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An introduction to SDEs with some examples

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Content

Itô formula I

Let

$$X(t) = X(0) + \int_0^t u(s)\,\mathrm{d}s + \int_0^t v(s)\,\mathrm{d}B(s)$$

be a n-dimensional Itô process (in matrix notation). Let $g(t,x) = (g_1(t,x), \ldots g_p(t,x))$, $p \in \mathbb{N}$, be a C^2 map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$.

Itô formula II

Then the process $Y(t,\omega) = g(t, X(t))$ is again an Itô process, whose k-th component is given by

$$\begin{aligned} Y_k(t) =& Y_k(0) + \int_0^t \left(\frac{\partial g_k}{\partial t}(s, X(s)) + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(s, X(s)) u_i(s) \right. \\ &\left. + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(s, X(s)) v_i(s) v_j(s)^\top \right) \mathrm{d}s \\ &+ \sum_{i=1}^{n} \int_0^t \frac{\partial g_k}{\partial x_i}(s, X(s)) v_i(s) \, \mathrm{d}B(s) \end{aligned}$$

where $v_i(s)$ denotes the i-th row of v.

Integration by parts

Let $f \in C^2([0,\infty])$. Then

$$\int_0^t f(s)\,\mathrm{d}B_s = f(t)B_t - \int_0^t f'(s)B_s\,\mathrm{d}s$$

Definition

A stochastic differential equation (SDE) is an equation of the form

$$X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$$

or in differential form

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t.$$

Example: Population Growth

Consider the stochastic differential equation

$$N_t = N_0 + \int_0^t rN_s \, \mathrm{d}s + \int_0^t \alpha N_s \, \mathrm{d}B_s$$

where $r \in \mathbb{R}$ and $\alpha, N_0 > 0$. With the Itô formula we get

$$\log N_t = \log N_0 + (r - \frac{1}{2}\alpha^2)t + \alpha B_t$$

and thus

$$N_t = N_0 \exp\left((r - \frac{1}{2}\alpha^2)t + \alpha B_t\right).$$

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For the solution N_t it holds that

$$\mathbb{E}N_t = N_0 e^{rt}$$

The law of iterated logarithm

For a Brownian Motion $\{B_t : t \geq 0\}$ it holds

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1 \quad \text{a. s.}$$

Asymptotic behaviour of the solution N_t

With the law of iterated logarithm it follows:

(i) If $r > \frac{1}{2}\alpha^2$ then $N_t \to \infty$ as $t \to \infty$, a.s.

(ii) If $r < \frac{1}{2}\alpha^2$ then $N_t \to 0$ as $t \to \infty$, a.s.

(iii) If $r = \frac{1}{2}\alpha^2$ then N_t will fluctuate between arbitrary large and arbitrary small (positive) values as $t \to \infty$, a.s.

Example: Electric Charge

Consider the 2-dimensional SDE

$$X(t) = X(0) + \int_0^t AX(s)\,\mathrm{d}s + \int_0^t H(s)\,\mathrm{d}s + \int_0^t K\,\mathrm{d}B(s)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}, H(t) = \begin{pmatrix} 0 \\ \frac{1}{L}G_t \end{pmatrix}, K = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}.$$

Here, α, C, L and R are positive constants and $G : [0,\infty) \to [0,\infty)\,, t \mapsto G_t$.

If we apply the 2-dimensional Itô formula with $g(t, x_1, x_2) = \exp(-At)(x_1, x_2)^\top$ and integrate by parts, we get the solution

$$X(t) = \exp(At)\Big(X(0) + \exp(-At)KB(t) + \int_0^t \exp(-As)\left(H(s) + AKB(s)\right) \mathrm{d}s \Big)$$

where $\exp(F) = \sum_{k=0}^{\infty} \frac{F^k}{k!}$ is the matrix exponential.

The function $X_t = \frac{B_t}{1+t}$ where $B_0 = 0$ solves

$$X_t = -\int_0^t \frac{1}{1+s} X_s \,\mathrm{d}s + \int_0^t \frac{1}{1+s} \,\mathrm{d}B_s.$$

The function $X_t = \sin B_t$ with $B_0 = a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ solves

$$X_t = \sin(a) - \int_0^t \frac{1}{2} X_s \, \mathrm{d}s + \int_0^t \sqrt{1 - X_s^2} \, \mathrm{d}B_s$$

for $0 \leq t < \inf\left\{s > 0 : B_s \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$.

References

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Stochastic differential equations.
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