Central Limit Theorems for Functionals of Stationary Germ-Grain Models

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Abstract. Conditions are derived for the asymptotic normality of a general class of vector–valued functionals of stationary Boolean models in the d-dimensional Euclidean space, where a Lindeberg–type central limit theorem (CLT) for m–dependent random fields is applied. These functionals can be used to construct joint estimators for the vector of specific intrinsic volumes of the underlying Boolean model. Extensions to functionals of more general germ–grain models satisfying some mixing and integrability conditions are also discussed.

Keywords. Random closed set; Boolean model; stationary random field; m–dependent random field; valuation; asymptotic normality; β–mixing; specific intrinsic volume; Euler number.

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1 Introduction

Consider a stationary random closed set $\Xi \subset \mathbb{R}^d$ such that $\Xi \cap K$ belongs to the convex ring $\mathcal{R}$ with probability one for any convex and compact test set $K \subset \mathbb{R}^d$. Assume that Ξ can be (indirectly) observed within a bounded observation window $W \subset \mathbb{R}^d$. Suppose that this indirect observation is done by measuring some "local" geometric features

$$Y(x) = f\left((\Xi - x) \cap K\right), \qquad x \in W \ominus \check{K} \tag{1.1}$$

of Ξ within a small scanning window $K \subset \mathbb{R}^d$, where $\ominus$ denotes Minkowski difference, $\check{K}$ is the reflection of K, and $f : \mathcal{R} \to \mathbb{R}$ is some real–valued functional possessing the properties of a valuation (see e.g. Section 3.4 of [12]). If f is invariant with respect to translations, then $Y(x) = f\left(\Xi \cap (K + x)\right)$ holds, where $K + x$ can be interpreted as local neighborhood of the measurement point x. A natural unbiased estimator for the mean $\mu = E\,Y(x)$ of the stationary random field $Y = \{Y(x),\ x \in \mathbb{R}^d\}$ is the weighted average

$$\widehat{\mu} = \int_W Y(x)\, G(W, x)\, dx\,, \tag{1.2}$$

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where $G(W, x)$ is a weighting kernel that integrates to one over W and vanishes for those x for which $Y(x)$ is not observable. The question to be answered is, which asymptotic properties the estimator $\widehat{\mu}$ has for an unboundedly increasing sequence of observation windows $W \uparrow \mathbb{R}^d$. It is well known from the general theory of stationary random fields (confer e.g. Section 1.7 of [8]) that the estimator given in (1.2), properly normalized, is asymptotically normally distributed under the assumption that $E|Y(x)|^{2+\delta} < \infty$ for some $\delta > 0$ and if additional Rosenblatt–type mixing conditions on Y are satisfied. Roughly speaking, these conditions assure that various mixing rates of Y expressing the dependence between $Y(x)$ and $Y(x+t)$ decrease in the order of $|t|^{-d-\varepsilon}$ for $|t| \to \infty$ and some $\varepsilon > 0$. Notice that these assumptions are dictated by the sectioning technique of Bernstein and the classical central limit theorem (CLT) in the form of Lyapunov used in the proofs.

However, in the context of random fields Y as defined in (1.1) and generated by random closed sets of the form $\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)$, where $\{X_i\}$ is a point process of "germs" and $\{M_i\}$ is a sequence of random compact "grains", mixing and integrability conditions on $\{X_i\}$ and $\{M_i\}$, respectively, can be used to show the asymptotic normality of the estimator $\widehat{\mu}$ given in (1.2). In particular, if $\{X_i\}$ is a Poisson process or a "Poisson–like" point process with finite range of correlation, a Lindeberg–type CLT developed in [4] for so–called m–dependent random fields is applicable.

We emphasize that this technique can be used to prove the asymptotic normality of $\widehat{\mu}$ for any conditionally bounded valuation f. Related results for another general class of functionals of germ–grain models have been derived in [7]. Furthermore, there exist various results in this direction for particular functionals f such as the empirical volume fraction, boundary length and convexity number; see e.g. [1], [5], [9] and references in [10], pp. 30–43.

The present paper is organized as follows. Section 2 contains preliminary results. In Section 2.1 some basic notions from stochastic geometry are recalled, like random closed sets, germ–grain models and, in particular, the Boolean model. Then, in Section 2.2, a quite general class of functionals of stationary random fields is introduced and an upper bound is derived for the moments of stationary random fields associated with these functionals. In Section 2.3, conditions for the mean–square consistency of the mean–value estimator $\widehat{\mu}$ are given. Some examples of valuations are discussed in Section 3. The corresponding random fields can be used to construct joint estimators for the vector of specific intrinsic volumes of stationary random sets; see [11] and [14]. In Section 4, we consider a Boolean model $\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)$ with convex and compact grains. In particular, we show in Section 4.1 that the covariance function $\mathrm{Cov}_Y(x)$ admits an integrable upper bound provided that

$$E\,|M_i \oplus \check{K}|^2 < \infty\,, \tag{1.3}$$

where $\oplus$ denotes the Minkowski sum and $|\cdot|$ is the d-dimensional Lebesgue measure. This bound depends on the distribution of the grains M_i, where the dependence is monotone with respect to inclusion. Using a truncation technique and the Lindeberg–type CLT for m–dependent random fields, we show in Sections 4.2 and 4.3 that the weighted average $\widehat{\mu}$ of Y over W is asymptotically normally distributed for any unboundedly increasing sequence of observation windows $W \uparrow \mathbb{R}^d$, which satisfies some additional regularity conditions. Using the well-known Cramér–Wold device, this result can be easily extended to a multidimensional setting. Conditions for the asymptotic normality of the estimator $\widehat{\mu}$ for more general germ–grain models are discussed in Section 5. Proceeding as in [7], a CLT for β-mixing random fields given in [6] is applied, together with an upper bound for the β-mixing coefficient of random measures associated with the germ–grain models. For this theorem, a stronger integrability condition is needed, which is $E\,D^{2d(1+\delta')}(M_i) < \infty$ for some $\delta' > 0$, where $D(M_i) = \sup\{|x| : x \in M_i\}$ is the "radius" of the grains. Notice that condition (1.3) is fulfilled if $E\,D^{2d}(M_i) < \infty$.

2 Mean–value estimators for stationary random fields

We first recall some basic notions from stochastic geometry which will be used in the present paper. Further details can be found e.g. in [13] and [15]. In the second part of this section, we consider a class of unbiased and consistent estimators for the mean value of certain stationary random fields.

2.1 Germ–grain models

Let $d \geq 2$ be an arbitrary fixed integer. For any two sets $B, B' \subset \mathbb{R}^d$, let $B \oplus B' = \{x+y : x \in B, y \in B'\}$ be the *Minkowski sum* of B and B' and write $B + x = B \oplus \{x\}$ for the *translation* of B by the vector $x \in \mathbb{R}^d$. Besides this, consider the *reflection* $\check{B} = \{-x : x \in B\}$ of B at the origin and denote the *Minkowski difference* of B and B' by $B \ominus B' = \{x : \check{B}' + x \subseteq B\}$.

Let $\mathcal{B}(\mathbb{R}^d)$ be the σ–algebra of Borel sets in $\mathbb{R}^d$ and let $\mathcal{B}_0(\mathbb{R}^d) \subset \mathcal{B}(\mathbb{R}^d)$ be the family of all bounded Borel sets. Furthermore, let $\mathcal{F} \subset \mathcal{B}(\mathbb{R}^d)$ denote the family of all closed sets and $\mathcal{K} \subset \mathcal{F}$ the family of all convex bodies, i.e. convex and compact sets in $\mathbb{R}^d$. For the *convex ring* we shall write $\mathcal{R}$. It is the family of all finite unions of sets from $\mathcal{K}$ which are sometimes also called *polyconvex sets*. The *extended convex ring* $\mathcal{S}$ is the family of Borel sets $B \in \mathcal{B}(\mathbb{R}^d)$ such that $B \cap K \in \mathcal{R}$ holds for any convex body $K \in \mathcal{K}$. A *random closed set* (RACS) Ξ in $\mathbb{R}^d$ is a $(\mathcal{A}, \sigma_{\mathcal{F}})$-measurable mapping from some probability space $(\Omega, \mathcal{A}, P)$ into $\mathcal{F}$ equipped with the σ-algebra $\sigma_{\mathcal{F}}$, which is generated by events $\{F \in \mathcal{F}, F \cap K \neq \emptyset\}$, $K \in \mathcal{F}$, K compact.

We say that Ξ is *stationary* if the distribution of the translated RACS $\Xi + x$ is equal to the distribution of Ξ for any $x \in \mathbb{R}^d$. In the following, we consider stationary RACS Ξ with realizations from the extended convex ring $\mathcal{S}$, i.e., $\Xi \cap K \in \mathcal{R}$ almost surely for any $K \in \mathcal{K}$. The RACS Ξ is said to be an (independently marked) *germ–grain model* if it can be represented in the form

$$\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)\,, \tag{2.1}$$

where the so–called *germs* X_i form a simple point process $X = \{X_i\}$ in $\mathbb{R}^d$ and the sequence $M = \{M_i\}$ of *grains* M_i is independent of $\{X_i\}$ and consists of independent copies of a non-empty compact RACS M_0. Notice that the infinite union of RACS $M_i + X_i$ at the right-hand side of (2.1) is almost surely closed and different from $\mathbb{R}^d$ if the point process X is stationary with finite intensity λ and if

$$E\,|M_0 \oplus \check{K}| < \infty, \tag{2.2}$$

for each $K \in \mathcal{K}$. This condition holds for instance if $E\,D^d(M_0) < \infty$, where $D(B) = \sup\{|x| : x \in B\}$ denotes the radius (or norm) of a Borel set $B \in \mathcal{B}(\mathbb{R}^d)$, and $|x|$ is the length of the vector $x \in \mathbb{R}^d$. Condition (2.2) and the stationarity of X imply that only finitely many translated grains $M_i + X_i$ have a non–empty intersection $(M_i + X_i) \cap K$ with any fixed convex body $K \in \mathcal{K}$. In other words, the random variable

$$N(\Xi \cap K) = \#\{i : (M_i + X_i) \cap K \neq \emptyset\} \tag{2.3}$$

is finite with probability one for each $K \in \mathcal{K}$, where $\#(B)$ denotes the cardinality of the set B. Let $g_{N(\Xi\cap K)}(s) = E\left(s^{N(\Xi\cap K)}\right)$, $s \in \mathbb{R}$, be the generating function of $N(\Xi \cap K)$.

If the point process $\{X_i\}$ of germs is a stationary Poisson process, then the stationary RACS Ξ defined by formula (2.1) is called a *Boolean model*. For Boolean models, it is not difficult to show that the random variable $N(\Xi \cap K)$ is Poisson distributed with parameter $\lambda\, E\,|M_0 \oplus \check{K}|$; see e.g. Section 4.1 of [3]. Thus, in this case, the generating function $g_{N(\Xi\cap K)}$ is given by

$$g_{N(\Xi\cap K)}(s) = e^{(s-1)\lambda\, E\,|M_0 \oplus \check{K}|}, \qquad s \in \mathbb{R}. \tag{2.4}$$

This means in particular that $g_{N(\Xi\cap K)}(s) < \infty$ for any $s \in \mathbb{R}$ as soon as (2.2) is fulfilled. Besides that, a Boolean model Ξ with non-empty polyconvex grains M_i can be represented as the union set of a Poisson (particle) process $\widetilde{M} = \{\widetilde{M}_i\}$ on $\mathcal{R}$, where $\widetilde{M}_i = M_i + X_i$, in other words, we have $\Xi = \bigcup_{i=1}^{\infty} \widetilde{M}_i$ (see e.g. Section 4.4 of [13]).

2.2 Random fields associated with germ–grain models

Let the functional $f : \mathcal{R} \to \mathbb{R}$ be a *valuation* on the convex ring $\mathcal{R}$. This means that $f(\emptyset) = 0$ and that f is measurable and additive, i.e.,

$$f(K_1 \cup K_2) = f(K_1) + f(K_2) - f(K_1 \cap K_2)$$

for any $K_1, K_2 \in \mathcal{R}$. Regarding the value $f(K_1 \cup \ldots \cup K_k)$ for the union of k sets $K_1, \ldots, K_k$ from $\mathcal{R}$, where $k \geq 2$, the general *inclusion-exclusion formula*

$$f(K_1 \cup \ldots \cup K_k) = \sum_{i=1}^{k} (-1)^{i-1} \sum_{j_1 < \ldots < j_i} f(K_{j_1} \cap \ldots \cap K_{j_i}) \tag{2.5}$$

easily follows from the additivity of f. Furthermore, we assume that f is *conditionally bounded* on $\mathcal{K}$, that is to say, for any pair $K, K' \in \mathcal{K}$ with $K' \subseteq K$ the inequality

$$|f(K')| \;\leq\; c(K)$$

holds for some finite bound $c(K)$. For any fixed convex body $K \in \mathcal{K}$ and for any RACS Ξ, consider the random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ given by

$$Y(x) = f\left((\Xi - x) \cap K\right)\,, \qquad x \in \mathbb{R}^d\,. \tag{2.6}$$

If Ξ is stationary, then the random field Y is stationary, i.e., its finite–dimensional distributions are invariant with respect to translations. In particular, we have $Y(x) \overset{\mathrm{d}}{=} Y(o)$ for any $x \in \mathbb{R}^d$, where $\overset{\mathrm{d}}{=}$ denotes equality in distribution and $o \in \mathbb{R}^d$ is the origin. Throughout this paper, we assume that the field Y given by (2.6) is of second order, which means that

$$E\,Y^2(x) < \infty\,, \qquad x \in \mathbb{R}^d\,.$$

This condition implies that the covariance $\mathrm{Cov}_Y(x) = \mathrm{Cov}\,(Y(o), Y(x))$ is well defined for any $x \in \mathbb{R}^d$. Notice that a sufficient condition for the existence of the second moment of Y can be provided in terms of the generating function $g_{N(\Xi\cap K)}(s)$ of the random variable $N(\Xi \cap K)$ defined in (2.3).

Lemma 2.1. *Let Ξ be a germ–grain model with $M_0 \in \mathcal{R}$ such that the minimal number of convex components of M_0 is bounded by some constant $n_0 < \infty$. Then, it holds that*

$$E\,|Y^p(x)| \le c^p(K) g_{N(\Xi\cap K)}(2^{n_0 p})$$

for any $p > 0$ and $x \in \mathbb{R}^d$, where $c(K)$ is an upper bound for $|f(K')|$ for all $K' \subseteq K, K' \in \mathcal{K}$.

Proof. We show the assertion only for the special case $n_0 = 1$, i.e., we assume that $M_0 \in \mathcal{K}$. For any integer $m \ge 0$ we put $I_m(x) = \{N((\Xi - x) \cap K) = m\}$ and $p_{N(\Xi\cap K)}(m) = P(I_m(o))$. Then, using the properties of valuations, we get

$$\begin{aligned}
E\,|Y^p(x)| &= \sum_{m=1}^{\infty} E\left(\left|f\left(\bigcup_{i=1}^{m}(M_i + X_i - x) \cap K\right)\right|^p \mid I_m(x)\right) p_{N(\Xi\cap K)}(m) \\
&= \sum_{m=1}^{\infty} E\left(\left|\sum_{k=1}^{m}(-1)^{k-1} \sum_{i_1<\ldots<i_k} f\left((M_{i_1} + X_{i_1} - x) \cap \ldots \cap (M_{i_k} + X_{i_k} - x) \cap K\right)\right|^p \mid I_m(x)\right) p_{N(\Xi\cap K)}(m) \\
&\le \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k} c(K)\right)^p p_{N(\Xi\cap K)}(m) = c^p(K) \sum_{m=0}^{\infty} 2^{mp} p_{N(\Xi\cap K)}(m) \;=\; c^p(K)\, g_{N(\Xi\cap K)}(2^p)\,,
\end{aligned}$$

where the inequality is due to the conditional boundedness of f. The proof of the general case is similar and therefore omitted. □

2.3 Unbiased and consistent estimation of the mean

Consider an unboundedly increasing sequence $\{W_n\}$ of bounded Borel sets $W_n \subset \mathbb{R}^d$ with

$$\lim_{n\to\infty} |W_n| = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{|\partial W_n \oplus B_r(o)|}{|W_n|} = 0 \tag{2.7}$$

for any $r > 0$. Here, $B_r(x) = \{y \in \mathbb{R}^d : |y - x| \le r\}$ is the closed ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $r > 0$, and ∂B is the boundary of a Borel set B. Notice that (2.7) implies

$$\lim_{n\to\infty} \frac{|W_n \oplus B_r(o)|}{|W_n|} = \lim_{n\to\infty} \frac{|W_n \ominus B_r(o)|}{|W_n|} = 1 \tag{2.8}$$

for any $r > 0$. Thus, without loss of generality, we can assume that $|W_n \ominus \check{K}| > 0$ for each $n \ge 1$.

Furthermore, let $G : \mathcal{B}_0(\mathbb{R}^d) \times \mathbb{R}^d \to [0,\infty)$ be some nonnegative function which is Borel-measurable in the second component such that for each $n \geq 1$

$$G(W_n, x) = 0 \quad \text{if } x \in \mathbb{R}^d \setminus (W_n \ominus \check{K}), \qquad \text{and} \qquad \int_{W_n} G(W_n, x)\, dx = 1\,. \tag{2.9}$$

Assume now that the RACS Ξ is stationary. Then, it follows from Fubini's theorem that

$$\widehat{\mu}_n = \int_{W_n} Y(x)\, G(W_n, x)\, dx \tag{2.10}$$

is an unbiased estimator for the expectation $\mu = E\,Y(o)$, where $Y(x)$ is given by (2.6). Moreover, the estimation variance $\operatorname{Var} \widehat{\mu}_n$ can be determined as follows.

Lemma 2.2. *For any $n \geq 1$, it holds*

$$\operatorname{Var} \widehat{\mu}_n = \int_{\mathbb{R}^d} \operatorname{Cov}_Y(x) R_{W_n}(x)\, dx\,,$$

where $R_{W_n}(x) = \int_{\mathbb{R}^d} G(W_n, y) G(W_n, x+y)\, dy$.

Proof. We have

$$\begin{aligned}
\operatorname{Var} \widehat{\mu}_n &= E\Big(\int_{W_n} (Y(u) - \mu)\, G(W_n, u)\, du \int_{W_n} (Y(v) - \mu)\, G(W_n, v)\, dv\Big) \\
&= \int_{W_n} \int_{W_n} E\left((Y(o) - \mu)(Y(v-u) - \mu)\right)\, G(W_n, u) G(W_n, v)\ dv\, du \\
&= \int_{W_n \oplus \check{W}_n} \operatorname{Cov}_Y(x) \int_{W_n \cap (W_n - x)} G(W_n, y) G(W_n, x+y)\, dy\ dx \\
&= \int_{\mathbb{R}^d} \operatorname{Cov}_Y(x) R_{W_n}(x)\, dx\,,
\end{aligned}$$

where the last equality follows from the fact that $W_n \cap (W_n - x) = \emptyset$ for any $x \notin W_n \oplus \check{W}_n$. □

To determine the asymptotic behavior of the estimation variance $\operatorname{Var} \widehat{\mu}_n$, we need some further conditions on the weighting function $G : \mathcal{B}_0(\mathbb{R}^d) \times \mathbb{R}^d \to [0,\infty)$. Besides (2.9), we additionally assume that there exist constants $c_1, c_2 < \infty$ such that

$$\sup_{y \in W_n} G(W_n, y) \leq \frac{c_1}{|W_n|} \quad \text{for any } n \geq 1 \qquad \text{and} \qquad \lim_{n\to\infty} |W_n| R_{W_n}(x) = c_2 \quad \text{for any } x \in \mathbb{R}^d. \tag{2.11}$$

Notice that (2.9) and (2.11) hold, for example, if $G(W_n, x) = \mathbb{I}(x \in W_n \ominus \check{K}) \,/\, |W_n \ominus \check{K}|$ for any $n \geq 1$ and $x \in \mathbb{R}^d$, where $\mathbb{I}(B)$ denotes the indicator function of event B. In this case, due to (2.8), we have $c_1 = 2$ and $c_2 = 1$. Furthermore, we assume that the covariance $\operatorname{Cov}_Y(x)$ of the stationary random field Y is integrable, i.e.,

$$\int_{\mathbb{R}^d} |\operatorname{Cov}_Y(x)|\, dx < \infty\,. \tag{2.12}$$

Lemma 2.3. *Let the conditions* (2.7), (2.9) *and* (2.11) – (2.12) *be fulfilled. Then, it holds that*

$$\lim_{n\to\infty} |W_n| \operatorname{Var} \widehat{\mu}_n = c_2 \int_{\mathbb{R}^d} \operatorname{Cov}_Y(x)\, dx\,.$$

Proof. Conditions (2.9) and (2.11) immediately imply that $|W_n| R_{W_n}(x) \le c_1$ holds for any $x \in \mathbb{R}^d$ and $n \ge 1$. Thus, using Lemma 2.2 and condition (2.12), the assertion follows from the Lebesgue dominated convergence theorem. □

Since $\lim_{n\to\infty} |W_n| = \infty$, Lemma 2.3 implies in particular that $\lim_{n\to\infty} \operatorname{Var} \widehat{\mu}_n = 0$, i.e., the unbiased estimator $\widehat{\mu}_n$ is also mean-square consistent for μ.

3 Examples

In this section, we briefly discuss some examples of stationary random fields which belong to the general class of random fields $Y = \{Y(x), x \in \mathbb{R}^d\}$ introduced in (2.6). They can be used to construct unbiased and mean-square consistent estimators for various morphological characteristics of stationary RACS. In the sequel, we assume that $\{W_n\}$ is an arbitrary sequence of bounded Borel sets with $|W_n| > 0$ for any $n \ge 1$ that satisfies (2.7).

3.1 Volume fraction

Let Ξ be a stationary RACS in $\mathbb{R}^d$ with *volume fraction* $p = P(o \in \Xi)$ and let $Z_d = \{Z_d(x), x \in \mathbb{R}^d\}$ be the random field given by $Z_d(x) = \mathbb{I}(x \in \Xi)$. Then,

$$\widehat{p}_n = \frac{1}{|W_n|} \int_{W_n} Z_d(x)\, dx \tag{3.1}$$

is an unbiased estimator for p. Since $\mathbb{I}(x \in \Xi) = \mathbb{I}\big((\Xi - x) \cap \{o\} \ne \emptyset\big)$ for any $x \in \mathbb{R}^d$, it is easy to see that Z_d is of the form considered in (2.6), where $K = \{o\}$ and the (bounded) valuation $f : \mathcal{R} \to \mathbb{R}$ is given by $f(K') = \mathbb{I}(K' \ne \emptyset)$. Clearly, the random field Z_d is of second order. If $\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)$ is a Boolean model with $E\,|M_0|^2 < \infty$, then it is well known that the covariance $\operatorname{Cov}_{Z_d}(x)$ of Z_d is integrable; see, e.g., the remarks after Corollary 4.2 of [1]. According to Lemma 2.3, the unbiased estimator $\widehat{p}_n$ is mean-square consistent for p.

3.2 Specific intrinsic volumes

Let Ξ be a stationary RACS such that $\Xi \in \mathcal{S}$ holds with probability 1. Then, for each $i = 0, \ldots, d$, the intrinsic volume $V_i(\Xi \cap K)$ of $\Xi \cap K$ is well defined for any convex body $K \in \mathcal{K}$, where, for instance, $V_d(\Xi \cap K) = |\Xi \cap K|$ is the usual volume and $V_0(\Xi \cap K)$ is the *Euler number* of the set $\Xi \cap K$ which is defined by the inclusion-exclusion formula (2.5) and $V_0(M) = \mathbb{I}(M \ne \emptyset)$, $M \in \mathcal{K}$; see also [12].

Assume that $E\,2^{\widetilde{N}(\Xi\cap[0,1]^d)} < \infty$ holds, where $\widetilde{N}(B)$ denotes the minimal number of convex components of the polyconvex set $B \in \mathcal{R}$. Then, for any sequence $\{K_n\}$ of convex bodies $K_n = nK_0$ with $K_0 \in \mathcal{K}$ such that $|K_0| > 0$ and $o \in \mathrm{int}(K_0)$, the limits

$$\overline{V}_i(\Xi) = \lim_{n\to\infty} \frac{E\,V_i(\Xi\cap K_n)}{|K_n|}\,, \qquad i = 0,\dots,d \tag{3.2}$$

exist and are called the *specific intrinsic volumes* of Ξ; see e.g. Section 5.1 of [13]. For some specific intrinsic volumes given in (3.2), estimators of several types are considered in the literature. Two indirect estimation methods have been proposed in [11] and [14], respectively. They have the advantage that *joint* estimators can be constructed for the vector $(\overline{V}_0(\Xi),\dots,\overline{V}_d(\Xi))$ of all $d+1$ specific intrinsic volumes.

The construction principle considered in [11] is based on Steiner's formula and makes use of the index of polyconvex sets; see Section 2.3 of [11]. The random field used therein is defined as follows. For $i = 0,\dots,d-1$, let $r_i > 0$ be any positive number and let the random field $Z_i = \{Z_i(x), x\in\mathbb{R}^d\}$ be given by

$$Z_i(x) = \sum_{q\in\partial\left((\Xi-x)\cap B_{r_i}(o)\right),\, q\neq 0} J((\Xi-x)\cap B_{r_i}(o), q, o)\,, \tag{3.3}$$

where the functional $J(K,q,x) = \mathbb{I}(q\in K)\big(1-\lim_{\delta\to+0}\lim_{\varepsilon\to+0} V_0\left(K\cap B_{|x-q|-\varepsilon}(x)\cap B_\delta(q)\right)\big)$ is called the *index* of $K\in\mathcal{R}$ at $x\in\mathbb{R}^d$. It is not difficult to see that Z_i is of the form considered in (2.6), where $K = B_{r_i}(o)$ and the valuation $f:\mathcal{R}\to\mathbb{R}$ is given by $f(K') = \sum_{q\in\partial K',q\neq 0} J(K',q,o)$. Here, the functional f is bounded on $\mathcal{K}$ with $f(K') = \mathbb{I}(o\notin K', K'\neq\emptyset)$ for any $K'\in\mathcal{K}$. If the covariance $\mathrm{Cov}_{Z_i}(x)$ of Z_i is integrable for any $i = 0,\dots,d$, it can be concluded from Lemma 2.3 that $\widehat{\mu}_{ni} = |W_n\ominus B_{r_i}(o)|^{-1}\int_{W_n} Z_i(x)\mathbb{I}(x\in W_n\ominus B_{r_i}(o))\,dx$ is an unbiased estimator for $\mu_i = E\,Z_i(o)$ which is mean-square consistent provided that $\{W_n\}$ satisfies (2.7). Then, assuming that $r_i \neq r_{i'}$ for any $i\neq i'$, the random vector $\widehat{v}_n = A^{-1}_{r_0,\dots,r_{d-1}}\left(\widehat{\mu}_{n0},\dots,\widehat{\mu}_{n(d-1)},\widehat{p}_n\right)^\top$ provides an unbiased, mean-square consistent estimator for $(\overline{V}_0(\Xi),\dots,\overline{V}_d(\Xi))$, where $\widehat{p}_n$ is the empirical volume fraction introduced in (3.1) and $A_{r_0\dots r_d}$ is a regular matrix of Vandermond type; see [11].

The estimator proposed in [14] employs the principal kinematic formula. Here, the construction principle is the following. For $i = 0,\dots,d$ introduce the random fields $\widetilde{Z}_i = \{\widetilde{Z}_i(x),\, x\in\mathbb{R}^d\}$ by $\widetilde{Z}_i(x) = V_0\left((\Xi-x)\cap B_{r_i}(o)\right)$. Each $\widetilde{Z}_i$ is of the form (2.6) with $f(K') = V_0(K')$, $K'\in\mathcal{R}$ and $K = B_{r_i}(0)$, where f is bounded on $\mathcal{K}$ with $f(K') = \mathbb{I}(K'\neq\emptyset)$ for all $K'\in\mathcal{K}$. For any $d+1$ pairwise different positive radii $r_0,\dots,r_d$ define

$$\widetilde{\mu}_{ni} = \int_{W_n\ominus B_{r_i}(o)} \frac{\widetilde{Z}_i(x)}{|W_n\ominus B_{r_i}(o)|}\,dx \quad\text{and}\quad \widetilde{A}_{r_0\dots r_d} = \begin{pmatrix} r_0^d\kappa_d & r_0^{d-1}\kappa_{d-1} & \dots & r_0^2\kappa_2 & r_0\kappa_1 & 1\\ r_1^d\kappa_d & r_1^{d-1}\kappa_{d-1} & \dots & r_1^2\kappa_2 & r_1\kappa_1 & 1\\ \dots & \dots & \dots & \dots & \dots & \dots\\ r_d^d\kappa_d & r_d^{d-1}\kappa_{d-1} & \dots & r_d^2\kappa_2 & r_d\kappa_1 & 1 \end{pmatrix},$$

where κ_i denotes the volume of the unit ball in $\mathbb{R}^i$, $i = 1,\dots,d$. Then, the random vector $\widetilde{v}_n = \widetilde{A}^{-1}_{r_0\dots r_d}\left(\widetilde{\mu}_{n0},\dots,\widetilde{\mu}_{nd}\right)^\top$ is an unbiased mean-square consistent estimator for $(\overline{V}_0(\Xi),\dots,\overline{V}_d(\Xi))$.

4 Asymptotic normality for functionals of Boolean models

Let $\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)$ be a Boolean model with compact and convex typical grain $M_0 \in \mathcal{K}' = \mathcal{K} \setminus \{\emptyset\}$. The aim of this section is to prove asymptotic normality of $\widehat{\mu}_n = \int_{W_n} Y(x) G(W_n, x) dx$ being an estimator for the mean value μ of the random field Y introduced in (2.6), where we assume that the conditions (2.7), (2.9) and (2.11) – (2.12) are fulfilled. More precisely, replacing (2.12) by a moment condition on M_0, we show that

$$\sqrt{|W_n|}\,(\widehat{\mu}_n - \mu) \quad \Longrightarrow \quad \mathcal{N}(0, \sigma^2)\,, \qquad n \to \infty, \tag{4.1}$$

where $\Longrightarrow$ denotes convergence in distribution and $\mathcal{N}(0, \sigma^2)$ is a Gaussian random variable with zero mean and variance $\sigma^2 = c_2 \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx$.

We concentrate on the case of the Boolean model for two reasons. First of all, the integrability of the covariance $\mathrm{Cov}_Y(x)$ of random field Y is quite generally tractable in this case; see Lemma 4.1. Secondly, one can make use of a central limit theorem for m-dependent random fields from [4] without imposing further conditions. A corresponding central limit theorem for more general germ–grain models is considered in Section 5.

4.1 Integrability of the covariance

The following lemma yields a simple sufficient condition for absolute integrability of the covariance $\mathrm{Cov}_Y(x), x \in \mathbb{R}^d$.

Lemma 4.1. *Assume that* $E\,|M_0 \oplus \check{K}|^2 < \infty$. *Then inequality* (2.12) *holds.*

Proof. For better readability, we use the representation of Ξ as the set-theoretic union of the generating Poisson particle process $\widetilde{M} = \{\widetilde{M}_i\}$ in $\mathcal{K}'$ with $\widetilde{M}_i = M_i + X_i$, and let Λ denote the intensity measure of $\widetilde{M}$. By Campbell's theorem for independently marked point processes on $\mathbb{R}^d$, see e.g. Section 3 of [13], the following representation for Λ holds

$$\Lambda(B) \;=\; \lambda \int_{\mathcal{K}} \int_{\mathbb{R}^d} \mathbb{I}((M_0 + y) \in B)\, dy\, d\mathbb{Q}(M_0) \tag{4.2}$$

for any set $B \subseteq \mathcal{K}$ with $B \in \sigma_{\mathcal{F}}$, where $\mathbb{Q}$ denotes the distribution of the typical grain M_0. Now, put $\mathcal{K}_x^* = \mathcal{K}_K \cap \mathcal{K}_{K+x}$ with $\mathcal{K}_K = \{K' \in \mathcal{K} : K' \cap K \neq \emptyset\}$ for any set $K \in \mathcal{K}'$, and let $B \triangle B' = (B \cup B') \setminus (B \cap B')$ be the symmetric difference between any sets B and B'. Considering the event $A = \{\widetilde{M}(\mathcal{K}_x^*) > 0\}$ and its complement A^c, where $\widetilde{M}(B)$ is the number of particles of $\widetilde{M}$ in a set $B \subseteq \mathcal{K}$, we can write

$$\mathrm{Cov}_Y(x) \;=\; E\big(Y(o)\,(Y(x) - \mu)\big) \;=\; E\big(Y(o) \cdot \mathbb{I}(A)\,(Y(x) - \mu)\big) + E\big(Y(o) \cdot \mathbb{I}(A^c)\,(Y(x) - \mu)\big)\,.$$

Using similar arguments as in the proof of Lemma 2.1, upper bounds for the absolute values of the summands in the above decomposition of $\mathrm{Cov}_Y(x)$ can be deduced in the following way. We have

$$\begin{aligned}
|E\big(Y(o)\,(Y(x) - \mu)\,\mathbb{I}(A)\big)| \;&\leq\; c^2(K)\, E\big(2^{\widetilde{M}(\mathcal{K}_K) + \widetilde{M}(\mathcal{K}_{K+x})}\, \mathbb{I}(A)\big) + c(K)\,|\mu|\, E\big(2^{\widetilde{M}(\mathcal{K}_K)}\, \mathbb{I}(A)\big) \\
&=\; c^2(K)\, E\big(2^{\widetilde{M}(\mathcal{K}_K \triangle \mathcal{K}_{K+x})}\big)\, E\big(2^{2\widetilde{M}(\mathcal{K}_x^*)}\, \mathbb{I}(A)\big) + c(K)\,|\mu|\, E\big(2^{\widetilde{M}(\mathcal{K}_K \setminus \mathcal{K}_x^*)}\big)\, E\big(2^{\widetilde{M}(\mathcal{K}_x^*)}\, \mathbb{I}(A)\big) \\
&\leq\; 2\, c^2(K)\, (E\, 2^{\widetilde{M}(\mathcal{K}_K)})^2 \cdot E\big(4^{\widetilde{M}(\mathcal{K}_x^*)}\, \mathbb{I}(A)\big)\,,
\end{aligned}$$

since the random variables $2^{\widetilde{M}(\mathcal{K}_K \triangle \mathcal{K}_{K+x})}$ and $2^{\widetilde{M}(\mathcal{K}_K \setminus \mathcal{K}_x^*)}$ are independent of $4^{\widetilde{M}(\mathcal{K}_x^*)}\, \mathbb{I}(A)$, and employing Lemma 2.1 and the stationarity of $\widetilde{M}$. For the second summand of the representation of $\mathrm{Cov}(x)$ define $Y_B(x) = f\big(\bigcup_{i:\widetilde{M}_i \in B}(\widetilde{M}_i - x) \cap K\big)$ for any $B \subseteq \mathcal{K}$. By the properties of valuation f it holds that $Y(o)\, \mathbb{I}(A^c) = Y_{\mathcal{K}\setminus\mathcal{K}_x^*}(o)\, \mathbb{I}(A^c)$ and $Y(x)\, \mathbb{I}(A^c) = Y_{\mathcal{K}_x\setminus\mathcal{K}_x^*}(x)\, \mathbb{I}(A^c)$, where $Y_{\mathcal{K}\setminus\mathcal{K}_x^*}(o)$, $Y_{\mathcal{K}_x\setminus\mathcal{K}_x^*}(x)$ and $\mathbb{I}(A^c)$ are mutually independent. Hence, one has

$$\begin{aligned}
|E\big(Y(o)\, \mathbb{I}(A^c)\, (Y(x) - \mu)\big)| &= |E\big(\, Y_{\mathcal{K}\setminus\mathcal{K}_x^*}(o)\, \mathbb{I}(A^c)\, (Y_{\mathcal{K}_x\setminus\mathcal{K}_x^*}(x) - \mu)\, \big)| \\
&= |E\big(Y(o)\, \mathbb{I}(A^c)\big)| \cdot |E\big(Y_{\mathcal{K}_x\setminus\mathcal{K}_x^*}(x) - \mu\big)| \\
&= |E\big(Y(o)\, \mathbb{I}(A^c)\big)| \cdot |E\big((Y_{\mathcal{K}_x\setminus\mathcal{K}_x^*}(x) - Y(x))\, \mathbb{I}(A)\big)| \\
&\le c(K)\, E\,|Y(o)|\, E\, 2^{\widetilde{M}(\mathcal{K}_K)} \cdot \big(E\, \mathbb{I}(A) + E\big(2^{\widetilde{M}(\mathcal{K}_x^*)}\, \mathbb{I}(A)\big)\big)\,,
\end{aligned}$$

where the inequality follows as before. Notice that $E\, s^{\widetilde{M}(\mathcal{K}_K)} = e^{(s-1)\,\Lambda(\mathcal{K}_K)} < \infty$ for any $s \in \mathbb{R}$, since $\Lambda(\mathcal{K}_K) = \lambda\, E\,|M_0 \oplus \check{K}| < \infty$ by condition (2.2) and formula (4.2). Thus, it suffices to show that $E(4^{\widetilde{M}(\mathcal{K}_x^*)}\, \mathbb{I}(A))$ is integrable with respect to $x \in \mathbb{R}^d$. Observe that with $\widetilde{M}$ being Poisson

$$E\big(4^{\widetilde{M}(\mathcal{K}_x^*)}\, \mathbb{I}(A)\big) \;=\; E\, 4^{\widetilde{M}(\mathcal{K}_x^*)} - E\, \mathbb{I}(A^c) \;=\; e^{3\Lambda(\mathcal{K}_x^*)}\big(1 - e^{-4\Lambda(\mathcal{K}_x^*)}\big) \;\le\; 4\, e^{3\lambda\, E\,|M_0 \oplus \check{K}|}\, \Lambda(\mathcal{K}_x^*),$$

where we used the estimate $1 - e^{-s} \le s$ for any $s \ge 0$ to obtain the latter inequality. By virtue of Campbell's and Fubini's formulae, we can finally conclude that

$$\begin{aligned}
\int_{\mathbb{R}^d} \Lambda(\mathcal{K}_x^*)\, dx &= \int_{\mathbb{R}^d} \Big(E \sum_{i=1}^{\infty} \mathbb{I}\big((M_i + X_i) \cap K \ne \emptyset, (M_i + X_i) \cap (K + x) \ne \emptyset\big)\Big)\, dx \\
&= \int_{\mathbb{R}^d} \Big(\lambda\, E \int_{\mathbb{R}^d} \mathbb{I}\big((M_0 + y) \cap K \ne \emptyset, (M_0 + y) \cap (K + x) \ne \emptyset\big)\, dy\Big)\, dx \\
&= \lambda\, E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{I}\big(y \in (\check{M}_0 \oplus K)\big)\, \mathbb{I}\big((y - x) \in (\check{M}_0 \oplus K)\big)\, dy\, dx \\
&= \lambda\, E\,|\check{M}_0 \oplus K|^2 \;=\; \lambda\, E\,|M_0 \oplus \check{K}|^2 \;<\; \infty\,.
\end{aligned}$$

□

Note that the proof of Lemma 4.1 provides an integrable upper bound $h(x) = h(x, M_0)$ of $|\mathrm{Cov}_Y(x)|$ which depends on the distribution of the typical grain M_0. This dependence is monotone with respect to set inclusion. Namely, if $M_0^{(1)} \subseteq M_0^{(2)}$, then it holds that $h(x, M_0^{(1)}) \le h(x, M_0^{(2)})$ for any $x \in \mathbb{R}^d$ with probability one.

4.2 Truncated germ–grain models

Let the conditions (2.7), (2.9) and (2.11) be fulfilled and assume that $E\,|M_0 \oplus \check{K}|^2 < \infty$. To prove the central limit theorem (4.1), we approximate the random field Y corresponding to Ξ by random fields Y_n induced by germ–grain models Ξ_n with truncated grains which are chosen in the following way. For any $n \ge 1$, let $A_n = [-a_n, a_n]^d$ for some $a_n > 0$ such that $\lim_{n\to\infty} a_n = \infty$. Introduce the auxiliary germ–grain model Ξ_n by

$$\Xi_n = \bigcup_{i=1}^{\infty} (M_{ni} + X_i)\,, \tag{4.3}$$

where $M_{ni} = M_i \cap A_n \in \mathcal{K}$ for any $i, n \in \mathbb{N}$. Accordingly, define the random field $Y_n = \{Y_n(x),\ x \in \mathbb{R}^d\}$ by $Y_n(x) = f\,((\Xi_n - x) \cap K)$, and let $\mu_n = E\,Y_n(o)$ and $\widehat{\mu}_n' = \int_{W_n} Y_n(x) G(W_n, x)\, dx$.

Lemma 4.2. *The random fields Y and Y_n are of second order. Moreover, $Y_n(x)$ converges in mean square to $Y(x)$ as $n \to \infty$, i.e. $\lim_{n\to\infty} E\,|Y(x) - Y_n(x)|^2 = 0$ for all $x \in \mathbb{R}^d$.*

Proof. Due to stationarity, we can assume that $x = o$. Since $g_{N(\Xi_n \cap K)}(4) \le g_{N(\Xi \cap K)}(4) < \infty$, both random fields Y and Y_n are of second order by Lemma 2.1. To show the second assertion, we put $N_{|A_n}(\Xi \cap K) = \#\{i : (M_i + X_i) \cap K \ne \emptyset, K \not\subseteq (A_n + X_i)\}$. Then, using similar arguments as in the proof of Lemma 2.1, we have

$$E\,|Y(o) - Y_n(o)|^2 \;\le\; 4c^2(K) E\Big(2^{2N(\Xi\cap K)}\, \mathbb{1}\big(N_{|A_n}(\Xi \cap K) > 0\big)\Big).$$

With $E\big(2^{2N(\Xi\cap K)}\big) = e^{3\lambda\, E\,|M_0 \oplus \check{K}|} < \infty$, it suffices to show that $\lim_{n\to\infty} P(N_{|A_n}(\Xi \cap K) > 0) = 0$. Apply Campbell's formula and Fubini's theorem to get

$$\begin{aligned}
P(N_{|A_n}(\Xi \cap K) > 0) \;&\le\; E\big(N_{|A_n}(\Xi \cap K)\big) \;=\; \lambda\, E \int_{\mathbb{R}^d} \mathbb{1}\big((M_0 + y) \cap K \ne \emptyset, K \not\subseteq (A_n + y)\big)\, dy \\
&\le\; \lambda\, E \int_{\mathbb{R}^d} \mathbb{1}\big(y \in (M_0 \oplus \check{K})\big) \mathbb{1}\big(y \notin (A_n \ominus \check{K})\big)\, dy \\
&\le\; \lambda\, E \int_{\mathbb{R}^d} \mathbb{1}\big(y \in (M_0 \oplus \check{K})\big) \mathbb{1}\big(|y| > a_n - D(K)\big)\, dy
\end{aligned}$$

for any $n \ge n_o$ such that $a_n - D(K) > 0$. By the dominated convergence theorem, the last expression on the right-hand side converges to zero with $\lim_{n\to\infty} a_n = \infty$, since $E\,|M_0 \oplus \check{K}| < \infty$. □

Next, we show that the asymptotic variance of the estimator $\widehat{\mu}_n' = \int_{W_n} Y_n(x)\, G(W_n, x)\, dx$ of μ_n is equal to the asymptotic variance of the estimator $\widehat{\mu}_n = \int_{W_n} Y(x)\, G(W_n, x)\, dx$ of the mean of Y.

Lemma 4.3. *The covariance $\mathrm{Cov}_{Y_n}(x)$ of the stationary random field Y_n is integrable and it holds that*

$$\lim_{n\to\infty} |W_n|\, \mathrm{Var}\, \widehat{\mu}_n' = c_2 \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx\,, \tag{4.4}$$

where the constant $c_2 > 0$ is defined in (2.11).

Proof. The integrability of the covariance $\mathrm{Cov}_{Y_n}(x)$ immediately follows from Lemma 4.1. By Lemma 2.2, we have $|W_n|\, \mathrm{Var}\, \widehat{\mu}_n' = \int_{\mathbb{R}^d} \mathrm{Cov}_{Y_n}(x)\, |W_n| R_{W_n}(x)\, dx$, where $\lim_{n\to\infty} |W_n| R_{W_n}(x) = c_2$ by (2.11). As mentioned at the end of Section 4.1, there exists an integrable function $h : \mathbb{R}^d \to \mathbb{R}_+$ such that $|\mathrm{Cov}_{Y_n}(x)| \le h(x, M_0 \cap A_n) \le h(x, M_0)$ for any $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Together with (2.11) this implies that $|\mathrm{Cov}_{Y_n}(x)|\,|W_n| R_{W_n}(x) \le c_1 h(x, M_0)$ for all $x \in \mathbb{R}^d$. Furthermore, it holds that $\lim_{n\to\infty} \mathrm{Cov}_{Y_n}(x) = \mathrm{Cov}_Y(x)$, since

$$\begin{aligned} |\,\mathrm{Cov}_{Y_n}(x) - \mathrm{Cov}_Y(x)\,| &\le \left|E\big(Y_n(o)Y_n(x)\big) - E\big(Y(o)Y(x)\big)\right| + \left|\mu_n^2 - \mu^2\right| \\ &= \left|E\big((Y_n(o) - Y(o))Y_n(x)\big) + E\big((Y_n(x) - Y(x))Y(o)\big)\right| + \left|\mu_n^2 - \mu^2\right| \\ &\le \Big(E\left(Y_n(o) - Y(o)\right)^2 \left(E\,Y_n^2(o) + E\,Y^2(o)\right)\Big)^{1/2} + \left|\mu_n^2 - \mu^2\right|, \end{aligned}$$

with $\lim_{n\to\infty} E\left(Y_n(o) - Y(o)\right)^2 = 0$ and $\lim_{n\to\infty} \left|\mu_n^2 - \mu^2\right| = 0$ by Lemma 4.2. Besides this, the expression $E\,Y_n^2(o) + E\,Y^2(o)$ is uniformly bounded in n. Consequently, the limit in (4.4) follows from the dominated convergence theorem. □

Let $||z|| = \max\{|z_i| : i = 1, \dots, d\}$ for any $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$ and let $m > 0$ be an arbitrary integer. A family of random variables $\{Z_z, z \in \mathbb{Z}^d\}$ is called an *m–dependent random field* if $(Z_z)_{z\in U}$ and $(Z_z)_{z\in U'}$ are independent random vectors for any finite sets $U, U' \subset \mathbb{Z}^d$ with $\inf\{||z - z'||,\, z \in U, z' \in U'\} > m$. Any stationary random field indexed over $\mathbb{Z}^d$ that satisfies an appropriate β–mixing condition is m-dependent; see e.g. Section 1.3.1 of [2].

The following central limit theorem for $\widehat{\mu}_n'$ is closely connected with a Lindeberg-type central limit theorem for m-dependent random fields as presented in Theorem 2 of [4].

Lemma 4.4. *Let $\{m_n, n \ge 1\}$ be an arbitrary sequence of positive integers such that $m_n \to \infty$ and let $\{U_n, n \ge 1\}$ be a sequence of finite subsets of $\mathbb{Z}^d$ with $\lim_{n\to\infty} \#(U_n) = \infty$. For each $n \ge 1$, let $\{Z_{nz}, z \in \mathbb{Z}^d\}$ be an m_n-dependent random field with $E\,Z_{nz} = 0$ for any $z \in U_n$ and $E\,(S_n^*)^2 = \sigma_n^2 \to \sigma^2$ as $n \to \infty$, where $S_n^* = \sum_{z\in U_n} Z_{nz}$. If there exists a constant $c > 0$ such that*

$$\sum_{z\in U_n} E\,Z_{nz}^2 \le c \tag{4.5}$$

for any $n \ge 1$ and if

$$\lim_{n\to\infty} m_n^{2d} \sum_{z\in U_n} E\left(Z_{nz}^2 \mathbb{1}(|Z_{nz}| \ge \varepsilon \cdot m_n^{-2d})\right) = 0 \tag{4.6}$$

holds for any $\varepsilon > 0$, then $S_n^* \Longrightarrow \mathcal{N}(0, \sigma^2)$ as $n \to \infty$.

The proof of Lemma 4.4 for $\sigma_n^2 = 1$ can be found e.g.in Section 3 of [4] and extended easily to the case $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 < \infty$.

Lemma 4.5. *If the truncation sequence $\{a_n\}$ satisfies*

$$\lim_{n\to\infty} a_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{a_n^{4d(1+\delta)/\delta}}{|W_n|} = 0 \tag{4.7}$$

for some $\delta > 0$, then the random variables $S_n' = \sqrt{|W_n|}\big(\widehat{\mu}_n' - \mu_n\big)$ are asymptotically normally distributed, that is to say,

$$S_n' \Longrightarrow \mathcal{N}(0, \sigma^2), \qquad n \to \infty,$$

where $\sigma^2 = c_2 \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\,dx$ with c_2 as defined in (2.11).

Proof. For $[z, z+e) = [z_1, z_1+1)\times\ldots\times[z_d, z_d+1)$, $z \in \mathbb{Z}^d$ consider the sets $U_n = \{z \in \mathbb{Z}^d : [z, z+e) \subseteq W_n\}$ and $W_n^- = \bigcup_{z\in U_n} [z, z+e)$. Furthermore, decompose S_n' into $S_n' = S_n^* + \widetilde{S}_n$ with

$$S_n^* = \sqrt{|W_n|}\int_{W_n^-} \big(Y_n(x) - \mu_n\big)\, G(W_n, x)\, dx\,, \quad \widetilde{S}_n = \sqrt{|W_n|}\int_{W_n\setminus W_n^-} \big(Y_n(x) - \mu_n\big)\, G(W_n, x)\, dx\,.$$

Condition (2.11) and the integrability of $\mathrm{Cov}_{Y_n}(x)$ imply that

$$\begin{aligned}
\lim_{n\to\infty} E\,\widetilde{S}_n^2 &= \lim_{n\to\infty}\int_{W_n\setminus W_n^-}\int_{W_n\setminus W_n^-} |W_n|\, E\,\big((Y_n(x)-\mu_n)\,(Y_n(y)-\mu_n)\big)\, G(W_n,x)\, G(W_n,y)\, dx\, dy \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^d} \mathrm{Cov}_{Y_n}(x)\, |W_n| \Big(\int_{W_n\setminus W_n^- \cap (W_n\setminus W_n^- - x)} G(W_n,y)\, G(W_n, x+y)\, dy\Big)\, dx \\
&\le \lim_{n\to\infty} \frac{c_1^2\cdot W_n\setminus W_n^-|}{|W_n|}\int_{\mathbb{R}^d} h(x)\, dx \;\le\; c\cdot\lim_{n\to\infty}\frac{\big|\partial W_n \oplus B_{2\sqrt{d}}(o)\big|}{|W_n|} \;=\; 0
\end{aligned}$$

for some constant $c > 0$, where the last equality follows from (2.7). Hence, the second component in the decomposition of S_n' converges to 0 in mean square. Using Slutsky's theorem, it is sufficient to show that $S_n^* \Longrightarrow \mathcal{N}(0, \sigma^2)$ as $n \to \infty$. We apply Lemma 4.4 to $S_n^* = \sum_{z\in U_n} Z_{nz}$, where

$$Z_{nz} = \begin{cases} \sqrt{|W_n|}\;\int_{[z,z+e)} \big(Y_n(x)-\mu_n\big)\, G(W_n,x)\, dx, & z \in U_n, \\ 0, & \text{otherwise.}\end{cases} \tag{4.8}$$

It is not difficult to see that the family of random variables $\{Z_{nz}, z \in \mathbb{Z}^d\}$ given in (4.8) forms an m_n-dependent random field for any $m_n \ge 2\,(a_n + D(K))$. By the definition of Z_{nz}, we have $E\,Z_{nz} = 0$ for any $z \in U_n$. Furthermore, it holds that $\lim_{n\to\infty} E\,(S_n^*)^2 = \sigma^2$ by Lemma 4.3 and since $E\,\widetilde{S}_n^2 \to 0$ as shown above. In order to complete the proof, it remains to show that the conditions (4.5) and (4.6) are fulfilled. Employ Fubini's theorem and (2.11) to get

$$\begin{aligned}
\sum_{z\in U_n} E\,Z_{nz}^2 &\le \sum_{z\in U_n} \frac{c_1^2}{|W_n|}\int_{[z,z+e)}\int_{[z,z+e)} |\mathrm{Cov}_{Y_n}(x-y)|\, dx\, dy \\
&\le \sum_{z\in U_n} \frac{c_1^2\ \mathrm{Var}\, Y_n(o)}{|W_n|} \;=\; c_1^2\ \mathrm{Var}\, Y_n(o)\,\frac{|W_n^-|}{|W_n|} \;\le\; c
\end{aligned}$$

for all sufficiently large n and some constant $c < \infty$, where the uniform bound of the latter expression follows from $\mathrm{Var}\, Y_n(o) \le h(o) < \infty$ and $|W_n^-|\,/\,|W_n| \le 1$ for any $n \ge 1$. Thus, (4.5) holds. Due to the stationarity of Ξ_n, the random variables

$$\widetilde{Z}_{nz} = \int_{[z,z+e)} \frac{|Y_n(x) - \mu_n|}{\sqrt{|W_n|}}\, dx\,, \qquad z \in U_n$$

are identically distributed. By the inequality in (2.11), we have in addition $|Z_{nz}| \le c_1 \widetilde{Z}_{nz}$. This yields the following estimates

$$
\begin{aligned}
m_n^{2d} \sum_{z\in U_n} E\left(Z_{nz}^2 \mathbb{I}\big(m_n^{2d}|Z_{nz}| \ge \varepsilon\big)\right) &\le m_n^{2d}\, |W_n^-|\, c_1^2\, E\left(\widetilde{Z}_{no}^2 \mathbb{I}\big(\widetilde{Z}_{no}^\delta \ge \frac{\varepsilon^\delta}{c_1^\delta\, m_n^{2d\delta}}\big)\right) \\
&\le \frac{c_1^{2+\delta}}{\varepsilon^\delta}\, m_n^{2d(1+\delta)}\, |W_n^-|\, E\,\widetilde{Z}_{no}^{2+\delta} \\
&\le \frac{c_1^{2+\delta}}{\varepsilon^\delta} \left(\frac{m_n^{4d(1+\delta)/\delta}}{|W_n|}\right)^{\delta/2} \frac{|W_n^-|}{|W_n|}\, E\,\widehat{Z}_{no}^{2+\delta}
\end{aligned}
\tag{4.9}
$$

for any $\delta > 0$, where $\widehat{Z}_{no} = \sqrt{|W_n|}\widetilde{Z}_{no}$. Since $m_n \ge 2(a_n + D(K))$, the second factor of the latter expression converges to zero as $n \to \infty$ if (4.7) holds for some $\delta > 0$. The remaining factors of (4.9) are uniformly bounded in n, because $|W_n^-| \,/\, |W_n| \le 1$ and

$$
\begin{aligned}
\big(E\,\widehat{Z}_{no}^{2+\delta}\big)^{\frac{1}{2+\delta}} &\le \left(E\left(\int_{[o,e)} |Y_n(x)| + |\mu_n|\, dx\right)^{2+\delta}\right)^{\frac{1}{2+\delta}} \le \left(E\int_{[o,e)} |Y_n(x)|^{2+\delta}\, dx\right)^{\frac{1}{2+\delta}} + |\mu_n| \\
&\le c(K)\big(g_{N(\Xi\cap K)}(2^{2+\delta})\big)^{\frac{1}{2+\delta}} + c(K) g_{N(\Xi\cap K)}(2) < \infty
\end{aligned}
$$

for any $n \ge 1$. Thus, condition (4.6) of Lemma 4.4 is fulfilled as well. □

Notice that condition (4.7) of Lemma 4.5 is satisfied, for example, if the truncation sequence $\{a_n\}$ is given by $a_n = r(W_n)^\eta$, where $\eta < \delta/\big(4(1+\delta)\big)$ and $r(W_n)$ denotes the radius of the largest disc that can be inscribed in W_n.

4.3 Asymptotic normality of mean–value estimators

The asymptotic normality proven in Lemma 4.5 for the mean–value estimator associated with the truncated germ–grain model implies an equivalent statement for the original functional.

Theorem 4.1. *Let the conditions* (2.7), (2.9) *and* (2.11) *be fulfilled and assume that* $E\,|M_0 \oplus \check{K}|^2 < \infty$. *Then it holds that*

$$
\sqrt{|W_n|}\,\big(\widehat{\mu}_n - \mu\big) \implies \mathcal{N}(0, \sigma^2)\,, \qquad n \to \infty
$$

for $\sigma^2 = c_2 \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\, dx$ *with* c_2 *as defined in* (2.11).

Proof. Under the above assumptions, Lemma 4.5 guarantees that $S_n' \implies \mathcal{N}(0,\sigma^2)$ as $n \to \infty$ provided that the truncation sequence $\{a_n\}$ satisfies the imposed conditions. Moreover, putting $S_n = \sqrt{|W_n|}\,(\widehat{\mu}_n - \mu)$, we see that the sequence of random variables $S_n - S_n'$ converges to zero in mean square as $n \to \infty$ for any truncation sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = \infty$. This assertion follows directly from Lemmas 2.3, 4.2 and 4.3 together with $E\left((Y(o) - \mu)(Y_n(x) - \mu_n)\right) \le h(x)$ for $h(x) \in L^1(\mathbb{R}^d)$ as derived in the proof of Lemma 4.1. The application of Slutsky's theorem completes the proof. □

Using similar arguments as for the one-dimensional setting, Theorem 4.1 can be easily extended to the multivariate case. For this, choose sets $K_i \in \mathcal{K}$, valuations $f_i : \mathcal{R} \to \mathbb{R}$, and random fields $Y_i = \{f_i((\Xi - x) \cap K_i), x \in \mathbb{R}^d\}$ for $i = 1, \dots, k$, which are defined on the basis of the same stationary RACS Ξ. In addition, let $G_i(W_n, .), i = 1, \dots, k$ be weight functions satisfying the conditions

$$G_i(W_n, x) = 0 \quad \text{if } x \in \mathbb{R}^d \setminus (W_n \ominus \check{K}_i)\,, \qquad \int_{W_n} G_i(W_n, x)\, dx = 1\,, \tag{4.10}$$

$$\sup_{y \in W_n} G_i(W_n, y) \le \frac{c_1}{|W_n|} \quad \forall\, n \ge 1, \qquad \text{and} \qquad \lim_{n \to \infty} |W_n| R_{W_n(ij)}(x) = c_{ij} \quad \forall\, x \in \mathbb{R}^d, \tag{4.11}$$

where $c_1, c_{ij} < \infty$ are some constants and

$$R_{W_n(ij)}(x) \;=\; \int_{\mathbb{R}^d} G_i(W_n, y) G_j(W_n, x + y)\, dy\,.$$

For each $i = 1, .., k$ define $\mu_i = E\, Y_i(o)$ and $\widehat{\mu}_{ni} = \int_{W_n} Y_i(x)\, G_i(W_n, x)\, dx$. Then, as in Lemma 2.2, we see that the cross–covariances $\text{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj})$ are given by

$$\text{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj}) \;=\; \int_{\mathbb{R}^d} \text{Cov}(Y_i(o), Y_j(x))\, R_{W_n(ij)}(x)\, dx\,.$$

Similar to Lemmas 2.3 and 4.1, the limits

$$\sigma_{ij} \;=\; \lim_{n \to \infty} |W_n|\, \text{Cov}(\widehat{\mu}_{ni}, \widehat{\mu}_{nj})$$

exist and are given by

$$\sigma_{ij} \;=\; c_{ij} \int_{\mathbb{R}^d} \text{Cov}(Y_i(o), Y_j(x))\, dx \tag{4.12}$$

for any $i, j = 1, \dots, k$ provided that $E\, |M_0 \oplus \check{K}_i|^2 < \infty$. We are now in a position to formulate a multidimensional analogue of Theorem 4.1.

Theorem 4.2. *Let the conditions* (2.7), (4.10) *and* (4.11) *be fulfilled and assume that* $E\, |M_0 \oplus \check{K}_i|^2 < \infty$ *for each* $i = 1, \dots, k$. *Then, it holds that*

$$\begin{pmatrix} \sqrt{|W_n|}\, (\widehat{\mu}_{n1} - \mu_1) \\ \vdots \\ \sqrt{|W_n|}\, (\widehat{\mu}_{nk} - \mu_k) \end{pmatrix} \Longrightarrow \mathcal{N}_k(o, \Sigma)\,, \qquad n \to \infty\,,$$

where $\mathcal{N}_k(o, \Sigma)$ *is a* k*–dimensional Gaussian random vector with zero mean vector and covariance matrix* $\Sigma = (\sigma_{ij})$, *whose entries are defined by* (4.12).

Proof. By the well-known Cramér–Wold device the assertion is true if and only if, for all $t \in \mathbb{R}^k \setminus \{o\}$,

$$\sqrt{|W_n|} \sum_{i=1}^{k} t_i (\widehat{\mu}_{ni} - \mu_i) = \sqrt{|W_n|} \int_{W_n} \sum_{i=1}^{k} t_i\ (Y_i(x) - \mu_i)\ G_i(W_n, x)\, dx \;\Longrightarrow \mathcal{N}(0, \sigma^2)\,,$$

where $\sigma^2 = t^\top \Sigma\, t$. The above convergence can be proven analogously to Theorem 4.1. $\square$

5 Asymptotic normality for β-mixing random measures

In the previous section, we considered germ–grain models driven by a Poisson point process. Now we show how the above results can be extended to a more general setting, where we do not assume that the point process $\{X_i\}$ of germs is necessarily Poisson, but that it satisfies some mixing condition.

Let us begin by recalling some basic notions of mixing; see e.g. [2] for further details. Consider the probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be two sub-σ-algebras of $\mathcal{A}$. The *β-mixing coefficient* (also called the *absolute regularity coefficient*) of $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined by

$$\beta(\mathcal{A}_1, \mathcal{A}_2) = \frac{1}{2} \sup \sum_k \sum_\ell |P(A_k \cap B_\ell) - P(A_k)P(B_\ell)|\,, \tag{5.1}$$

where the supremum is taken over all pairs of finite partitions $\{A_k\}$ and $\{B_\ell\}$ of Ω with $A_k \in \mathcal{A}_1$ for all k and $B_\ell \in \mathcal{A}_2$ for all ℓ. Furthermore, for any pair of bounded Borel sets C_1, $C_2 \in \mathcal{B}_0(\mathbb{R}^d)$, let $\rho(C_1, C_2) = \inf\{|x_1 - x_2| : x_1 \in C_1, x_2 \in C_2\}$ denote the distance between C_1 and C_2. For any $s > 0$, the *β-mixing rate* $\beta_X(s)$ of a point process $X = \{X_i\}$ in $\mathbb{R}^d$, is defined by

$$\beta_X(s) = \sup\{\beta\big(\sigma(N_X(C_1)), \sigma(N_X(C_2))\big) :\ C_1, C_2 \in \mathcal{B}_0(\mathbb{R}^d),\ \rho(C_1, C_2) \geq s\}\,,$$

where $\sigma(N_X(C))$ is the σ-algebra generated by $N_X(C) = \#\{i :\ X_i \in C\}$.

Let Ξ be a germ–grain model of the form (2.1) for some stationary point process $X = \{X_i\}$ and convex grains, and let the random field $Y = \{Y(x), x \in \mathbb{R}^d\}$ be given by (2.6). If X is a Poisson process, then we have $\beta_X(s) = 0$ for all $s > 0$ and Theorem 4.1 implies that $\sqrt{|W_n|}\,(\widehat{\mu}_n - \mu)$ converges weakly to a Gaussian random variable if $E\,|M_0 \oplus \check{K}| < \infty$. Using similar arguments, we can even consider a slightly more general case, where $\beta_X(s) = 0$ for all $s \geq s_0$ and some $s_0 > 0$, as is true, for example, if X is a Matérn cluster process; see e.g. [15] for a definition. Then, the assertions of Lemmas 4.2 and 4.3 hold if $g_{N(\Xi\cap K)}(4) < \infty$ and if there exists an integrable function $h : \mathbb{R}^d \to \mathbb{R}_+$ such that $|\mathrm{Cov}_{Y_n}(x)| \leq h(x)$ for any $x \in \mathbb{R}^d$ and $n \geq n_0$, where $n_0 \geq 0$ is some integer. Moreover, Theorem 4.1 remains valid, where the range of dependency $m_n \geq 2(a_n + D(K))$ in the proof of Lemma 4.5 has to be replaced by $m'_n \geq 2(a_n + D(K)) + s_0$.

In the remaining part of this section, we briefly discuss a different technique in order to show that the normal convergence (4.1) holds. Notice that this technique has been used in [7] for another general class of functionals of germ–grain models. Let $\{Z_z, z \in \mathbb{Z}^d\}$ be a stationary random field and let $\{U_n, n \geq 1\}$ be a sequence of finite subsets of $\mathbb{Z}^d$ with $\lim_{n\to\infty} \#(U_n) = \infty$. Assuming that there exist functions b^*_Z and b^{**}_Z on $[0, \infty)$ such that

$$\beta(\sigma(Z_z, |z| < p+1), \sigma(Z_z, |z| \geq p+q)) \;\leq\; \begin{cases} b^*_Z(q), & q > p = 0, \\ p^{d-1}\, b^{**}_Z(q), & p \geq q \geq 1, \end{cases}$$

the following central limit theorem for absolutely regular random fields holds; see Theorem 6.1 of [6].

Lemma 5.1. *Let $\{Z_z, z \in \mathbb{Z}^d\}$ satisfy the following conditions*

$$E\, Z_0 = 0 \qquad \textit{and} \qquad E\,|Z_0|^{2+\delta} < \infty \quad \textit{for some } \delta > 0 \tag{5.2}$$

$$\sum_{q\geq 1} q^{d-1} \big(b^*_Z(q)\big)^{\delta/(2+\delta)} < \infty \qquad \textit{and} \qquad \lim_{q\to\infty} q^{2d-1}\, b^{**}_Z(q) = 0\,. \tag{5.3}$$

Then $\#(U_n)^{-\frac{1}{2}} S_n \Longrightarrow \mathcal{N}(0,\sigma^2)$, *where* $S_n = \sum_{z\in U_n} Z_z$ *and* $\sigma^2 = \sum_{z\in\mathbb{Z}^d} E(Z_0 Z_z)$ *is absolutely convergent.*

Now, let W_n be the d-dimensional cube $[-n,n)^d$, put $W_n^K = [-n + D'(K), n - D'(K))^d$, where $D'(K)$ is the smallest integer greater than the norm $D(K)$, and define $G(W_n,x) = \mathbb{1}(x \in W_n^K)/|W_n^K|$ for any $x \in \mathbb{R}^d$ and all n large enough such that $|W_n^K| > 0$. Furthermore, let $\Xi = \bigcup_{i=1}^{\infty} (M_i + X_i)$ be a germ–grain model, where the stationary point process $X = \{X_i\}$ has the following mixing property. Like in [7] we assume that there exists a non-increasing function $b_X(\cdot)$ on $[1,\infty)$ such that for all $a, \Delta \geq 1$

$$\beta\big(\sigma\big(N_X([-a,a]^d)\big), \sigma\big(N_X(\mathbb{R}^d \setminus [-a-\Delta, a+\Delta]^d)\big)\big) \;\leq\; b_X(\Delta)\big(a/\min\{a,\Delta\}\big)^{d-1}. \tag{5.4}$$

Theorem 5.1. *Let* Ξ *be a stationary germ–grain model satisfying* (5.4) *with non-empty typical grain* $M_0 \in \mathcal{R}$. *If there exist some* $\delta, \varepsilon > 0$ *such that*

$$E\,|Y(o)|^{2+\delta} < \infty, \qquad E\,D^{2d(\frac{\delta+1}{\delta})+\varepsilon}(M_0) < \infty \tag{5.5}$$

and

$$\sum_{n=1}^{\infty} n^{d-1}\, b_X(n)^{\delta/(2+\delta)} < \infty, \tag{5.6}$$

then it holds that

$$\sqrt{|W_n|}\int_{W_n} (Y(x)-\mu) G(W_n,x)\,dx \;\Longrightarrow\; \mathcal{N}(0,\sigma^2)\,, \qquad n \to \infty, \tag{5.7}$$

where $\sigma^2 = \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\,dx$.

The *proof* of Theorem 5.1 is similar to that of Theorem 6.2 of [7]. Hence, we merely sketch the main steps. We consider the set function $\eta : \mathcal{B}_0(\mathbb{R}^d) \to \mathbb{R}$ with

$$\eta(B) \;=\; \int_B (Y(x)-\mu)\,dx, \qquad B \in \mathcal{B}_0(\mathbb{R}^d)$$

and show that $|W_n^K|^{-1/2}\,\eta(W_n^K) \Longrightarrow \mathcal{N}(0,\sigma^2)$ as $n \to \infty$ using Lemma 5.1. Namely, the latter expression can be written as

$$|W_n^K|^{-1/2}\,\eta(W_n^K) \;=\; \#(U_n)^{-1/2}\sum_{z\in U_n} Z_z, \qquad U_n = \{z \in \mathbb{Z}^d : [z, z+e) \subseteq W_n^K\}$$

for the stationary random field $Z = \{Z_z, z \in \mathbb{Z}^d\}$ with $Z_z = \int_{[z,z+e)} (Y(x)-\mu)\,dx$. Then, we have $E\,Z_0 = 0$ and $E\,|Z_0|^{2+\delta} \leq E\,|Y(0)-\mu|^{2+\delta}$, where the latter bound is finite by assumption (5.5), and finally

$$\sigma^2 \;=\; \sum_{z\in\mathbb{Z}^d} E(Z_0 Z_z) \;=\; \sum_{z\in\mathbb{Z}^d}\int_{[o,e)}\int_{[z,z+e)} \mathrm{Cov}_Y(x-y)\,dx\,dy \;=\; \int_{\mathbb{R}^d} \mathrm{Cov}_Y(x)\,dx\,.$$

To check the conditions in (5.3), let $\sigma_\eta(B)$ be the σ-algebra generated by $\{\eta(B'), B' \subseteq B, B' \in \mathcal{B}_0(\mathbb{R}^d)\}$. For any $p, q \in \mathbb{N}$ there exist $a, \Delta \geq 0$ satisfying

$$\beta\big(\sigma(Z_z, |z| < p+1), \sigma(Z_z, |z| \geq p+q)\big) \;\leq\; \beta\big(\sigma_\eta([-a,a)^d), \sigma_\eta(\mathbb{R}^d \setminus [-a-\Delta, a+\Delta)^d)\big)\,.$$

The right hand side of the last estimate can be bounded by the β-mixing coefficient of the underlying point process X and certain moments of $D(M_0)$. This can be seen following the proofs of Lemmas 5.1 and 5.2 of [7], where

$$\begin{aligned}
&\beta\big(\sigma_\eta([-a,a)^d),\sigma_\eta(\mathbb{R}^d\setminus[-a-\Delta,a+\Delta)^d)\big)\\
&\quad\le\ \beta\Big(\sigma\big(N_X([-a-\frac{\Delta}{4},a+\frac{\Delta}{4})^d)\big),\sigma\big(N_X(\mathbb{R}^d\setminus[-a-\frac{3\Delta}{4},a+\frac{3\Delta}{4})^d)\big)\Big)\\
&\qquad+\lambda\, d\, 2^{d+1}\Big(\Big(\frac{\Delta+4a}{\Delta}\Big)^{d+1}+\Big(\frac{\Delta+12a}{\Delta}\Big)^{d+1}\Big)\, E\big(D^d(M_0\oplus\check{K})\,\mathbb{I}(D(M_0\oplus\check{K})>\Delta/4)\big)\\
&\quad\le\ \Big(\frac{a}{\min\{a,\Delta\}}\Big)^{d-1}\Big(c_1(d)\,b_X(\Delta/2)+c_2(d)\,E\big(D^d(M_0\oplus\check{K})\,\mathbb{I}(D(M_0\oplus\check{K})>\Delta/4)\big)\Big)
\end{aligned}$$

for some finite constants $c_1(d), c_2(d)$ employing assumption (5.4) and $E\,D^d(M_0\oplus\check{K})<\infty$. In the proofs of the above-mentioned lemmas, let $\Xi_B=\bigcup_{X_i\in B}(M_i+X_i)$ denote the germ–grain model restricted to germs within $B\in\mathcal{B}(\mathbb{R}^d)$, define $Y_B(x)=f((\Xi_B-x)\cap K)$ and let $\eta_B(B')=\int_{B'}Y_B(x)\,dx$ for any $B'\in\mathcal{B}_0(\mathbb{R}^d)$. Next, define

$$b_\eta(\Delta)=c_1(d)\,b_X(\Delta/2)+c_2(d)\,E\big(D^d(M_0\oplus\check{K})\,\mathbb{I}(D(M_0\oplus\check{K})>\Delta/4)\big)$$

and put $b_Z^*(\Delta)=b_\eta(\Delta)$ and $b_Z^{**}(\Delta)=b_\eta(\Delta)/\Delta^{d-1}$. Hence, the proof of Theorem 5.1 is completed by noting that (5.5) and (5.6) imply

$$\begin{aligned}
\lim_{\Delta\to\infty}\Delta^d\, b_Z^{**}(\Delta)\ &\le\ c_1(d)\lim_{\Delta\to\infty}E\big(D(M_0\oplus\check{K})\,\mathbb{I}(D(M_0\oplus\check{K})>\Delta)\big)\lim_{\Delta\to\infty}\Delta^{d-1}\,b_X^{\delta/(2+\delta)}(\Delta/2)\\
&\quad+4^d\,c_2(d)\lim_{\Delta\to\infty}E\big(D^{2d}(M_0\oplus\check{K})\,\mathbb{I}(D(M_0\oplus\check{K})>\Delta/4)\big)\ =\ 0\,,
\end{aligned}$$

and that there exists some Δ_0 such that

$$\begin{aligned}
\sum_{\Delta\ge\Delta_0}\Delta^{d-1}\big(b_Z^*(\Delta)\big)^{\delta/(2+\delta)}\ &\le\ 2^{d-1}\,c_1^{\delta/(2+\delta)}(d)\sum_{\Delta\ge1}\Delta^{d-1}\big(b_X(\Delta)\big)^{\delta/(2+\delta)}\\
&\quad+4^{d-1}\,c_2^{\delta/(2+\delta)}(d)\sum_{\Delta\ge1}\Delta^{d-1}E\big(D^d(M_0\oplus\check{K})\,\mathbb{I}(D(M_0\oplus\check{K})>\Delta)\big)^{\delta/(2+\delta)}\\
&\le\ \widetilde{c}_1(d)+\widetilde{c}_2(d)E\big(D^{2d(\frac{\delta+1}{\delta})+\varepsilon}(M_0\oplus\check{K})\big)\sum_{\Delta\ge1}\Delta^{-(1+\varepsilon')}\ <\ \infty
\end{aligned}$$

with $\widetilde{c}_1(d)=2^{d-1}\,c_1^{\delta/(2+\delta)}(d)\sum_{\Delta\ge1}\Delta^{d-1}\big(b_X(\Delta)\big)^{\delta/(2+\delta)}<\infty$ and $\widetilde{c}_2(d)=4^{d-1}\,c_2^{\delta/(2+\delta)}(d)<\infty$.

For the Boolean model $\Xi=\bigcup_{i=1}^\infty(M_i+X_i)$ with $M_0\in\mathcal{R}\setminus\{\emptyset\}$ such that the minimal number of convex components of M_0 is bounded by some finite constant, the conditions of Theorem 5.1 are fulfilled if $E\,D^{2d(\frac{\delta+1}{\delta})+\varepsilon}(M_0)<\infty$ holds for some $\delta,\varepsilon>0$. Notice that this integrability condition is stronger than the assumption $E\,|M_0\oplus\check{K}|^2$ made in Theorems 4.1 and 4.2. Further examples of point processes X satisfying conditions (5.4) and (5.6) can be found in [7]. We also remark that in the special case $Y=Z_r$ considered in (3.3) with arbitrary X and $M_0\in\mathcal{K}'$, the integrability condition $E\,N^{(2+\delta)d}(\Xi\cap B_1(o))<\infty$ implies $E\,|Y(x)|^{2+\delta}<\infty$ for $\delta>0$, provided that there is almost surely no boundary point of Ξ where more than d germs overlap; see Section 4 of [11].

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