The Markov Property

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Motion of a small particle in a moving liquid

- \( b(t,x) \in \mathbb{R}^3 \) velocity of the fluid
- \( X_t \in \mathbb{R}^3 \) position of particle at time \( t \)
- \( \sigma(t, x) \in \mathbb{R}^{3 \times 3} \)
- \( B_t \) 3-dim. Brownian motion

\[
dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t
\]
Definition - A (time-homogeneous) Itô diffusion

is a stochastic process $X_t(\omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$ satisfying a SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geq s, \quad X_s = x$$

where $B_t$ is a m-dim. Brownian motion, and $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ satisfying the condition

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|; \quad x, y \in \mathbb{R}^n$$
Remarks

- We denote the (unique) solution of the SDE by $X_t = X_t^{s,x}$ if $s > 0$ ( $X_t^x$ if $s=0$), $t \geq s$

- The Itô diffusion has the property of being time-homogeneous, i.e. for $s \geq 0$
  $$\{X_s^{s,x}\}_{h \geq 0}, \text{ and } \{X_h^x\}_{h \geq 0}$$ have the same $P^0$-distribution.
Let $\mathcal{M}_\infty$ be the $\sigma$-Algebra generated by the Itô diffusion $X_t(\omega)$.
Define $Q^x$ by

$$Q^x[X_{t_1} \in E_1, \ldots, X_{t_k} \in E_k] = P^0[X_{t_1}^x \in E_1, \ldots, X_{t_k}^x \in E_k]$$

where $E_i \subset \mathbb{R}^n$ are Borel sets;
$P^0$ the probability law of $B_t$ starting in 0.
Furthermore let $\mathcal{F}_t^{(m)}$ be the $\sigma$-Algebra generated by $\{B_r; r \leq t\}$ and
$\mathcal{M}_t$ the $\sigma$-Algebra generated by $\{X_r; r \leq t\}$
Theorem - The Markov Property

Let \( f \) be a bounded Borel function from \( \mathbb{R}^n \) to \( \mathbb{R} \) and \( X_t \) an Itô diffusion. Then for \( t, h \geq 0 \) it holds a.s. (w.r.t. \( P^0 \))

\[
\mathbb{E}^x [f(X_{t+h}) | \mathcal{F}_t^{(m)}](\omega) = \mathbb{E}^{X_t(\omega)}[f(X_h)]
\]

\( \mathbb{E}^x \) denotes the expectation w.r.t. \( Q^x \).

\( \mathbb{E}^y[f(X_h)] \) means \( \mathbb{E}[f(X_h^y)] \), where \( \mathbb{E} \) denotes the expectation w.r.t. \( P^0 \).
Remarks

- Definition: A (time-continuous) stochastic Process \( \{X_t : t \geq 0\} \) is called a (time-continuous) Markov Process, if it fulfills the Markov Property.

- Since \( \mathcal{M}_t \subseteq \mathcal{F}_t^{(m)} \), \( X_t \) is also a Markov Process w.r.t. the \( \sigma \)-algebras \( \{\mathcal{M}_t\}_{t \geq 0} \)
Definition - (strict) stopping time

Let \( \{ \mathcal{N}_t \}_{t \geq 0} \) be an increasing family of \( \sigma \)-algebras (of subsets of \( \Omega \)).
A function \( \tau : \Omega \to [0, \infty] \) is called a (strict) stopping time w.r.t. \( \{ \mathcal{N}_t \} \), if

\[
\{ \omega; \tau(\omega) \leq t \} \in \mathcal{N}_t, \text{ for all } t \geq 0.
\]
Example - first exit time

Let $X_t$ be an Itô diffusion and $U \subset \mathbb{R}^n$. Define $\tau_U = \inf\{t > 0; X_t \notin U\}$. Then $\tau_U$ is a stopping time (w.r.t. $\mathcal{M}_t$).
Definition - $\mathcal{N}_\tau$, $\mathcal{M}_\tau$ and $\mathcal{F}_\tau^{(m)}$

Let $\tau$ be a stopping time w.r.t. $\{\mathcal{N}_t\}$ and let $\mathcal{N}_\infty$ be the smallest $\sigma$-algebra containing $\mathcal{N}_t$ for all $t \geq 0$. Then $\mathcal{N}_\tau$ consists of all sets $N \in \mathcal{N}_\infty$ such that

$$N \cap \{\tau \leq t\} \in \mathcal{N}_t \text{ for all } t \geq 0.$$ 

In the case when $\mathcal{N}_t = \mathcal{M}_t$

$$\mathcal{M}_\tau = \text{the } \sigma \text{- algebra generated by } \{X_{\min(s,\tau)}; s \geq 0\}$$

In the case when $\mathcal{N}_t = \mathcal{F}_t^{(m)}$

$$\mathcal{F}_\tau^{(m)} = \text{the } \sigma \text{- algebra generated by } \{B_{\min(s,\tau)}; s \geq 0\}$$
Theorem - The Strong Markov Property

Let $f$ be a bounded Borel function on $\mathbb{R}^n$, $X_t$ an Itô diffusion and $\tau$ a stopping time w.r.t. $\mathcal{F}_t^{(m)}$, $\tau < \infty$ a.s. Then it holds a.s. (w.r.t. $P^0$)

$$\mathbb{E}^X[f(X_{\tau+h})|\mathcal{F}_{\tau}^{(m)}] = \mathbb{E}^{X_{\tau}}[f(X_h)] \text{ for all } h \geq 0.$$
Literature