Functional Central Limit Theorem for the Volume of Excursion Sets Generated by Associated Random Fields

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Abstract

We prove a functional central limit theorem for the volume of the excursion sets generated by a stationary and associated random field with smooth realizations.

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1 Introduction

Associated random fields form an important class of dependent systems and were first introduced in [5]. Their main advantage compared to mixing systems is that the conditions of limit theorems are easier to verify.

Definition 1. A finite collection $X = (X_1, \ldots, X_n)$ of real-valued random variables X_k, $k = 1, \ldots, n$, is called *associated* if $\mathrm{Cov}\left(f\left(X\right), g\left(X\right)\right) \geq 0$ for any coordinate-wise non-decreasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$, whenever the covariance exists. An infinite family of random variables is associated if this is valid for every finite sub-family.

In statistical physics, this property is known as the FKG-inequalities, see [6]. Associated systems are also encountered in mathematical statistics, reliability theory, random measures and so forth. Starting with [9], many limit theorems like CLTs, invariance principles, etc. have been proven for associated and related random fields, see [3] and references therein.

Definition 2. The *excursion set* of a random field X at the level $a \in \mathbb{R}$ is the random set $\left\{t \in \mathbb{R}^d : X_t \geq a\right\}$. The *level set* of X at the level a is the set $\left\{t \in \mathbb{R}^d : X_t = a\right\}$.

It is natural to consider the volume of excursion sets determined by levels $a \in \mathbb{R}$ in a bounded observation window as a random process and to study its limit behaviour as the window size grows. The aim of the present paper is to provide a functional central limit theorem for the volume of excursion sets generated by stationary associated random fields.
A related functional central limit theorem is proven in [7]. There we consider stationary and isotropic Gaussian random fields with a.s. C^1 realizations whose covariance function decays rapidly and prove a functional central limit theorem for the $(d-1)$-dimensional Hausdorff measure of their level sets.
First, we introduce some notation. Let X be a real-valued, strictly stationary, square-integrable, associated random field in $\mathbb{R}^d$ with a.s. continuous trajectories such that the components X_t have density bounded by $M < \infty$. Further, assume that for some $c_0 > 0$ and some $\lambda > 9d$ it holds that

$$\left|\mathrm{Cov}\left(X_0, X_t\right)\right| \leq c_0 \left(1 + \|t\|_\infty\right)^{-\lambda},$$

where $\|\cdot\|_\infty$ denotes the maximum norm.
For $n \in \mathbb{N}$ and $a \in \mathbb{R}$, denote by

$$Y_n\left(a\right) = \frac{1}{n^{d/2}} \left(\nu\left(\left\{t \in [0, n]^d : X_t \geq a\right\}\right) - n^d \mathbb{P}\left(X_0 \geq a\right)\right)$$

the centered and normalized volume of the excursion set of the field X in the set $[0, n]^d$ at the level a, where $\nu\left(\cdot\right)$ denotes the Lebesgue measure in $\mathbb{R}^d$.

2 Main Result and Proof

Theorem 1. *Under the assumptions above, the distributions of the random processes $\left\{Y_n\right\}_{n\in\mathbb{N}}$ converge in the Skorokhod space $D\left(\mathbb{R}\right)$, as $n \to \infty$, to the distribution of the centered Gaussian process Y with covariance function*

$$\mathrm{Cov}\left(Y\left(a\right), Y\left(b\right)\right) = \int_{\mathbb{R}^d} \left(\mathbb{P}\left(X_0 > a, X_t > b\right) - \mathbb{P}\left(X_0 > a\right)\mathbb{P}\left(X_0 > b\right)\right) dt.$$

Remark 1. A Gaussian random field is associated whenever its covariance function is nonnegative ([3, Theorem 1.2.1]). So, the theorem holds for stationary Gaussian fields with exponentially decreasing correlations. Another important example is an autoregression field with positive dependence, i.e. it has a density, though the innovations can be discrete random variables.

Remark 2. In general it is possible that the conditions on the field X are satisfied but the trajectories of Y are not continuous. Consider for example a strictly stationary associated sequence $Z = (Z_n, n \in \mathbb{Z})$ such that Z_0 has bounded density and the covariance function decreases exponentially, for example Z could be Gaussian. For $t \in [n - 1/3, n + 1/3] \setminus \{n\}$, define $Z_t = Z_n$ and on $[n + 1/3, n + 2/3]$, define Z_t by linear interpolation, here $n \in \mathbb{Z}$. Then let $U \sim \mathcal{U}(0, 1)$ be independent of Z. The random process $X_t = Z_{t+U}$ then satisfies the conditions of the theorem but the corresponding trajectories of Y have jumps.

For any $a, b \in \mathbb{R}$ we have $\mathbb{E}\left(Y(a) - Y(b)\right)^2 \leq K\left|b - a\right|^{1-3d/\lambda}$, where the constant K depends on d, c_0, λ, and M. Hence we get the following Large Deviation Principle from Theorem 1 and [10, Theorem 2.8.1].

Corollary 1. *Let $T \subset \mathbb{R}$ with $\nu(T) > 0$ and let X be such that Y is a.s. continuous. Then there exists a constant C (depending only on d, c_0, λ, and M) such that for any $S \subset T$*

$$\mathbb{P}\left(\max_{a \in S} \left|Y(a)\right| > u\right) \leq C\nu(T)\, u^{2d\lambda/(\lambda-3d)}\, \Psi\left(u / \sup_{a \in S} \sqrt{\operatorname{Var}(Y(a))}\right),$$

as $u \to \infty$, where Ψ denotes the tail probability function of the standard Gaussian distribution.

To prove Theorem 1, we start with some notation and some auxiliary lemmas.
Following [3, p. 88], the partial Lipschitz constants $\operatorname{Lip}_i(f)$ of a function $f : \mathbb{R}^d \to \mathbb{R}$ are defined by

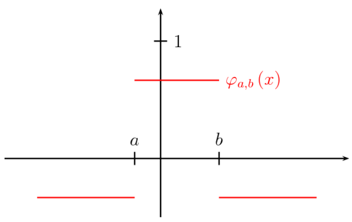
$$\operatorname{Lip}_i(f) = \sup_{\substack{x_1, \ldots, x_d \in \mathbb{R} \\ y_i \in \mathbb{R} \\ x_i \neq y_i}} \frac{\left|f(x_1, \ldots, x_d) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_d)\right|}{\left|x_i - y_i\right|}$$

for $i = 1, \ldots, d$. Clearly, it holds that $\operatorname{Lip}_i(f) \leq \operatorname{Lip}(f)$, for all $i = 1, \ldots, d$. The following covariance inequality with the name *quasi-association inequality* can be found in [2], see also [3, p. 89]. For a finite set $I = \{t_1, \ldots, t_k\} \subset \mathbb{R}^d$, denote the cardinality of I by $|I|$ and let $X_I = (X_{t_1}, \ldots, X_{t_k})^\top \in \mathbb{R}^k$.

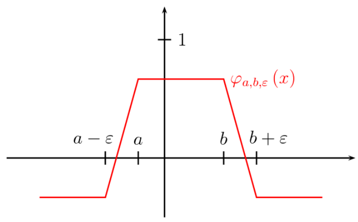
Lemma 1. *Let X be an associated random field on $\mathbb{Z}^d$ such that $\mathbb{E}X_j^2 < \infty$ for all $j \in \mathbb{Z}^d$. Let $I = \{t_1, \ldots, t_k\}, J = \{s_1, \ldots, s_l\} \subset \mathbb{Z}^d$ be two finite* *subsets of* $\mathbb{Z}^d$*. Then, for any Lipschitz functions* $f : \mathbb{R}^k \to \mathbb{R}$ *and* $g : \mathbb{R}^l \to \mathbb{R}$*, it holds that*

$$\left|\operatorname{Cov}\left(f\left(X_I\right), g\left(X_J\right)\right)\right| \le \sum_{i=1}^{k} \sum_{j=1}^{l} \operatorname{Lip}_i(f) \operatorname{Lip}_j(g) \left|\operatorname{Cov}\left(X_{t_i}, X_{t_j}\right)\right|.$$

Random fields with this property are called *quasi-associated*.
To formulate the next lemma, let $a, b \in \mathbb{R}$ with $a < b$. Define the function $\varphi_{a,b} : \mathbb{R} \to \mathbb{R}$ by $\varphi_{a,b}(x) = \mathbb{1}(a < x \le b) - \mathbb{P}(a < X_0 \le b)$ where $\mathbb{1} : \mathbb{R} \to \{0, 1\}$ denotes the indicator function. For $\varepsilon > 0$, let $\varphi_{a,b,\varepsilon} : \mathbb{R} \to \mathbb{R}$ be a continuous approximation of $\varphi_{a,b}$ which is linear on the intervals $[a - \varepsilon, a]$ and $[b, b + \varepsilon]$ and coincides with $\varphi_{a,b}$ outside these intervals, see Figure 1. Clearly, $\varphi_{a,b,\varepsilon}$ has Lipschitz constant $1/\varepsilon$.



(a) $\varphi_{a,b}$



(b) $\varphi_{a,b,\varepsilon}$

Figure 1: The functions $\varphi_{a,b}$ and $\varphi_{a,b,\varepsilon}$.

Lemma 2. *Let* $T = \{t_1, \dots, t_k\}$ *be a set of distinct points in* $\mathbb{R}^d$ *and let* $T_1 \cup T_2 = T$ *be a partition of* T*, i.e.* $T_1 \ne \emptyset$*,* $T_2 \ne \emptyset$ *and* $T_1 \cap T_2 = \emptyset$*. Then it holds that*

$$\operatorname{Cov}\left(\prod_{t \in T_1} \varphi_{a,b}(X_t), \prod_{s \in T_2} \varphi_{a,b}(X_s)\right) \le \left(\frac{k^2}{4} + 8Mk\right) c_0^{1/3} (1 + r)^{-\lambda/3}$$

where $r := \min\left\{\|t - s\|_\infty, t \in T_1, s \in T_2\right\}$ *is the distance between* T_1 *and* T_2*.*

Proof. It holds that

$$
\begin{aligned}
&\operatorname{Cov}\left(\prod_{t\in T_1}\varphi_{a,b}\left(X_t\right),\prod_{s\in T_2}\varphi_{a,b}\left(X_s\right)\right)\\
&=\operatorname{Cov}\left(\prod_{t\in T_1}\varphi_{a,b,\varepsilon}\left(X_t\right),\prod_{s\in T_2}\varphi_{a,b,\varepsilon}\left(X_s\right)\right) \qquad (1)\\
&\quad+\mathbb{E}\prod_{t\in T_1}\varphi_{a,b}\left(X_t\right)\left(\prod_{s\in T_2}\varphi_{a,b}\left(X_s\right)-\prod_{s\in T_2}\varphi_{a,b,\varepsilon}\left(X_s\right)\right)\\
&\quad+\mathbb{E}\left(\prod_{t\in T_1}\varphi_{a,b}\left(X_t\right)-\prod_{t\in T_1}\varphi_{a,b,\varepsilon}\left(X_t\right)\right)\prod_{s\in T_2}\varphi_{a,b,\varepsilon}\left(X_s\right)\\
&\quad+\mathbb{E}\prod_{t\in T_1}\varphi_{a,b}\left(X_t\right)\left(\mathbb{E}\prod_{s\in T_2}\varphi_{a,b,\varepsilon}\left(X_s\right)-\mathbb{E}\prod_{s\in T_2}\varphi_{a,b}\left(X_s\right)\right)\\
&\quad+\left(\mathbb{E}\prod_{t\in T_1}\varphi_{a,b,\varepsilon}\left(X_t\right)-\mathbb{E}\prod_{t\in T_1}\varphi_{a,b}\left(X_t\right)\right)\mathbb{E}\prod_{s\in T_2}\varphi_{a,b,\varepsilon}\left(X_s\right).
\end{aligned}
$$

By the quasi-association inequality (Lemma 1), we have

$$
\operatorname{Cov}\left(\prod_{t\in T_1}\varphi_{a,b,\varepsilon}\left(X_t\right),\prod_{s\in T_2}\varphi_{a,b,\varepsilon}\left(X_s\right)\right)\leq\frac{k^2c_0}{4\varepsilon^2}\left(1+r\right)^{-\lambda},
$$

where we used that $\operatorname{Cov}\left(X_t,X_s\right)\leq c_0\left(1+r\right)^{-\lambda}$. To estimate the other summands in (1), we need inequalities of the following type: For any $m=1,\ldots,k$, it holds that

$$
\begin{aligned}
&\mathbb{E}\prod_{j=1}^{m-1}\varphi_{a,b}\left(X_{t_j}\right)\prod_{j=m}^{k}\varphi_{a,b,\varepsilon}\left(X_{t_j}\right)-\mathbb{E}\prod_{j=1}^{m}\varphi_{a,b}\left(X_{t_j}\right)\prod_{j=m+1}^{k}\varphi_{a,b,\varepsilon}\left(X_{t_j}\right)\\
&\qquad\leq\mathbb{E}\left(\varphi_{a,b,\varepsilon}\left(X_{t_m}\right)-\varphi_{a,b}\left(X_{t_m}\right)\right)\leq 2M\varepsilon, \qquad (2)
\end{aligned}
$$

where we used the fact that X_t has a density which is bounded by M. Hence, it holds that

$$
\mathbb{E}\prod_{t\in T_1}\varphi_{a,b}\left(X_t\right)\left(\prod_{s\in T_2}\varphi_{a,b}\left(X_s\right)-\prod_{s\in T_2}\varphi_{a,b,\varepsilon}\left(X_s\right)\right)\leq 2Mk\varepsilon.
$$

The third, fourth, and fifth summand in (1) can be estimated in exactly the same way. The Lemma follows with $\varepsilon=c_0^{1/3}\left(1+r\right)^{-\lambda/3}$. □

In what follows, K is a positive number which depends on d, c_0, λ, and M only and also may change from line to line.

Lemma 3. *For any $a, b \in \mathbb{R}$ with $0 < a < b < 1$, it holds that*

$$\mathbb{E}\left(Y_n(a) - Y_n(b)\right)^4 \leq K\left(\frac{(b-a)^{1-9d/\lambda}}{n^d} + (b-a)^{2-6d/\lambda}\right).$$

Proof. To shorten the notation, denote $\mathbf{t} = (t_1, t_2, t_3, t_4) \in [0, n]^{4d}$. By Fubini's theorem, it holds that

$$\mathbb{E}\left(Y_n(a) - Y_n(b)\right)^4 = \frac{1}{n^{2d}} \int_{[0,n]^{4d}} \mathbb{E} \prod_{i=1}^{4} \varphi_{a,b}\left(X_{t_i}\right) d\mathbf{t}.$$

This integral is split into three parts and each one is estimated separately. For that, define the function $h : [0, n]^{4d} \to \mathbb{R}$ by

$$h(\mathbf{t}) = \max_{Q \subset \{1,2,3,4\}} \min_{\substack{i \in Q \\ j \notin Q}} \left\|t_i - t_j\right\|_\infty .$$

The function h can be interpreted in the following way: For any $\mathbf{t} \in [0, n]^{4d}$, it is possible to connect the points by three line segments such that the longest segment (in the supremum norm) has length $h(\mathbf{t})$. Let $c > 1$ be a number to be specified later. The sets

$$A_1 = \left\{\mathbf{t} \in [0, n]^{4d} : h(\mathbf{t}) \leq c\right\},$$

$$A_2 = \left\{\mathbf{t} \in [0, n]^{4d} : h(\mathbf{t}) \geq c \text{ and } \left|Q^*_{\mathbf{t}}\right| = 1\right\},$$

and

$$A_3 = \left\{\mathbf{t} \in [0, n]^{4d} : h(\mathbf{t}) \geq c \text{ and } \left|Q^*_{\mathbf{t}}\right| = 2\right\},$$

form a partition of $[0, n]^{4d}$, where

$$Q^*_{\mathbf{t}} = \operatorname*{argmax}_{Q \subset \{1,2,3,4\}} \min_{\substack{i \in Q \\ j \notin Q}} \left\|t_i - t_j\right\|_\infty .$$

Examples for quadruples of points in the sets A_1, A_2 and A_3 can be seen in Figures 2 – 4. We define

$$I_j = \frac{1}{n^{2d}} \int_{A_j} \mathbb{E} \prod_{i=1}^{4} \varphi_{a,b}\left(X_{t_i}\right) d\mathbf{t}.$$

For all $(t_1, t_2, t_3, t_4) \in A_1$, it holds that $\max_{i,j=1,2,3,4} \left\|t_i - t_j\right\|_\infty \leq 3c$.

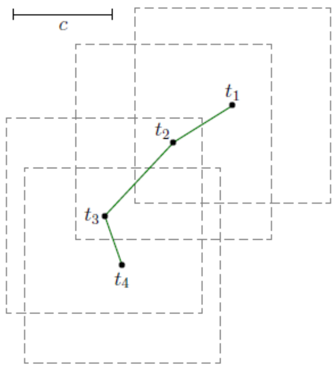


Figure 2: Example for a point $\mathbf{t} \in A_1$. There is no subset of the set of four points with distance greater than c from the other points (in supremum norm).

Hence, $\nu(A_1) \le n^d (3c)^{3d}$. By the Cauchy-Schwarz inequality it holds that

$$\mathbb{E} \prod_{i=1}^{4} \varphi_{a,b}(X_{t_i}) \le (b-a) M.$$

Putting things together, we have

$$I_1 \le K(b-a) c^{3d}/n^d.$$

For I_2, denote the distance between two sets Q and $\widetilde{Q}$ (in the supremum norm) by $d_\infty(Q, \widetilde{Q})$ where we put $d_\infty(Q, \emptyset) = 0$. W.l.o.g., let $Q_{\mathbf{t}}^* = \{t_1\}$ in the definition of the set A_2. Then, by Lemma 2, it holds that

$$I_2 \le \frac{K}{n^{2d}} \int_{A_2} (1 + d_\infty(\{t_1\}, \{t_2, t_3, t_4\}))^{-\lambda/3} \, d\mathbf{t}.$$

The set of points $(t_1, t_2, t_3, t_4) \in [0, n]^{4d}$ for which all distances between pairs t_i and t_j are equal, has Lebesgue measure zero. Hence, we assume in the following that not all distances between pairs of points are equal. Further, w.l.o.g., we assume that

$$d_\infty(\{t_1\}, \{t_2, t_3, t_4\}) = \|t_1 - t_2\|_\infty \le \min\{\|t_1 - t_3\|_\infty, \|t_1 - t_4\|_\infty\}.$$

One of the points t_3, t_4 belongs to the ball with radius $\|t_1 - t_2\|_\infty$ centered at t_2 (in supremum norm). Say, it is t_3, then t_4 in turn belongs to the union of balls of radius $\|t_1 - t_2\|_\infty$ centered at the points t_2 and t_3. Thus

$$\begin{aligned} I_2 &\le \frac{K}{n^{2d}} \int\int_{\substack{(t_1,t_2)\in[0,n]^{2d} \\ \|t_1-t_2\|_\infty \ge c}} (1 + \|t_1 - t_2\|_\infty)^{-\lambda/3} \|t_1 - t_2\|_\infty^{2d} \, dt_1 dt_2 \\ &\le \frac{K}{n^d} \int_c^\infty (1+r)^{-\lambda/3} r^{2d} r^{d-1} dr \le \frac{K}{n^d} c^{3d-\lambda/3}. \end{aligned}$$

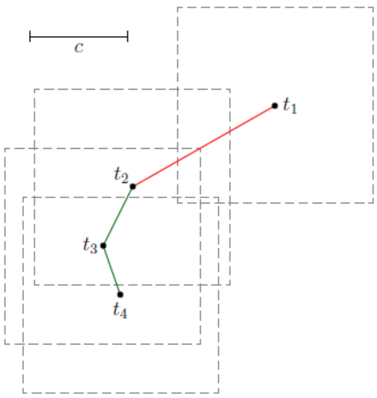


Figure 3: Example for a quadruple of points $\mathbf{t} \in A_2$. After connecting each point with its nearest neighbour, there are two groups. One group contains three points, while the other group contains the remaining point. The distance between the two sets is greater than c (in the supremum norm).

For I_3 we consider only points $\mathbf{t} \in [0,n]^{4d}$ for which $Q^*_{\mathbf{t}} = \{t_1, t_2\}$ in the definition of A_3. Then it holds that

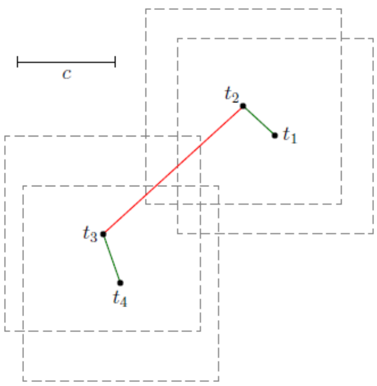


Figure 4: Example for a quadruple of points $\mathbf{t} \in A_3$. After connecting each point with its nearest neighbour, there are two groups each of which contains two points. The distance between the two sets is greater than c (in the supremum norm).

$$
\begin{aligned}
I_3 = \frac{1}{n^{2d}} \int_{A_3} &\operatorname{Cov}\left(\prod_{i=1}^{2} \varphi_{a,b}\left(X_{t_i}\right), \prod_{i=3}^{4} \varphi_{a,b}\left(X_{t_i}\right) \right) d\mathbf{t} \\
&+ \frac{1}{n^{2d}} \int_{A_3} \mathbb{E} \prod_{i=1}^{2} \varphi_{a,b}\left(X_{t_i}\right) \mathbb{E} \prod_{i=3}^{4} \varphi_{a,b}\left(X_{t_i}\right) d\mathbf{t} = J_1 + J_2.
\end{aligned}
$$

The integral J_1 is estimated in the same way as I_2 yielding

$$J_1 \leq Kc^{3d-\lambda/3}/n^d.$$

The second integral is estimated with the help of Lemma 2. By the Cauchy-Schwarz inequality we get

$$J_2 \leq \frac{K}{n^{2d}} \left(\int\int_{(t_1,t_2)\in[0,n]^{2d}} (b-a) \wedge (1 + \|t_1 - t_2\|_\infty)^{-\lambda/3}\, dt_1 dt_2 \right)^2,$$

where $x \wedge y = \min\{x, y\}$. The double integral can be bounded by

$$\begin{aligned} &\int\int_{(t_1,t_2)\in[0,n]^{2d}} (b-a) \wedge (1 + \|t_1 - t_2\|_\infty)^{-\lambda/3}\, dt_1 dt_2 \\ &\leq \int\int_{\substack{(t_1,t_2)\in[0,n]^{2d} \\ \|t_1-t_2\|_\infty \leq \gamma}} (b-a)\, dt_1 dt_2 + \int\int_{\substack{t_1,t_2\in[0,n]^{2d} \\ \|t_1-t_2\|_\infty > \gamma}} (1 + \|t_1 - t_2\|_\infty)^{-\lambda/3}\, dt_1 dt_2 \\ &\leq (b-a)\,(2n\gamma)^d + n^d \kappa_d \frac{\gamma^{d-\lambda/3}}{\lambda/3 - d}, \end{aligned}$$

with some $\gamma > 1$. Choosing $\gamma = (b-a)^{-3/\lambda}$ yields

$$J_2 \leq K\,(b-a)^{2-6/\lambda}.$$

The lemma follows with

$$c = \left(\frac{\lambda/3 - 3d}{3d\,(b-a)} \right)^{3/\lambda}.$$

□

Finally, we turn to the proof of Theorem 1.

Proof of Theorem 1. The convergence of the finite-dimensional distributions follows from the CLT for associated random fields by standard arguments. It is proved in a recent paper ([4, Theorem 2]). As for the tightness, is is enough to prove it for the restriction of the processes Y_n onto an arbitrary segment. W.l.o.g. we use the segment $[0, 1]$. Note that for $a_1, a_2 \in \mathbb{R}$ with $a_1 < a_2$, $b \in (a_1, a_2)$ and $n \in \mathbb{N}$ one has estimates

$$Y_n(b) \leq Y_n(a_1) + Mn^{d/2}(a_2 - a_1)$$

and

$$Y_n(b) \geq Y_n(a_2) - Mn^{d/2}(a_2 - a_1)$$

because X_t has density bounded by M. This bound allows to replace the supremum with a maximum over a finite set of points, plus some additional non-random summand. Namely, for any $a \in [0,1]$, $k \in \mathbb{N}$ and $\delta > 0$, we have

$$\sup_{b\in[a,a+\delta]} |Y_n(b) - Y_n(a)| \leq \max_{i=1,\dots,\lceil\delta k\rceil} \sup_{b\in[a+(i-1)/k,a+i/k]} |Y_n(b) - Y_n(a)|$$
$$\leq \max_{i=1,\dots,\lceil\delta k\rceil} \left(\left|Y_n\left(a+\frac{i-1}{k}\right) - Y_n(a)\right| \vee \left|Y_n\left(a+\frac{i}{k}\right) - Y_n(a)\right|\right) + \frac{M}{k}n^{d/2}$$
$$\leq \max_{i=1,\dots,\lceil\delta k\rceil} \left|Y_n\left(a+\frac{i}{k}\right) - Y_n(a)\right| + \frac{M}{k}n^{d/2},$$

where $x \vee y = \max\{x, y\}$. Then, for any $\varepsilon > 0$, it holds that

$$\mathbb{P}\left(\sup_{b\in[a,a+\delta]} |Y_n(b) - Y_n(a)| > \varepsilon\right)$$
$$\leq \frac{1}{\varepsilon^4}\mathbb{E}\left(\max_{i=1,\dots,\lceil\delta k\rceil} \left|Y_n\left(a+\frac{i}{k}\right) - Y_n(a)\right| + \frac{M}{k}n^{d/2}\right)^4$$
$$\leq \frac{8}{\varepsilon^4}\mathbb{E}\max_{i=1,\dots,\lceil\delta k\rceil} \left(Y_n\left(a+\frac{i}{k}\right) - Y_n(a)\right)^4 + \frac{8M^4}{\varepsilon^4 k^4}n^{2d},$$

where we used Chebyshev's inequality for the fourth moment and the fact that $(p+q)^{2^k} \leq 2^{2^k-1}\left(p^{2^k} + q^{2^k}\right)$ for any $p, q, k \in \mathbb{N}$.
In the following, let $\zeta > 0$ be a small number and select

$$n^{d/2+\zeta} < k = k(n) < n^{\lambda d/(3d+\lambda)}.$$

With $i_1, i_2 \in \mathbb{N}$, $i_1 < i_2 \leq \delta k$, it follows from Lemma 3 that

$$\mathbb{E}\left(Y_n\left(a+\frac{i_1}{k}\right) - Y_n\left(a+\frac{i_2}{k}\right)\right)^4$$
$$\leq K\left(\frac{1}{n^d}\left(\frac{i_2-i_1}{k}\right)^{1-9d/\lambda} + \left(\frac{i_2-i_1}{k}\right)^{2-6d/\lambda}\right).$$

The inequality $k < n^{\lambda d/(3d+\lambda)}$ can be written as $n^d > k^{1+3d/\lambda}$. Hence, it holds that

$$\frac{1}{n^d}\left(\frac{i_2-i_1}{k}\right)^{1-9d/\lambda} < \left(\frac{i_2-i_1}{k}\right)^{2-6d/\lambda}$$

and

$$\mathbb{E}\left(Y_n\left(a+\frac{i_1}{k}\right) - Y_n\left(a+\frac{i_2}{k}\right)\right)^4 < K\left(\frac{i_2-i_1}{k}\right)^{2-6d/\lambda}.$$

By the Móricz theorem ([8, Theorem 2]), it holds that

$$\mathbb{E}\max_{j=1,\dots,\lceil\delta k\rceil} (Y_n(a+j/k) - Y_n(a))^4 \leq K\frac{\lfloor\delta k\rfloor^{2-6d/\lambda}}{k^{2-6d/\lambda}} \leq K\delta^{2-6d/\lambda}.$$

By dividing the unit interval into smaller subintervals of length δ, we get

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{\substack{a,b\in[0,1]\\|a-b|<\delta}}|Y_n(a)-Y_n(b)|>2\varepsilon\right)$$

$$=\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left(\max_{1\le m\le\lfloor 1/\delta\rfloor}\sup_{\substack{a,b\in[(m-1)\delta,(m+1)\delta\wedge 1]\\|a-b|<\delta}}|Y_n(a)-Y_n(b)|>2\varepsilon\right)$$

$$\le\lim_{\delta\to 0}\limsup_{n\to\infty}\left\lfloor\frac{1}{\delta}\right\rfloor\sup_{a\in[0,1]}\mathbb{P}\left(\sup_{b\in[a,a+2\delta]}|Y_n(a)-Y_n(b)|>\varepsilon\right)$$

$$\le\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{2}{\delta}\left(K\delta^{2-6d/\lambda}+\frac{8M^4}{\varepsilon^4k^4}n^{2d}\right)=0,$$

as $n^{2d}/k^4\to 0$ for $n\to\infty$ because $k>n^{d/2+\zeta}$. This is the tightness condition in [1, Theorem 15.5]. $\square$

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