Stochastics II | **Universität Ulm**
WS 2019/2020 | Prof. Dr. Evgeny Spodarev
October 16, 2019 | Dr. Vitalii Makogin

Exercise sheet 1 (total – 18 points) till October 23, 2019

Exercise 1-1 (2 points)

Let ξ be a random variable with distribution function F. Prove that $\{X(t), t \in \mathbb{R}\}$ is a stochastic process, if

1. $X(t) = \max(\xi, t), t \in \mathbb{R}$.
2. $X(t) = \xi t, t \in \mathbb{R}$.

Draw the sample paths of the process X. Find one-dimensional distributions of the process X.

Exercise 1-2 (3 points)

Let τ be a random variable with uniform distribution on [0,1] and $\{X(t), t \in [0,1]\}$ be a waiting process corresponding to τ, that is, $X(t) = \mathbb{1}(\tau \leq t), t \in \mathbb{R}$. Draw the sample paths of the process X. Find all one-dimensional, two-dimensional, m-dimensional distributions of the process X.

Exercise 1-3 (3 points)

The number of real users in a social network "Handbook" grows continuously with constant rate $c_1 > 0$ (increases each day on c_1 users). Simultaneously, fake-users are created with constant rate $c_2 > c_1$. The researchers of "Handbook" scan the activity in the network in order to detect fake-accounts. Write down the formulas for the total number of network users at time moment t and draw the sample paths of the number of real and fake-users if

(a) the time intervals $\tau_i, i \geq 1$ between successive detections are independent, continuous positive random variables. After each detection the fixed number $L > 0$ of fake users are deleted.

(b) During every $i-$th successive period of $T > 0$ days, the researches detect independent random number X_i of fake users, which are deleted at the end of $i-th$ period. $X_i, i \geq 1$ follow negative binomial law $\text{NB}(r, p)$.

There were $K > 0$ of real users and no fake-accounts at time $t = 0$. For the case (b) determine the distribution of the number of fake accounts at moment $t > 0$ and write down the formula of the probability of "network's collapse", that is the portion of real users becomes less than 50%.

Exercise 1-4 (4 points)

Prove the following result (based on Kolmogorov's theorem).

Proposition 1. *The family of measures* $\mathbf{P}_{t_1,\dots,t_d}$ *on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), (t_1, \dots, t_d) \in T^d, d \geq 1$, *satisfies the conditions of symmetry and consistency iff for all* $d \geq 2, (s_1, \dots, s_d) \in \mathbb{R}^d$ *and* $(t_1, \dots, t_d) \in T^d$ *it holds* $\varphi_{\mathbf{P}_{t_1,\dots,t_d}}((s_1, \dots, s_d)) = \varphi_{\mathbf{P}_{t_{i_1},\dots,t_{i_d}}}((s_{i_1}, \dots, s_{i_d}))$ *for any permutation* $(1, \dots, d) \to (i_1, \dots, i_d)$, *and* $\varphi_{\mathbf{P}_{t_1,\dots,t_{d-1}}}((s_1, \dots, s_{d-1})) = \varphi_{\mathbf{P}_{t_1,\dots,t_d}}((s_1, \dots, s_{d-1}, 0))$.

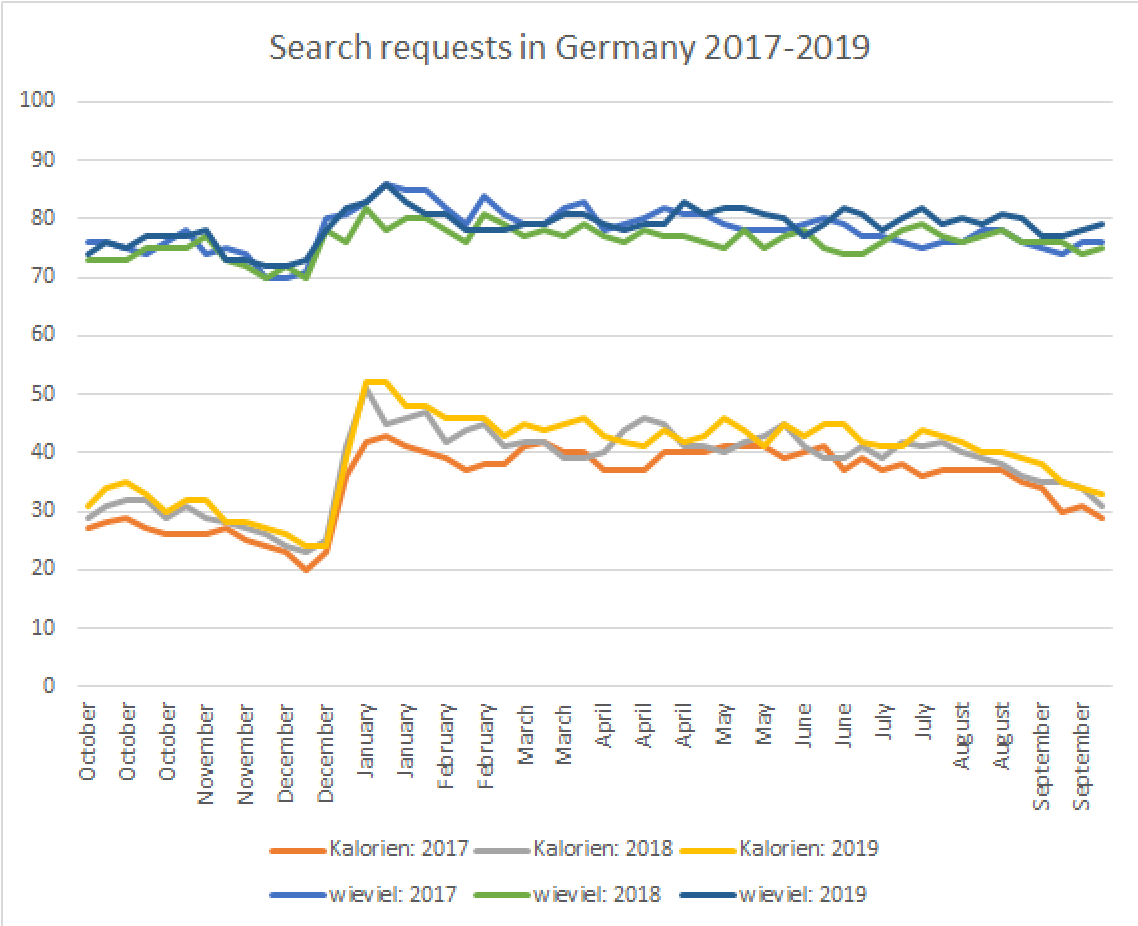

Figure 1: Numbers represent search interest relative to the highest point on the chart for the given time

Exercise 1-5 (4 points)

1. (2 points) Let $Z = (\xi_1, \ldots, \xi_n)^T$ be a random vector with independent $N(0,1)$-distributed components. Let A be a $n \times n$ matrix, $\bar{\mu} = (\mu_1, \ldots, \mu_n)^T$. Calculate the characteristic function of $n-$dimensional Gaussian random vector $Y = AZ + \bar{\mu}$. Find its mean vector and covariance matrix $\Sigma = (\sigma_{i,j})_{i,j}^n$.

2. (2 point) Show the existence of a Gaussian random function. Specify spaces $(S_{t_1,\ldots,t_n}, \mathcal{B}_{t_1,\ldots,t_n})$.

Exercise 1-6 (2 points)

In Figure 1, the search interest in Germany of "Kalorien" and "wieviel" are presented (the numbers of search requests relative to the highest point due to Google Trends[1]). The data are grouped in the period of October-September starting from October 2016.

- (1 point) Propose a stochastic model for such search requests.
- (1 point) Are the sequences of search requests "Kalorien" and "wieviel" independent during the observation time? Please argue your reasoning.

[1] see more datails on https://trends.google.com/trends/?geo=DE