# The Markov Property

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# Motion of a small particle in a moving liquid

- b(t,x) $\in \mathbb{R}^3$ velocity of the fluid
- $X_t \in \mathbb{R}^3$ position of particle at time t
- $\sigma(t,x) \in \mathbb{R}^{3\times 3}$
- $B_t$ 3-dim. Brownian motion

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

# Definition - A (time-homogeneous) Itô diffusion

is a stochastic process $X_t(\omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$ satisfying a SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geq s, \; X_s = x$$

where $B_t$ is a m-dim. Brownian motion, and $b : \mathbb{R}^n \to \mathbb{R}^n, \sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ satisfying the condition $|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|; x, y \in \mathbb{R}^n$

# Remarks

- We denote the (unique) solution of the SDE by $X_t = X_t^{s,x}$ *if* $s > 0$ ( $X_t^x$ if s=0), $t \geq s$
- The Itô diffusion has the property of being time-homogeneous, i.e. for s $\geq 0$ $\{X_{s+h}^{s,x}\}_{h\geq 0},$ *and* $\{X_h^x\}_{h\geq 0}$ have the same $P^0$-distribution.

Let $\mathcal{M}_\infty$ be the $\sigma$-Algebra generated by the Itô diffusion $X_t(\omega)$.
Define $Q^x$ by

$$Q^x[X_{t_1} \in E_1, ..., X_{t_k} \in E_k] = P^0[X^x_{t_1} \in E_1, ..., X^x_{t_k} \in E_k]$$

where $E_i \subset \mathbb{R}^n$ are Borel sets;
$P^0$ the probability law of $B_t$ starting in 0.
Furthermore let
$\mathcal{F}_t^{(m)}$ be the $\sigma$-Algebra generated by $\{B_r; r \leq t\}$ and
$\mathcal{M}_t$ the $\sigma$-Algebra generated by $\{X_r; r \leq t\}$

# Theorem - The Markov Property

Let f be a bounded Borel function from $\mathbb{R}^n$ to $\mathbb{R}$ *and* $X_t$ an Itô diffusion. Then for $t, h \geq 0$ it holds a.s. (w.r.t. $P^0$ )

$$\mathbb{E}^x[f(X_{t+h})|\mathcal{F}_t^{(m)}](\omega) = \mathbb{E}^{X_t(\omega)}[f(X_h)]$$

$\mathbb{E}^x$ denotes the expectation w.r.t. $Q^x$.
$\mathbb{E}^y[f(X_h)]$ means $\mathbb{E}[f(X_h^y)]$, where $\mathbb{E}$ denotes the expectation w.r.t. $P^0$

# Remarks

- Definition: A (time-continuous) stochstic Process $\{X_t : t \geq 0\}$ is called a (time-continuous) Markov Process, if it fulfills the Markov Property.
- Since $\mathcal{M}_t \subseteq \mathcal{F}_t^{(m)}, X_t$ is also a Markov Process w.r.t. the $\sigma$-algebras $\{\mathcal{M}_t\}_{t \geq 0}$

## Definition - (strict) stopping time

Let $\{\mathcal{N}_t\}_{t\geq 0}$ be an increasing family of $\sigma$-algebras (of subsets of $\Omega$).
A function $\tau : \Omega \to [0, \infty]$ is called a (strict) stopping time w.r.t. $\{\mathcal{N}_t\}$, if

$$\{\omega; \tau(\omega) \leq t\} \in \mathcal{N}_t, \text{ for all } t \geq 0.$$

## Example - first exit time

Let $X_t$ be an Itô diffusion and U $\subset \mathbb{R}^n$.
Define $\tau_U = inf\{t > 0; X_t \notin U\}$.
Then $\tau_U$ is a stopping time (w.r.t. $\mathcal{M}_t$ ).

## Definition - $\mathcal{N}_\tau,\ \mathcal{M}_\tau$ *and* $\mathcal{F}_\tau^{(m)}$

Let $\tau$ be a stopping time w.r.t. $\{\mathcal{N}_t\}$ and let $\mathcal{N}_\infty$ be the smallest $\sigma$-algebra containing $\mathcal{N}_t$ for all t $\geq$ 0.
Then $\mathcal{N}_\tau$ consists of all sets $N \in \mathcal{N}_\infty$ such that

$$N \cap \{\tau \leq t\} \in \mathcal{N}_t \text{ for all } t \geq 0.$$

In the case when $\mathcal{N}_t = \mathcal{M}_t$

$$\mathcal{M}_\tau = \text{the } \sigma - \text{algebra generated by } \{X_{min(s,\tau)}; s \geq 0\}$$

In the case when $\mathcal{N}_t = \mathcal{F}_t^{(m)}$

$$\mathcal{F}_\tau^{(m)} = \text{the } \sigma - \text{algebra generated by } \{B_{min(s,\tau)}; s \geq 0\}$$

# Theorem - The Strong Markov Property

Let f be a bounded Borel function on $\mathbb{R}^n, X_t$ an Itô diffusion and $\tau$ a stopping time w.r.t. $\mathcal{F}_t^{(m)}, \tau < \infty$ a.s. Then it holds a.s. (w.r.t. $P^0$)

$$\mathbb{E}^x[f(X_{\tau+h})|\mathcal{F}_\tau^{(m)}] = \mathbb{E}^{X_\tau}[f(X_h)] \textit{ for all } h \geq 0.$$

# Literature

Bernt Øksendal, Stochastic differential equations, Springer, 2003