \varphi(y) dy \\
= -G(y_0) \int_{1}^{\infty} (y^2 - 1) \varphi(y) dy + G(1) \int_{1}^{\infty} (y^2 - 1) \varphi(y) dy \\
= (G(1) - G(y_0)) \int_{1}^{\infty} (y^2 - 1) \varphi(y) dy > 0.
\]
For nonnegative nonincreasing \( G \), we can use the estimate
\[
\int_{0}^{\infty} G(y)(y^2 - 1)\varphi(y)dy \leq G(y_0) \int_{0}^{1} (y^2 - 1)\varphi(y)dy
\]
\[
+ G(1) \int_{1}^{\infty} (y^2 - 1)\varphi(y)dy = (G(y_0) - G(1)) \int_{0}^{1} (y^2 - 1)\varphi(y)dy < 0.
\]
If \( G(y) \leq 0 \) just multiply it by \(-1\). This proves that the Hermite rank of \( G(|y|) \) is 2.

Now compute the Hermite rank of \( \tilde{\zeta}_{G,1,u} \) for any \( u \in \mathbb{R} \). Since \( \tilde{\zeta}_{G,1,u} \) is even rank \((\tilde{\zeta}_{G,1,u}) > 1 \). Assuming w.l.o.g. that \( G \) is nonnegative and nondecreasing we calculate
\[
\langle \tilde{\zeta}_{G,1,u}, H_2 \rangle \varphi = \mathbb{E}[(Y^2 - 1)1\{G(|Y|) > u\}]
\]
\[
= \int_{\mathbb{R}} (y^2 - 1)1\{|y| > G^{-}(u)\}\varphi(y)dy = 2 \int_{G^{-}(u)}^{\infty} (y^2 - 1)\varphi(y)dy \neq 0
\]
due to (32) and \( G^{-}(u) \neq 0 \). So rank \( \tilde{\zeta}_{G,1,u} = 2 \). For general \( Z \), we note that \( \tilde{\zeta}_{G,Z,u} \) is even, so rank \((\tilde{\zeta}_{G,Z,u}) > 1 \). If \( G \) is non–negative then
\[
\langle \tilde{\zeta}_{G,Z,u}, H_2 \rangle \varphi = \int_{\mathbb{R}} \bar{F}_Z(u/G(|y|))H_2(y)\varphi(y)dy \neq 0
\]
by the first part of the proof of 2) since \( \bar{F}_Z(u/G(|y|)) \) is a monotone even function of \( y \). Modifications of the proof for \( G \leq 0 \) or \( G \) nonincreasing are obvious.

\[\square\]

**Proof of Theorem 4.3.** Let \( \mathcal{Y} \) be the \( \sigma \)–algebra generated by the entire random field \( \{Y_t, t \in T\} \). Then
\[
\int_{W_n} g(X_t) dt = \int_{W_n} (g(X_t) - \mathbb{E}[g(X_t) | \mathcal{Y}]) dt + \int_{W_n} \mathbb{E}[g(X_t) | \mathcal{Y}] dt = M_n + K_n ,
\]
where
\[
M_n = \int_{W_n} (g(X_t) - \mathbb{E}[g(X_t) | \mathcal{Y}]) dt = \int_{W_n} m(Y_t, Z_t) dt
\]
and
\[
K_n = \int_{W_n} \mathbb{E}[g(X_t) | \mathcal{Y}] dt = \int_{W_n} \xi(Y_t) dt .
\]
The above decomposition is allowed by (24). The limiting behaviour of the integral depends on an interplay between \( M_n \) and \( K_n \). First, we state the limiting results for \( M_n \) and \( K_n \) separately.
Lemma 6.1. Under the assumptions of Theorem 4.3, it holds

$$\tilde{M}_n := n^{-d/2}M_n \overset{d}{\to} N(0, \sigma^2),$$

where $$\sigma^2 = \mathbb{E}[\chi(Y_0)]\nu_d(W) > 0.$$

Proof. Without loss of generality assume $$W = [-1/2, 1/2]^d$$. Let

$$\{I_j : j = (j_1, \ldots, j_d), j_i \in (1, n] \cap \mathbb{N}, \ i = 1, \ldots, d\}$$

be a disjoint partition of $$W$$ into $$n^d$$ cubes congruent to $$W$$. We calculate

$$\mathbb{E}\left[\exp\left\{iz\tilde{M}_n\right\} | \mathcal{Y}\right] = \mathbb{E}\left[\exp\left\{\frac{iz}{n^{d/2}} \sum_j \int_{t \in I_j} m(Y_t, Z_t) dt\right\} | \mathcal{Y}\right]$$

$$=: \mathbb{E}\left[\exp\left\{\frac{iz}{n^{d/2}} \sum_j V_j\right\} | \mathcal{Y}\right],$$

where $$V_j = \int_{t \in I_j} m(Y_t, Z_t) dt$$. Note that, due to stationarity of $$Y$$ and $$Z$$, the random variables $$V_j$$ are identically distributed and conditionally independent, given $$\mathcal{Y}$$. Therefore,

$$\mathbb{E}\left[\exp\left\{iz\tilde{M}_n\right\} | \mathcal{Y}\right] = \mathbb{E}\left[\exp\left\{\frac{iz}{n^{d/2}} \sum_j V_j\right\} | \mathcal{Y}\right]$$

$$= \prod_j \mathbb{E}\left[\exp\left\{\frac{iz}{n^{d/2}} V_j\right\} | \mathcal{Y}\right].$$

The standard inequality,

$$|\exp(itz) - (1 + itz - t^2z^2/2)| \leq \min\{|tz|^2, |tz|^3\}$$

yields

$$\left|\mathbb{E}\left[\exp\left\{\frac{iz}{n^{d/2}} V_j\right\} | \mathcal{Y}\right] - \mathbb{E}\left[\left(1 + \frac{izV_j}{n^{d/2}} - \frac{1}{2} \frac{z^2V_j}{n^d}\right) | \mathcal{Y}\right]\right|$$

$$\leq \mathbb{E}\left[\min\left\{\frac{|z|^2V_j^2}{n^d}, \frac{|z|^3|V_j|^3}{n^{3d/2}}\right\} | \mathcal{Y}\right] =: \mathbb{E}[V_{j,n} | \mathcal{Y}].$$

For complex numbers $$z_1, \ldots, z_m, w_1, \ldots, w_m$$ of modulus at most 1, we have

$$\left|\prod_{i=1}^m z_i - \prod_{i=1}^m w_i\right| \leq \sum_{i=1}^m |z_i - w_i|.$$
Hence
\[ A_n(Y) := \prod_j \mathbb{E} \left[ \exp \left\{ \frac{iz}{n^{d/2}} V_j \right\} | Y \right] - \prod_j \mathbb{E} \left[ \left( 1 + \frac{izV_j}{n^{d/2}} - \frac{1}{2} \frac{z^2V_j^2}{n^d} \right) | Y \right] \]
\[ \leq \sum_j \mathbb{E} \left[ \exp \left\{ \frac{iz}{n^{d/2}} V_j \right\} | Y \right] - \mathbb{E} \left[ \left( 1 + \frac{izV_j}{n^{d/2}} - \frac{1}{2} \frac{z^2V_j^2}{n^d} \right) | Y \right] \]
\[ \leq \sum_j \mathbb{E}[V_{j,n} | Y]. \]

We argue that
\[ A_n(Y) \to 0 \quad (33) \]
in probability. If this is the case, then the conditional characteristic function
\[ \mathbb{E} \left[ \exp \{iz\tilde{M}_n\} | Y \right] \]
and
\[ B_n(Y) := \prod_j \mathbb{E} \left[ \left( 1 + \frac{izV_j}{n^{d/2}} - \frac{1}{2} \frac{z^2V_j^2}{n^d} \right) | Y \right] \]
have the same limit in probability. Applying the log to the above expression and \( \log(1 - x) = -x + O(x^3) \) we have
\[
\log B_n(Y) = \sum_j \log \mathbb{E} \left[ \left( 1 + \frac{izV_j}{n^{d/2}} - \frac{1}{2} \frac{z^2V_j^2}{n^d} \right) | Y \right] \\
= \frac{iz}{n^{d/2}} \sum_j \mathbb{E}[V_j | Y] - \frac{z^2}{2n^d} \sum_j \mathbb{E}[V_j^2 | Y] \\
+ O(1) \frac{|z|^3}{n^{3d/2}} \sum_j (\mathbb{E}[V_j | Y])^3 + O(1) \frac{z^6}{n^{3d}} \sum_j (\mathbb{E}[V_j^2 | Y])^3.
\]
The expression in the last line is \( o_P(1) \) by (26). By the definition, \( \mathbb{E}[m(y, Z_t)] = 0 \) and hence \( \mathbb{E}[V_j | Y] = 0 \). By Fubini theorem and conditional independence of \( m(Y_t, Z_t), m(Y_s, Z_s) \), given \( Y, t \neq s \), we evaluate
\[
\mathbb{E}[V_j^2 | Y] = \mathbb{E} \left[ \int_{t \in I_j} \int_{s \in I_j} m(Y_t, Z_t)m(Y_s, Z_s) dt ds | Y \right] \\
= \mathbb{E} \left[ \int_{t \in I_j} m^2(Y_t, Z_t) dt | Y \right] = \int_{t \in I_j} \chi(Y_t) dt.
\]
Therefore,
\[
\log B_n(Y) = -\frac{z^2}{2n^d} \sum_j \int_{t \in I_j} \chi(Y_t) dt + o_p(1).
\]
Since \(\chi\) is measurable, the ergodic theorem ([46, p. 339]) implies that
\[
\frac{1}{n^d} \sum_j \int_{t \in I_j} \chi(Y_t) dt \xrightarrow{P} \mathbb{E}[\chi(Y_0)] \nu_d(W), \quad n \to +\infty,
\]
whenever the covariance of the field \(\chi(Y_t)\) goes to zero as \(\|t\| \to +\infty\). To check the latter property, we use Lemma 3.4 to conclude
\[
|\text{Cov}(\chi(Y_0), \chi(Y_t))| \leq |\rho(t)| \sum_{k=1}^{\infty} \frac{\langle \chi, H_k \rangle^2}{k!} \to 0
\]
as \(\|t\| \to +\infty\), since the infinite series in the last expression is finite due to \(\text{Var}(\chi(Y_0)) < \infty\); cf. (26). Hence, \(\log B_n(Y) \to -z^2\sigma^2/2\) in probability. By continuous mapping theorem, it holds
\[
\mathbb{E}\left[ \exp\{iz\tilde{M}_n\} \mid Y \right] \xrightarrow{P} e^{-z^2\sigma^2/2}, \quad n \to +\infty.
\]
Since \(\mathbb{E}\left[ \exp\{iz\tilde{M}_n\} \mid Y \right] \leq 1\) for all \(n \in \mathbb{N}\) this sequence is uniformly integrable. Using the property of \(L^1\)-convergence of uniformly integrable sequences we get
\[
\mathbb{E}\left[ \exp\{iz\tilde{M}_n\} \right] \to e^{-z^2\sigma^2/2}, \quad n \to +\infty,
\]
and we are done. \(\square\)

**Lemma 6.2.** Under the assumptions of Theorem 4.3, it holds
\[
n^{q\eta/2-d}L^{-q/2}(n)K_n \xrightarrow{d} R, \quad n \to \infty.
\]

**Proof.** Consider the random variable
\[
K_n(q) = \sum_{m=q}^{\infty} \frac{J(m)}{m!} \int_{W_n} H_m(Y_t) dt.
\]
According to [26, Theorem 4] the random variables
\[
\frac{K_n}{\sqrt{\text{Var}K_n}}, \quad \frac{K_n(q)}{\sqrt{\text{Var}K_n(q)}}
\]
have the same limiting distributions as $n \to +\infty$. Furthermore, if $\eta \in (0, d/q)$ we have by [26, Theorem 5] that
\[
n^{q\eta/2-d}L^{-q/2}(n) \int_{W_n} H_q(Y_t)dt\]
converges in distribution to random variable $R$.  

If $\xi(y) \equiv 0$, the long memory part $K_n$ is not present and we apply Lemma 6.1. If $\xi(y) \not\equiv 0$, we note that the rate of convergence in Lemma 6.2 is slower than in Lemma 6.1, whenever $\eta \in (0, d/q)$. \hfill \Box

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**References**


