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The Existence and Uniqueness of Strong Solutions

Thomas Stöwer | June 15, 2015 | Seminar on Stochastic Geometry and its applications

Theorem (Existence and uniqueness for SDEs)

Let $T > 0$ and $b\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|) \quad (1)$$

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y| \quad (2)$$

for all $x, y \in \mathbb{R}^n,\ t \in [0, T]$ and for some constants C, D. Let Z be a RV which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(\cdot)$, $s \geq 0$ and such that $\mathbb{E}[|Z|^2] < \infty$.

Theorem - continued

Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T,\ X_0 = Z$$

has a unique t-continous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(\cdot)$; $s \leq t$ and it holds that

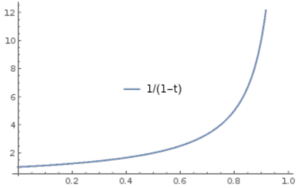
$$\mathbb{E}\left[\int_0^T |X_t|^2 \, \mathrm{d}t\right] < \infty.$$

Example 1

The ordinary differential equation

$$u'(t) = u(t)^2, \quad u(0) = 1$$

has the solution



$$u(t) = \frac{1}{1-t}, \quad 0 \leq t < 1.$$

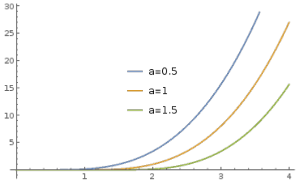
It is impossible to find a global, t-continous solution in this case.

Example 2

The ordinary differential equation

$$u'(t) = 3u(t)^{2/3}, \quad u(0) = 0$$

is solved by



$$u(t) = \begin{cases} 0 & \text{if } t \leq a \\ (t-a)^3 & \text{if } t > a \end{cases}$$

for any $a > 0$.

Uniqueness

Let $X_t(\omega)$ and $\hat{X}_t(\omega)$ be solutions with $X_0 = \hat{X}_0 = Z$, which satisfy all the conditions of the theorem. Then

$$\mathbb{P}\left[|X_t(\omega) - \hat{X}_t(\omega)| = 0 \text{ for all } t \in [0, T]\right] = 1.$$

Existence

Lemma

Define the process $Y_t^{(k)}$ *inductively by*

$$Y_t^{(0)} = X_0$$
$$Y_t^{(k+1)} = X_0 + \int_0^t b\left(s, Y_s^{(k)}\right) \mathrm{d}s + \int_0^t \sigma\left(s, Y_s^{(k)}\right) \mathrm{d}B_s.$$

Then there is a constant $A(C, D, T, \mathbb{E}[|X_0|^2])$ *such that*

$$\mathbb{E}\left[\left|Y_t^{(k+1)} - Y_t^{(k)}\right|^2\right] \leq \frac{A^{k+1} t^{k+1}}{(k+1)!}, \quad k \geq 0, t \in [0, T].$$

Existence

Lemma

The sequence $\left(Y_t^{(n)}(\omega)\right)_{n\in\mathbb{N}}$ *is uniformly convergent in* $[0, T]$ *for a.a.* ω. *We denote the limit by* X_t.

Existence

X_t of the Lemma above is adapted to the filtration F_t^Z, solves the stochastic differential equation and it holds that

$$\mathbb{E}\left[\int_0^T |X_t|^2 \, \mathrm{d}t\right] < \infty.$$

References

B. Øksendal, *Stochastic differential equations*, Springer, 2003.