Functional Central Limit Theorem for the Measure of Level Sets Generated by a Gaussian Random Field

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Abstract

The level sets of a stationary and isotropic Gaussian random field with smooth realizations are studied. We establish a Hilbert space functional central limit theorem for their Hausdorff measure.

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1 Introduction

The geometrical properties of level and excursion sets of Gaussian random fields have been investigated for many years. This is motivated by their applications to modeling of random geometrical structures, see, e.g., [4]. While the volume of an excursion set is a relatively simple feature, other geometric functionals (such as surface area, Euler characteristics etc.) are rather difficult to study due to their complex dependence on realizations. First results were achieved for Gaussian processes on the real line, starting from the classical Rice formula for the expected number of crossings [2, p. 263]. Moment inequalities (including optimal ones) were given by Ito, Cramer, Belyaev, Geman, Wschebor etc., and central limit theorems by Malevich, Cuzick, Piterbarg, Slud, see the review by Kratz [8] and references therein.

The results for random fields on a multidimensional index space (and, more generally, on smooth manifolds) have appeared in more recent years. Adler [1] and Wschebor [13] gave versions of the Rice formula for the expectations of various geometrical characteristics of excursion sets. Its further generalizations can be found in [2, Ch. 11–13]. Kratz and León [9] developed a general method of proving limit theorems for non-smooth functionals of trajectories, based on the Hermite expansions. In particular, they established the asymptotic normality of the length of the level curve of a Gaussian random field on $\mathbb{R}^2$. Iribarren [7] proved a functional limit theorem for the integrals on the level set of mixing random fields.

It is natural to consider the surface areas of excursion sets determined by levels $x \in \mathbb{R}$ as a random process, and to study its limit behavior. The aim of the present paper is to provide a functional central limit theorem for the measure of level sets generated by Gaussian random fields. In 2-dimensional case, this measure is just the length of the level curve.

A related functional central limit theorem is proven in [10]. There we consider a stationary, square-integrable, associated random field with a.s. continuous trajectories such that the components X_t have bounded density and the covariance function decays rapidly and prove a functional central limit theorem for the volume of their excursion sets.

We start with some notation. Let $d > 1$ and $X = \{X_s, s \in \mathbb{R}^d\}$ be a centered stationary and isotropic Gaussian random field whose realizations are a.s. C^1. Suppose that the covariance function R of X satisfies the inequality

$$|R(s)| + \frac{1}{\sqrt{1-R(s)}} \sum_{j=1}^{d} \left| \frac{\partial R(s)}{\partial s_j} \right| + \sum_{j,q=1}^{d} \left| \frac{\partial^2 R(s)}{\partial s_j \partial s_q} \right| < g(s)$$

for all $s \neq 0$, where g is bounded, continuous, $g(s) \to 0$ as $s \to \infty$ and $\int_{\mathbb{R}^d} \sqrt{g(s)} ds < \infty$. Note that for $j = 1, \ldots, d$ the function

$$\frac{1}{\sqrt{1-R(s)}} \frac{\partial R(s)}{\partial s_j}$$

is bounded near the origin, so the requirement concerns only the behavior at infinity. We require also that for any $s \in \mathbb{R}^d \setminus \{0\}$ the Gaussian random vector $(X_0, X_s, \nabla X_0, \nabla X_s)$ is nondegenerate. We will also normalize X so that $R(0) = 1$.

Definition 1. The *level set* of a random field X at level $x \in \mathbb{R}$ is the random set $\{s \in \mathbb{R}^d : X_s = x\}$. The *excursion set* at level x is the set $\{s \in \mathbb{R}^d : X_s \geq x\}$.

Remark 1. These definitions are closely connected. Indeed, the boundary of the excursion set is included in the level set at the same level. On the other hand if $X_s = x$, then either s is in the boundary of excursion set or $\nabla X_s = 0$. If, e.g., the field X is C^2, then there is at most a countable number of points satisfying the last equality [2, Cor. 11.2.2].

In what follows, we denote by $|I|$ the cardinality of a finite set I, and by $\mathcal{H}_r(B)$ the r-dimensional Hausdorff measure of a set B, so $\mathcal{H}_d$ is the usual Lebesgue measure. For a vector $u \in \mathbb{R}^m$, its Euclidean norm is $\|u\|$.
Denote by

$$N_t(x) = \frac{\mathcal{H}_{d-1}\{s \in [0,t]^d : X_s = x\} - \mathsf{E}\mathcal{H}_{d-1}\{s \in [0,t]^d : X_s = x\}}{\sqrt{t^d}},$$

for $t > 0$ and $x \in \mathbb{R}$, the centered and normalized Hausdorff measure of the level set. Let μ be the standard Gaussian measure on $\mathbb{R}$. The next theorem is the invariance principle for the family $\{N_t, t > 0\}$ in the space $L^2(\mathbb{R}, \mu)$.

2 Main Result and Proof

Theorem 1. *Under the conditions above, the family of random elements* $\{N_t, t > 0\}$ *converges weakly in the Hilbert space* $L^2(\mathbb{R}, \mu)$*, as* $t \to \infty$*, to the process* $\lambda^{1-d/2}N$*, where* $\lambda^2 = -\partial^2 R(0)/\partial s_1^2$ *and* N *is the centered Gaussian random element in* $L^2(\mathbb{R}, \mu)$ *with covariance operator given by the equality*

$$\mathsf{Var}(N, f)_{L^2(\mathbb{R},\mu)} = \\ = \frac{1}{2\pi} \int_{\mathbb{R}^d} cov\left(f(X_0) e^{-X_0^2/2} \|\nabla X_0\|, f(X_s) e^{-X_s^2/2} \|\nabla X_s\| \right) ds, \quad (1)$$

$f \in L^2(\mathbb{R}, \mu)$.

We will consider only the case $\lambda = 1$. For, if the theorem is true in that case and X is a random field not satisfying this extra condition, then the theorem is proved for X by considering the random field $\widetilde{X}$, where $\widetilde{X}(s) = X(\lambda^{-1}s),\ s \in \mathbb{R}^d$.
According to [11, Lemma 1.6], weak convergence of $\{N_t, t > 0\}$ to N in the Hilbert space $L^2(\mathbb{R}, \mu)$ is equivalent to the tightness of $\{N_t,\, t > 0\}$ and the relation

$$\mathbb{E} \exp\left(i\,(N_t, f)\right) \to \mathbb{E} \exp\left(i\,(N, f)\right) \quad (2)$$

as $t \to \infty$ for any $f \in L^2(\mathbb{R}, \mu)$. We start from tightness, which is more difficult, and then prove convergence of characteristic functions.

Let H_k, $k \geq 0$, be the probabilists' Hermite polynomial of order k. The functions $\left\{H_k/\sqrt{k!}\right\}$ form an orthonormal base in $L^2(\mathbb{R},\mu)$. Hence we should prove ([11, Theorem 1.13]) that

$$\lim_{M\to\infty}\sup_{t>1}\sum_{k=M}^{\infty}\mathsf{E}\left(N_t, H_k/\sqrt{k!}\right)^2_{L^2(\mathbb{R},\mu)} = 0. \tag{3}$$

By the coarea formula ([6, Theorem 3.2.12]) one almost surely has

$$\begin{aligned}
&\left(N_t, H_k/\sqrt{k!}\right)_{L^2(\mathbb{R},\mu)} = \frac{1}{\sqrt{k!}}\int_{\mathbb{R}} N_t(x)H_k(x)d\mu \\
&= \frac{1}{\sqrt{2\pi t^d k!}}\int_{[0,t]^d} H_k(X_s)e^{-X_s^2/2}\|\nabla X_s\|ds - \\
&\qquad - \frac{1}{\sqrt{2\pi t^d k!}}\int_{[0,t]^d} \mathsf{E}H_k(X_s)e^{-X_s^2/2}\|\nabla X_s\|ds \\
&= \frac{1}{\sqrt{2\pi t^d k!}}\int_{[0,t]^d} H_k(X_s)e^{-X_s^2/2}\overline{\|\nabla X_s\|}ds + \\
&\qquad + \frac{\mathsf{E}\|\nabla X_0\|}{\sqrt{2\pi t^d k!}}\int_{[0,t]^d}\left(H_k(X_s)e^{-X_s^2/2} - \mathsf{E}H_k(X_s)e^{-X_s^2/2}\right)ds \\
&=: D_1(t,k) + D_2(t,k).
\end{aligned} \tag{4}$$

Here and in what follows we denote $\overline{\|\nabla X_s\|} = \|\nabla X_s\| - \mathsf{E}\|\nabla X_s\|$ for brevity. We have also used that X_s and ∇X_s are independent. Note that for $M \geq 0$

$$\begin{aligned}
&\sup_{t>1}\sum_{k=M}^{\infty}\mathsf{E}\left(N_t, H_k/\sqrt{k!}\right)^2_{L^2(\mathbb{R},\mu)} \leq \\
&\qquad \leq 2\sup_{t>1}\sum_{k=M}^{\infty}\mathsf{E}D_1^2(t,k) + 2\sup_{t>1}\sum_{k=M}^{\infty}\mathsf{E}D_2^2(t,k).
\end{aligned} \tag{5}$$

We will split the proof in two parts: in the first we work with D_1 (which is more involved), and in the second one with D_2.

Part 1

First we prove that the series in (5) containing D_1 converges for each $t > 0$. The idea is to interchange the sum over k and the spatial integral and verify that it is legal.

A measurable function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is called *translation-invariant* if $f(x+h, y+h) = f(x,y)$ for all $x, y, h \in \mathbb{R}^d$.

Lemma 2. *If the function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is translation-invariant and $\int_{\mathbb{R}^d}|f(0,y)|dy < \infty$, then*

$$t^{-d}\int_{[0,t]^d}\int_{[0,t]^d} f(s,u)duds = \int_{\mathbb{R}^d} f(0,y)\psi(t,y)dy,$$

where $\psi(t,y) = t^{-d}\prod_{j=1}^{d}(t-|y_j|)^+$.

Proof. Use the substitution $y = u - s$ and then change the order of integration. □

Take arbitrary $z \in (0,1)$ and consider the power series

$$
\begin{aligned}
&\sum_{k=0}^{\infty} z^k \mathsf{E} D_1^2(t,k) \\
&= \sum_{k=0}^{\infty} \mathsf{E} \int_{\mathbb{R}^d} \frac{z^k}{k!} H_k(X_0) H_k(X_s) e^{-X_0^2/2 - X_s^2/2} \overline{\|\nabla X_0\| \|\nabla X_s\|} \psi(t,s) ds \\
&= \sum_{k=0}^{\infty} \int_{\mathbb{R}^d} \mathsf{E} \frac{z^k}{k!} H_k(X_0) H_k(X_s) e^{-X_0^2/2 - X_s^2/2} \overline{\|\nabla X_0\| \|\nabla X_s\|} \psi(t,s) ds \\
&= \int_{\mathbb{R}^d} \mathsf{E} \sum_{k=0}^{\infty} \frac{z^k}{k!} H_k(X_0) H_k(X_s) e^{-X_0^2/2 - X_s^2/2} \overline{\|\nabla X_0\| \|\nabla X_s\|} \psi(t,s) ds. \quad (6)
\end{aligned}
$$

Here the first equality is due to Lemma 2. The second equality is obvious, since we take the expectation of the integral of a continuous random field over a bounded set. To verify the third one, note that by the Cauchy-Schwarz inequality

$$
\mathsf{E}|H_k(X_0) H_k(X_s) e^{-X_0^2/2 - X_s^2/2} \overline{\|\nabla X_0\| \|\nabla X_s\|}| \le \mathsf{E} H_k^2(X_0) \overline{\|\nabla X_0\|}^2 \le dk!.
$$

Hence the series obtained by multiplying the left hand side by $z^k/k!$ is summable, and by Fubini theorem the third equality in (6) also holds.

Lemma 3. ([12]). *For any* $x, y \in \mathbb{R}$ *and* $z \in (0,1)$ *one has*

$$
\sum_{k=0}^{\infty} \frac{z^k}{k!} H_k(x) H_k(y) e^{-x^2/2 - y^2/2} = \frac{1}{\sqrt{1-z^2}} \exp\left\{ -\frac{x^2 + y^2 - 2xyz}{2 - 2z^2} \right\}.
$$

Note that in the right hand side here we have the density of a Gaussian random vector (ξ, η) with mean zero and covariance matrix $\begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix}$ (up to a multiple 2π). Thus,

$$
\begin{aligned}
&\sum_{k=0}^{\infty} z^k \mathsf{E} D_1^2(t,k) = \quad (7) \\
&= \int_{\mathbb{R}^d} \frac{1}{2\pi\sqrt{1-z^2}} \mathsf{E} \exp\left\{ -\frac{X_0^2 + X_s^2 - 2X_0 X_s z}{2 - 2z^2} \right\} \overline{\|\nabla X_0\| \|\nabla X_s\|} \psi(t,s) ds.
\end{aligned}
$$

Our next goal is to evaluate the limit of the right hand side in (7) when $z \nearrow 1$.

Let $p(x,y,u,v)$ be the density of random vector $(X_0, X_s, \nabla X_0, \nabla X_s)$, $p(x,y)$ be the density of (X_0, X_s) and $q_z(x,y)$ be the density of (ξ,η). With the notation $\overline{\|u\|} = \|u\| - \mathsf{E}\|\nabla X_0\|$ and $\overline{\|v\|} = \|v\| - \mathsf{E}\|\nabla X_s\|$ (for $u, v \in \mathbb{R}^d$) we obtain for any $s \in \mathbb{R}^d$, $s \neq 0$, that

$$\begin{aligned}
&\frac{1}{\sqrt{1-z^2}}\mathsf{E}\exp\left\{-\frac{X_0^2+X_s^2-2X_0X_sz}{2-2z^2}\right\}\overline{\|\nabla X_0\|\|\nabla X_s\|} \\
&\qquad = 2\pi\int_{\mathbb{R}^{2d+2}} q_z(x,y)\overline{\|u\|\|v\|}p(x,y,u,v)dudvdydx \\
&\qquad = 2\pi\mathsf{E}\int_{\mathbb{R}^{2d}} \overline{\|u\|\|v\|}p(\xi,\eta,u,v)dudv \\
&\qquad \to 2\pi\mathsf{E}\int_{\mathbb{R}^{2d}} \overline{\|u\|\|v\|}p(\xi,\xi,u,v)dudv \\
&\qquad = \sqrt{2\pi}\int_{\mathbb{R}} e^{-x^2/2}\int_{\mathbb{R}^{2d}} \overline{\|u\|\|v\|}p(x,x,u,v)dudvdx \qquad (8)
\end{aligned}$$

as $z \nearrow 1$, because $(\xi,\eta) \to (\xi,\xi)$ in distribution, and the function

$$(x,y) \mapsto \int_{\mathbb{R}^{2d}} \overline{\|u\|\|v\|}p(x,y,u,v)dudv$$

is continuous and bounded. Indeed, for random vectors W_1 and W_2 having common density one has

$$p_{W_1|W_2=w_2}(w_1) = \frac{p_{W_1,W_2}(w_1,w_2)}{p_{W_2}(w_2)} \qquad (9)$$

provided that the denominator is positive. Hence

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{2d}} \overline{\|u\|\|v\|}p(x,y,u,v)dudv\right| \\
&\qquad = p(x,y)\left|\mathsf{E}\left(\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = x, X_s = y\right)\right| \\
&\qquad\qquad \le p(x,y)\mathsf{E}\left(\|\nabla X_0\|^2 + \|\nabla X_s\|^2|X_0 = x, X_s = y\right),
\end{aligned}$$

but the latter conditional expectation is of quadratic order in x, y, while $p(x,y)$ decreases exponentially as $x^2 + y^2 \to \infty$.
Using (8) we see that the expression inside the integral in the right hand side (7) has a limit as $z \nearrow 1$. In order to prove the convergence of integrals, we should bound the same expression with an integrable function, uniformly in $z \in (0,1)$. We proceed with $\|\cdot\|$ instead of $\overline{\|\cdot\|}$, and similar expressions

containing $\mathsf{E}\|\nabla X_0\|$ are estimated analogously. We have

$$\frac{1}{2\pi\sqrt{1-z^2}}\mathsf{E}\exp\left\{-\frac{X_0^2+X_s^2-2X_0X_sz}{2-2z^2}\right\}\|\nabla X_0\|\|\nabla X_s\|$$
$$= \mathsf{E}q_z(X_0,X_s)\|\nabla X_0\|\|\nabla X_s\|$$
$$= \int_{\mathbb{R}^2} q_z(x,y)\int_{\mathbb{R}^{2d}}\|u\|\|v\|p(x,y,u,v)dudvdxdy$$
$$\leq \int_{\mathbb{R}^2} q_z(x,y)p(x,y)\mathsf{E}\left(\|\nabla X_0\|^2|X_0=x,X_s=y\right)dxdy$$
$$\leq \sup_{x,y\in\mathbb{R}^2} p(x,y)\mathsf{E}\left(\|\nabla X_0\|^2|X_0=x,X_s=y\right)$$
$$\leq d\sup_{x,y\in\mathbb{R}^2} p(x,y)\mathsf{E}\left(\left(\frac{\partial X_0}{\partial s_1}\right)^2|X_0=x,X_s=y\right).$$

Lemma 4. *For any $s\neq 0$, the random vector $\nabla X_0-\alpha(s)X_0-\beta(s)X_s$ does not depend on (X_0,X_s), where the vector functions α and β are*

$$\alpha(s)=-\frac{R(s)}{1-R^2(s)}\nabla R(s),$$

$$\beta(s)=\frac{1}{1-R^2(s)}\nabla R(s).$$

Proof. Straightforward computations.
By Lemma 4 and standard Gaussian argument, for any $x,y\in\mathbb{R}$ and $s\neq 0$ one has

$$\mathsf{E}\left(\left(\frac{\partial X_0}{\partial s_1}\right)^2|X_0=x,X_s=y\right)$$
$$=\mathsf{E}\left(\frac{\partial X_0}{\partial s_1}-\alpha_1(s)X_0-\beta_1(s)X_s\right)^2+\left(\alpha_1(s)x+\beta_1(s)y\right)^2$$
$$\leq 1+(\alpha_1(s)x+\beta_1(s)y)^2.$$

Therefore

$$p(x,y)\mathsf{E}\left(\left(\frac{\partial X_0}{\partial s_1}\right)^2|X_0=x,X_s=y\right)$$
$$\leq\frac{1}{\sqrt{1-R^2(s)}}+\frac{(\partial R(s)/\partial s_1)^2}{(1-R^2(s))^{5/2}}\sup_{x,y\in\mathbb{R}^2}(y-R(s)x)^2\exp\left\{-\frac{x^2+y^2-2xyR(s)}{2-2R^2(s)}\right\}$$
$$\leq\frac{1}{\sqrt{1-R^2(s)}}+\frac{(\partial R(s)/\partial s_1)^2}{(1-R^2(s))^{3/2}}=:\rho(s). \quad (10)$$

This function does not depend on z and its product with $\psi(t,s)$ is integrable over $\mathbb{R}^d$. Thus, (7)–(8) yield that for any $t>0$

$$\int_{\mathbb{R}^d} \frac{1}{\sqrt{1-z^2}} \mathsf{E} \exp\left\{-\frac{X_0^2+X_s^2-2X_0X_sz}{2-2z^2}\right\} \overline{\|\nabla X_0\|\|\nabla X_s\|}\psi(t,s)ds \\ \to \int_{\mathbb{R}^d} \sqrt{2\pi} \int_{\mathbb{R}} e^{-x^2/2} \int_{\mathbb{R}^{2d}} \overline{\|u\|\|v\|} p(x,x,u,v)dudvdx\psi(t,s)ds \quad (11)$$

when $z \to 1$. Recall an elementary proposition.

Lemma 5. *Suppose that the sequence $\{a_k, k \geq 0\}$ is nonnegative and $f(z) = \sum_{k=0}^{\infty} a_k z^k < \infty$ for all $z \in (0,1)$. Then, if $\lim_{z \nearrow 1} f(z) = L < \infty$, then $\sum_{k=0}^{\infty} a_k = L$.*

Applying Lemma 5, (6) and (11) we come to equality

$$\sum_{k=0}^{\infty} \mathsf{E} D_1^2(t,k) \\ = \sum_{k=0}^{\infty} \frac{1}{2\pi k!} \int_{\mathbb{R}^d} \mathsf{E} H_k(X_0) H_k(X_s) e^{-X_0^2/2-X_s^2/2} \overline{\|\nabla X_0\|\|\nabla X_s\|}\psi(t,s)ds \\ = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} \int_{\mathbb{R}^{2d}} \overline{\|u\|\|v\|} p(x,x,u,v)dudvdx\psi(t,s)ds. \quad (12)$$

Thus we have established the (possibly non-uniform) convergence of the first series in (5). To obtain the uniform convergence we estimate the function under the left integral in (11) uniformly in t.

Lemma 6. *For random vectors $\zeta_1, \zeta_2, \zeta_3$ having common density, such that dimensions of ζ_1 and ζ_2 are the same, one has*

$$p_{\zeta_1,\zeta_3|\zeta_1=\zeta_2}(w_1,w_3) = \frac{p_{\zeta_1,\zeta_2,\zeta_3}(w_1,w_1,w_3)}{p_{\zeta_1-\zeta_2}(0)}.$$

provided that the denominator is positive.

Proof. By (9) it holds that

$$p_{\zeta_1,\zeta_3|\zeta_1=\zeta_2}(w_1,w_3) = p_{\zeta_1,\zeta_3|\zeta_1-\zeta_2=0}(w_1,w_3) = \frac{p_{\zeta_1,\zeta_1-\zeta_2,\zeta_3}(w_1,0,w_3)}{p_{\zeta_1-\zeta_2}(0)},$$

but the numerator equals $p_{\zeta_1,\zeta_2,\zeta_3}(w_1,w_1,w_3)$. □
So, using that $X_0 - X_s \sim N(0, 2-2R(s))$, we have

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} e^{-x^2/2} \int_{\mathbb{R}^{2d}} \overline{\|u\|\|v\|} p(x,x,u,v)dudvdx\psi(t,s)ds \\ = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{1-R(s)}} \mathsf{E}(e^{-X_0^2/2} \overline{\|\nabla X_0\|\|\nabla X_s\|} | X_0 = X_s)\psi(t,s)ds. \quad (13)$$

Lemma 7. *The distribution of $(X_0, \nabla X_0, \nabla X_s)$ conditioned over $\{X_0 = X_s\}$ is Gaussian with mean zero and covariance matrix*

$$\begin{pmatrix} \frac{1}{2}(1+R(s)) & -\frac{1}{2}\nabla R(s) & -\frac{1}{2}\nabla R(s) \\ -\frac{1}{2}\nabla R(s) & I_d - \frac{1}{2-2R(s)}\nabla R(s)\nabla^T R(s) & -\nabla^2 R(s) - \frac{1}{2-2R(s)}\nabla R(s)\nabla^T R(s) \\ -\frac{1}{2}\nabla R(s) & -\nabla^2 R(s) - \frac{1}{2-2R(s)}\nabla R(s)\nabla^T R(s) & I_d - \frac{1}{2-2R(s)}\nabla R(s)\nabla^T R(s) \end{pmatrix}.$$

Proof. Straightforward calculations. □

Lemma 8. *There exists some $C = C(d)$ such that*

$$\mathsf{E}(e^{-X_0^2/2}\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = X_s) \le C\sqrt{g(s) + g^2(s)}.$$

Proof. Let $A > 1$ and $\gamma_A(x) = (|x| \wedge A)sgn(x)$. We have

$$\begin{aligned}
&\mathsf{E}\left(e^{-X_0^2/2}\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = X_s\right) \\
&= \mathsf{E}\left(e^{-X_0^2/2}(\overline{\|\nabla X_0\|} - \gamma_A(\overline{\|\nabla X_0\|}))\overline{\|\nabla X_s\|}\Big|X_0 = X_s\right) \\
&\qquad + cov\left(e^{-X_0^2/2}\gamma_A(\overline{\|\nabla X_0\|}), \overline{\|\nabla X_s\|}\Big|X_0 = X_s\right) \\
&\qquad + \mathsf{E}\left(e^{-X_0^2/2}\gamma_A(\overline{\|\nabla X_0\|})\Big|X_0 = X_s\right)\mathsf{E}\left(\overline{\|\nabla X_s\|}\Big|X_0 = X_s\right) \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

First, by Cauchy-Schwarz and Markov inequalities it holds that

$$\begin{aligned}
|J_1| &\le \mathsf{E}\left(|\overline{\|\nabla X_0\|}| \cdot |\overline{\|\nabla X_s\|}| I(|\overline{\|\nabla X_0\|}| > A)\Big|X_0 = X_s\right) \\
&\le A^{-1}\left(\mathsf{E}\left(\overline{\|\nabla X_0\|}^4\Big|X_0 = X_s\right)\mathsf{E}\left(\overline{\|\nabla X_0\|}^2\Big|X_0 = X_s\right)\right)^{1/2}.
\end{aligned}$$

Lemma 7, $\nabla X(0)$ has Gaussian conditional distribution with mean zero and covariance matrix $I_d - \frac{1}{2-2R(s)}\nabla R(s)\nabla^T R(s)$. All diagonal entries of this matrix are non greater than 1, so

$$|J_1| \le \sqrt{3}M^{-1}d^{3/2}.$$

The quantity J_2 is estimated via the quasi-association inequality ([5, Th. 1.5.3]). Namely, the function $(x, y) \mapsto e^{-x^2/2}\gamma_A(\overline{\|y\|})$ has Lipschitz constant not exceeding $2A$. Thus by Lemma, it holds that 7

$$\begin{aligned}
|J_2| &= \left|cov\left(e^{-X_0^2/2}\gamma_A(\overline{\|\nabla X_0\|}), \overline{\|\nabla X_s\|}\Big|X_0 = X_s\right)\right| \\
&\le 2A\sum_{j=1}^d \left|cov\left(X_0, \frac{\partial X(s)}{\partial s_j}\Big|X_0 = X_s\right)\right| \\
&\qquad + 2A\sum_{j,q=1}^d \left|cov\left(\frac{\partial X(0)}{\partial s_j}, \frac{\partial X(s)}{\partial s_q}\Big|X_0 = X_s\right)\right| \\
&\le 2A(g(s) + g^2(s)).
\end{aligned}$$

Finally we have to bound J_3. The first conditional expectation is bounded uniformly in s, since $\mathsf{E}(\|\nabla X_0\|^2|X_0 = X_s)$ is (see the argument when estimating J_1). As for the second, we have

$$\begin{aligned}\mathsf{E}\left(\overline{\|\nabla X_s\|}\Big|X_0 = X_s\right) &= \mathsf{E}\left(\|\nabla X_s\|\Big|X_0 = X_s\right) - \mathsf{E}\|\nabla X_s\| \\ &= \mathsf{E}\|\nabla X_s - r(s)(X_0 - X_s)\| - \mathsf{E}\|\nabla X_s\|\end{aligned}$$

by standard Gaussian argument, here $r(s) \in \mathbb{R}^d$ is such that $\nabla X_s - r(s)(X_0 - X_s)$ and $X_0 - X_s$ are independent. Clearly $r(s) = \nabla R(s)/(2 - 2R(s))$, so

$$\begin{aligned}\left|\mathsf{E}\left(\overline{\|\nabla X_s\|}\Big|X_0 = X_s\right)\right| &\leq \mathsf{E}\|r(s)(X_0 - X_s)\| \leq \|r(s)\|\sqrt{2 - 2R(s)} \\ &\leq \|r(s)\|\sqrt{2 - 2R(s)} \leq \frac{\|\nabla R(s)\|}{\sqrt{1 - R(s)}}.\end{aligned}$$

Consequently, $|J_3| \leq dg(s)$. Combining the estimates and taking $A := (g(s) + g^2(s))^{-1/2}$, we come to the desired statement. □

Using this lemma we again estimate the expression inside the integral at the right hand side of (7), but now we want to obtain a bound integrable at infinity. Let s be such that $\|s\| > 1$. Again

$$\begin{aligned}&\frac{1}{2\pi\sqrt{1 - z^2}}\mathsf{E}\exp\left\{-\frac{X_0^2 + X_s^2 - 2X_0X_sz}{2 - 2z^2}\right\}\overline{\|\nabla X_0\|\|\nabla X_s\|} \\ &\qquad = \int_{\mathbb{R}^2} q_z(x, y)p(x, y)\mathsf{E}(\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = x, X_s = y)dxdy. \qquad (14)\end{aligned}$$

The conditional expectation is calculated by standard Gaussian argument and quasi-association inequality for Gaussian variables:

$$\begin{aligned}&\mathsf{E}(\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = x, X_s = y) \\ &\leq cov(\|\nabla X_0 - \alpha(s)X_0 - \beta(s)X_s + \alpha(s)x + \beta(s)y\|, \\ &\qquad \|\nabla X_s + \alpha(s)X_s + \beta(s)X_0 - \alpha(s)y - \beta(s)x\|) \\ &\qquad + (\mathsf{E}\|\nabla X_0 - \alpha(s)X_0 - \beta(s)X_s + \alpha(s)x + \beta(s)y\| - \mathsf{E}\|\nabla X_0\|)^2 \\ &\leq 2g(s) + 2d^2(\alpha_0(s)^2 + \beta_0(s)^2)g(s) + 3d^2(\alpha_0(s)^2 + \beta_0(s)^2)(2 + x^2 + y^2)\end{aligned}$$

where $\alpha_0(s) = \max_j |\alpha_j(s)|$ and β_0 is defined similarly. Analogous estimate holds from below. Inserting this estimate into (14), recalling that $p(x, y)$ is the density of (X_0, X_s) we see that for any $z \in (0, 1)$ the absolute value of the expression in (14) does not exceed

$$\begin{aligned}&p(0, 0)\int_{\mathbb{R}^{2d}} q_z(x, y)\left(2g(s) + d^2(\alpha_0(s)^2 + \beta_0(s)^2)(2g(s) + 6 + 3x^2 + 3y^2)\right)dxdy \\ &\leq (1 - R^2(s))^{-1/2}\left(2g(s) + 2d^2(\alpha_0(s)^2 + \beta_0(s)^2)(g(s) + 6)\right) =: h(s). \qquad (15)\end{aligned}$$

The right hand side is integrable over $\{s : \|s\| > 1\}$.
Combining (12) and the equation (13), we get

$$\sum_{k=0}^{\infty} \mathsf{E}D_1^2(t,k) \\ = \int_{\mathbb{R}^d} \frac{1}{2\sqrt{\pi(1-R(s))}} \mathsf{E}(e^{-X_0^2/2} \overline{\|\nabla X_0\| \|\nabla X_s\|} | X_0 = X_s) \psi(t,s) ds. \quad (16)$$

Denote the summand at the left hand side by $\varphi_k(t)$ and the integral at the right by $F(t)$. Recall that all φ_k are nonnegative functions (being the expectations of a square of some random variable). They are also continuous in $t \in [1, \infty)$. The same can be said about $F(t)$ (e.g. by Lemma 8). Our next goal is to understand what happens for $t = +\infty$. Note that $\psi(t,s) \to 1$ as $t \to \infty$, so in the limit the domain of integration becomes unbounded. We need the following

Lemma 9. *Suppose that $v : \mathbb{R}^{d+1} \to \mathbb{R}$ is a measurable function such that $\mathsf{E}v^2(X_0, \nabla X_0) < \infty$. Then for any $s \neq 0$ one has*

$$|cov(v(X_0, \nabla X_0), v(X_s, \nabla X_s))| \leq \mathsf{E}v^2(X_0, \nabla X_0)(1 \wedge g(s)).$$

Proof. If s is such that $g(s) < 1$, then this follows from the Arcones inequality [3, Lemma 1], otherwise use the Cauchy-Schwarz inequality. □

By Lemma 9 and the relation $\mathsf{E}H_k^2(X_0)\|\nabla X_0\|^2 = dk!$, for any $k \in \mathbb{N}$ there exists a limit $\lim_{t\to\infty} \varphi_k(t) =: \varphi_k(\infty)$. Due to Lemma 8 and dominated convergence, there exists a finite $\lim_{t\to\infty} F(t) = F(\infty)$.
By Fatou's lemma it easily follows that

$$\sum_{k=0}^{\infty} \varphi_k(\infty) \leq F(\infty).$$

If we manage to prove the converse inequality, then we will have that, on the compact set $[1, +\infty]$, the series $\sum_k \varphi_k$ of nonnegative continuous functions converges to a continuous limit. Then by Dini theorem it converges uniformly, which is what we need. Obviously

$$\sum_{k=0}^{\infty} \varphi_k(\infty) = \lim_{z \nearrow 1} \sum_{k=0}^{\infty} z^k \varphi_k(\infty) \\ = \lim_{z \nearrow 1} \int_{\mathbb{R}^d} \sum_{k=0}^{\infty} \frac{z^k}{2\pi k!} \mathsf{E}H_k(X_0)H_k(X_s) e^{-X_0^2/2 - X_s^2/2} \overline{\|\nabla X_0\| \|\nabla X_s\|} ds \\ = \lim_{z \nearrow 1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} q_z(x,y) p(x,y) \mathsf{E}(\overline{\|\nabla X_0\| \|\nabla X_s\|} | X_0 = x, X_s = y) dx dy ds. \quad (17)$$

where the change of order (in the second equality) was done due to Lemma 9.
Fix some arbitrary $\delta > 0$. Take $t_0 > 1$ so large that $\int_{s:\|s\|>t_0} h(s)ds < \delta$ and after that select $t_1 > t_0$ such that $|F(t_1) - F(\infty)| < \delta$ and

$$1 - \psi(t_1, s) < \frac{\delta}{1 + \int_{s:\|s\|\le t_0} \rho(s)ds} \text{ for } \|s\| \le t_0.$$

Then it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^2} q_z(x,y)p(x,y)\mathsf{E}(\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = x, X_s = y)dxdyds \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^2} q_z(x,y)p(x,y)\mathsf{E}(\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = x, X_s = y)\psi(t_1,s)dxdyds \\
&\quad + \int_{\|s\|\le t_0}\int_{\mathbb{R}^2} q_z(x,y)p(x,y)\mathsf{E}(\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = x, X_s = y)\times \\
&\qquad\qquad \times (1 - \psi(t_1,s))dxdyds \\
&\quad + \int_{\|s\|>t_0}\int_{\mathbb{R}^2} q_z(x,y)p(x,y)\mathsf{E}(\overline{\|\nabla X_0\|\|\nabla X_s\|}|X_0 = x, X_s = y)\times \\
&\qquad\qquad \times (1 - \psi(t_1,s))dxdyds \\
&\qquad\qquad\qquad =: I_1 + I_2 + I_3.
\end{aligned}$$

We know that

$$\lim_{z\nearrow 1} I_1 = F(t_1).$$

Due to (10) one has

$$|I_2| \le \delta d.$$

Finally,

$$|I_3| \le \int_{\|s\|>t_0} h(s)ds < \delta.$$

Therefore,

$$\sum_{k=0}^{\infty} \varphi_k(\infty) \ge F(t_1) - \delta d - \delta \ge F(\infty) - 2\delta - \delta d,$$

so we have proved the opposite inequality, as was desired.

Part 2

Now we have to consider

$$\sum_{k=0}^{\infty} \mathsf{E} D_2^2(t,k)$$
$$= \sum_{k=0}^{\infty} \frac{(\mathsf{E}\|\nabla X_0\|)^2}{2\pi k!} \int_{\mathbb{R}^d} \psi(t,s) cov\left(H_k(X_0)e^{-X_0^2/2}, H_k(X_s)e^{-X_s^2/2}\right) ds$$
$$= \int_{\mathbb{R}^d} \sum_{k=0}^{\infty} \frac{z^k}{2\pi k!} cov\left(H_k(X_0)e^{-X_0^2/2}, H_k(X_s)e^{-X_s^2/2}\right) \psi(t,s) ds \quad (18)$$

by virtue of Lemma 2 and argument similar to (6), here $t > 0$ and $k \in \mathbb{N}$. Opposite to the Part 1, here we can compute the function inside the integral. Let $\zeta \sim N(0,1)$ be independent of X. As before, let q_z stand for the normal density with mean zero and covariance matrix $\begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix}$, while p and $p_\perp$ denote the densities of (X_0, X_s) and (X_0, ζ) respectively. Then, since by Cauchy-Schwarz inequality the series are absolutely summable, using Lemma 3 we have

$$\sum_{k=0}^{\infty} \frac{z^k}{2\pi k!} cov\left(H_k(X_0)e^{-X_0^2/2}, H_k(X_s)e^{-X_s^2/2}\right)$$
$$= \mathsf{E} \sum_{k=0}^{\infty} \frac{z^k}{2\pi k!} H_k(X_0)e^{-X_0^2/2} H_k(X_s)e^{-X_s^2/2}$$
$$- \mathsf{E} \sum_{k=0}^{\infty} \frac{z^k}{2\pi k!} H_k(X_0)e^{-X_0^2/2} H_k(Y_s)e^{-Y_s^2/2}$$
$$= \int_{\mathbb{R}^2} q_z(x,y)(p(x,y) - p_\perp(x,y)) dx\, dy$$
$$= \frac{1}{8\pi^3}\left(\frac{1}{\sqrt{4-(R(s)+z)^2}} - \frac{1}{\sqrt{4-z^2}}\right).$$

To calculate the last integral notice that for a positive-definite matrix T of order n one has

$$\int_{\mathbb{R}^n} e^{-(Tx,x)/2} dx = \left((2\pi)^n \det T\right)^{-1/2}.$$

Hence, selecting $A > 0$ such that $\|s\| > A$ implies $|R(s)| < 1/2$, we can estimate the expression inside the integral in (18) by the integrable function

$$I\{\|s\| \le A\} \frac{2}{\sqrt{3}\sqrt{4-(R(s)+1)^2}} + I\{\|s\| > A\} \frac{\sqrt{3}|R(s)|}{\sqrt{7}\sqrt{4-(R(s)+1)^2}}.$$

This bound is uniform both in z and in t. Hence, for each t, we can use Lemma 5 and pass to the limit in (18) as $z \nearrow 1$, to obtain

$$\sum_{k=0}^{\infty} \mathsf{E} D_2^2(t,k) = \frac{(\mathsf{E}\|\nabla X_0\|)^2}{8\pi^3} \int_{\mathbb{R}^d} \left(\frac{1}{\sqrt{4-(R(s)+1)^2}} - \frac{1}{\sqrt{3}} \right) \psi(t,s)\, ds \quad (19)$$

Now we have to establish the uniform convergence in the same way as in Part 1. Denote $\varphi_k(t) = \mathsf{E} D_2^2(t,k)$ and let $F(t)$ stand for the right hand side of (19). Obviously φ_k and F are continuous on the compact set $[1,+\infty]$. So we again need only to prove that $\sum_{k=1}^{\infty} \varphi_k(\infty) \geq F(\infty)$. But Lemma 5 implies that

$$\sum_{k=1}^{\infty} \varphi_k(\infty) = \lim_{z\nearrow 1} \sum_{k=1}^{\infty} z^k \varphi_k(\infty) = \lim_{z\nearrow 1} \int_{\mathbb{R}^d} \frac{(\mathsf{E}\|\nabla X_0\|)^2}{8\pi^3} \left(\frac{1}{\sqrt{4-(R(s)+z)^2}} - \frac{1}{\sqrt{4-z^2}} \right) ds,$$

which is equal to $F(\infty)$. The tightness is established.

Finally we return to proving (2). Note that by the already proved tightness property it suffices to prove it for any smooth function f with finite support as such functions form a dense subset in $L^2(\mathbb{R},\mu)$. By the coarea formula one has

$$(N_t, f) = \frac{1}{\sqrt{2\pi t^d}} \int_{[0,t]^d} \left(f(X_s) e^{-X_s^2/2} \|\nabla X_s\| - \mathsf{E} f(X_s) e^{-X_s^2/2} \|\nabla X_s\| \right) ds.$$

Applying Lemma 9 one sees that for a bounded Borel set $B \subset \mathbb{R}^d$ and a measurable function $v : \mathbb{R}^{d+1} \to \mathbb{R}$ such that $\mathsf{E} v^2(X_0, \nabla X_0) < \infty$, one has

$$\mathsf{Var} \int_B v(X_s, \nabla X_s) ds \leq \mathcal{H}_d(B) \mathsf{E} v^2(X_0, \nabla X_0) \int_{\mathbb{R}^d} \min\{1, g(s)\} ds. \quad (20)$$

Lemma 10. *Suppose that $v : \mathbb{R}^{d+1} \to \mathbb{R}$ is a Lipschitz function such that $\mathsf{E} v^2(X_0, \nabla X_0) < \infty$ and $\mathsf{E} v(X_0, \nabla X_0) = 0$. Then*

$$\frac{1}{\sqrt{t^d}} \int\limits_{[0,t]^d} v(X_s, \nabla X_s) ds \to N\left(0, \int_{\mathbb{R}^d} cov(v(X_0, \nabla X_0), v(X_s, \nabla X_s)) ds\right)$$

in distribution, as $t \to \infty$.

Proof. For $j \in \mathbb{Z}^d$, denote Q_j to be the unit cube $(j_1, j_1+1] \times \ldots \times (j_d, j_d+1]$ and set $Y_j = \int_{Q_j} v(X_s, \nabla X_s) ds$. The random field $\{Y_j, j \in \mathbb{Z}^d\}$ is centered, square-integrable and (BL, θ)-dependent (see [5, p. 94]) with the sequence $\theta_r = 2Lip^2(v) \int_{\|s\| \geq r/d} g(s) ds$. To prove the last statement, for $k \in \mathbb{N}$ and $j \in \mathbb{Z}^d$ let $Q_j^{(k)} \subset Q_j$ be the finite set of points $x = j + \left(\frac{q_1}{k}, \ldots, \frac{q_d}{k}\right)$, $q_1, \ldots, q_d \in \{1, \ldots, k\}$. Now take finite disjoint sets $I, J \subset \mathbb{Z}^d$ and bounded Lipschitz functions $F : \mathbb{R}^{|I|} \to \mathbb{R}$, $G : \mathbb{R}^{|J|} \to \mathbb{R}$. Using the Riemann approximation of integrals and the quasi-association inequality for vector-valued Gaussian fields, we have

$$
\begin{aligned}
& cov(F(Y_i, i \in I), G(Y_j, j \in J)) \\
&= cov\left(F\left(\int_{Q_i} v(X_s, \nabla X_s) ds, i \in I\right), G\left(\int_{Q_j} v(X_s, \nabla X_s) ds, j \in J\right)\right) \\
&\qquad = \lim_{k \to \infty} cov\left(F\left(k^{-d} \sum_{s \in Q_i^{(k)}} v(X_s, \nabla X_s), i \in I\right),\right. \\
&\qquad\qquad \left. G\left(k^{-d} \sum_{s \in Q_j^{(k)}} v(X_s, \nabla X_s), j \in J\right)\right) \\
&\quad \leq 2Lip(F)Lip(G)Lip^2(v) \lim_{k \to \infty} k^{-2d} \sum_{i \in I, j \in J} \sum_{s \in Q_i^{(k)}, u \in Q_j^{(k)}} g(s-u) \\
&\quad = 2Lip(F)Lip(G)Lip^2(v) \sum_{i \in I, j \in J} \int_{Q_i} \int_{Q_j} g(s-u) du ds \\
&\qquad \leq 2Lip(F)Lip(G)Lip^2(v)(|I| \wedge |J|) \int_{s: \|s\| \geq dist(I,J)} g(s) ds.
\end{aligned}
$$

Here the distance $dist(\cdot, \cdot)$ is with respect to the Euclidean norm. Thus, by the central limit theorem for (BL, θ)-dependent fields [5, Theorem 3.1.12], one infers the convergence

$$
\frac{1}{\sqrt{[t]^d}} \int_{[0,[t]]^d} v(X_s, \nabla X_s) ds \to N\left(0, \int_{\mathbb{R}^d} cov(v(X_0, \nabla X_0), v(X_s, \nabla X_s)) ds\right)
$$

in distribution as $t \to \infty$, here $[t]$ is the integer part of t. Now the lemma follows from (20), since $\mathcal{H}_d([0,t]^d \setminus [0,[t]]^d) \leq 2^d t^{d-1}$. $\square$

Let $A > 0$ be some fixed number and denote

$$
\begin{aligned}
& Z_1(t) := \\
& \frac{1}{\sqrt{2\pi t^d}} \int_{[0,t]^d} \left(f(X_s) e^{-X_s^2/2} \gamma_A(\|\nabla X_s\|) - \mathsf{E} f(X_s) e^{-X_s^2/2} \gamma_A(\|\nabla X_s\|)\right) ds
\end{aligned}
$$

and $Z_2(t) = (N_t, f) - Z_1(t)$. Note that for any $p \geq 1$ one has $\gamma_A(\|\nabla X_0\|) \to \|\nabla X_0\|$ in L^p, when $A \to \infty$. Thus, due to (20) $\mathsf{Var} Z_2(t) \to 0$ as $A \to \infty$ uniformly in t. By the Lemma 10 we obtain that $Z_1(t)$ is asymptotically Gaussian with limit variance given by (1) in which $\|\cdot\|$ is changed to $\gamma_A(\|\cdot\|)$. Now the Lemma 9 and the dominated convergence theorem ensure that this limit variance tends (when $A \to \infty$) to the given in (1), which proves the theorem.

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