An introduction to SDEs with some examples

Lukas Petrich | June 8, 2015 | Seminar on Stochastic Geometry and its applications
Content

Introduction

Example: Population Growth

Example: Electric Charge

Further Examples
Itô formula I

Let

\[ X(t) = X(0) + \int_0^t u(s) \, ds + \int_0^t v(s) \, dB(s) \]

be a n-dimensional Itô process (in matrix notation). Let \( g(t, x) = (g_1(t, x), \ldots g_p(t, x)), p \in \mathbb{N}, \) be a \( C^2 \) map from \([0, \infty) \times \mathbb{R}^n \) into \( \mathbb{R}^p. \)
Itô formula II

Then the process $Y(t, \omega) = g(t, X(t))$ is again an Itô process, whose $k$-th component is given by

$$Y_k(t) = Y_k(0) + \int_0^t \left( \frac{\partial g_k}{\partial t}(s, X(s)) + \sum_{i=1}^n \frac{\partial g_k}{\partial x_i}(s, X(s)) u_i(s) \right. \left. + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 g_k}{\partial x_i \partial x_j}(s, X(s)) v_i(s) v_j(s)^\top \right) \, ds$$

$$+ \sum_{i=1}^n \int_0^t \frac{\partial g_k}{\partial x_i}(s, X(s)) v_i(s) \, dB(s)$$

where $v_i(s)$ denotes the $i$-th row of $v$. 
Integration by parts

Let $f \in C^2([0, \infty])$. Then

$$
\int_0^t f(s) \, dB_s = f(t)B_t - \int_0^t f'(s)B_s \, ds
$$
Definition

A stochastic differential equation (SDE) is an equation of the form

\[ X_t = X_0 + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s \]

or in differential form

\[ dX_t = b(t, X_t) \, dt + \sigma(t, X_t) \, dB_t. \]
Example: Population Growth

Consider the stochastic differential equation

\[ N_t = N_0 + \int_0^t rN_s \, ds + \int_0^t \alpha N_s \, dB_s \]

where \( r \in \mathbb{R} \) and \( \alpha, N_0 > 0 \). With the Itô formula we get

\[ \log N_t = \log N_0 + (r - \frac{1}{2} \alpha^2) t + \alpha B_t \]

and thus

\[ N_t = N_0 \exp \left( (r - \frac{1}{2} \alpha^2) t + \alpha B_t \right). \]
Example: Population Growth

Consider the stochastic differential equation

\[ N_t = N_0 + \int_0^t r N_s \, ds + \int_0^t \alpha N_s \, dB_s \]

where \( r \in \mathbb{R} \) and \( \alpha, N_0 > 0 \). With the Itô formula we get

\[ \log N_t = \log N_0 + (r - \frac{1}{2} \alpha^2) t + \alpha B_t \]

and thus

\[ N_t = N_0 \exp \left( (r - \frac{1}{2} \alpha^2) t + \alpha B_t \right). \]
For the solution $N_t$ it holds that

$$\mathbb{E}N_t = N_0 e^{rt}$$
The law of iterated logarithm

For a Brownian Motion \( \{ B_t : t \geq 0 \} \) it holds

\[
\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1 \quad \text{a. s.}
\]
Asymptotic behaviour of the solution $N_t$

With the law of iterated logarithm it follows:

(i) If $r > \frac{1}{2} \alpha^2$ then $N_t \to \infty$ as $t \to \infty$, a.s.

(ii) If $r < \frac{1}{2} \alpha^2$ then $N_t \to 0$ as $t \to \infty$, a.s.

(iii) If $r = \frac{1}{2} \alpha^2$ then $N_t$ will fluctuate between arbitrary large and arbitrary small (positive) values as $t \to \infty$, a.s.
Example: Electric Charge

Consider the 2-dimensional SDE

\[ X(t) = X(0) + \int_0^t A X(s) \, ds + \int_0^t H(s) \, ds + \int_0^t K \, dB(s) \]

where

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & -R \\ CL & L \end{pmatrix}, \quad H(t) = \begin{pmatrix} 0 \\ 1 \\ L \end{pmatrix} G_t, \quad K = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Here, \( \alpha, C, L \) and \( R \) are positive constants and \( G : [0, \infty) \to [0, \infty), \ t \mapsto G_t. \)
If we apply the 2-dimensional Itô formula with $g(t, x_1, x_2) = \exp(-At)(x_1, x_2)^\top$ and integrate by parts, we get the solution

$$X(t) = \exp(At) \left( X(0) + \exp(-At)KB(t) + \int_0^t \exp(-As) \left( H(s) + AKB(s) \right) ds \right)$$

where $\exp(F) = \sum_{k=0}^{\infty} \frac{F^k}{k!}$ is the matrix exponential.
The function \( X_t = \frac{B_t}{1+t} \) where \( B_0 = 0 \) solves

\[
X_t = -\int_0^t \frac{1}{1+s} X_s \, ds + \int_0^t \frac{1}{1+s} \, dB_s.
\]
The function $X_t = \sin B_t$ with $B_0 = a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ solves

$$X_t = \sin(a) - \int_0^t \frac{1}{2} X_s \, ds + \int_0^t \sqrt{1 - X_s^2} \, dB_s$$

for $0 \leq t < \inf \left\{ s > 0 : B_s \not\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}$. 
References

Bernt Øksendal.  
*Stochastic differential equations.*  