Selected topics in the theory of spatial stationary flat processes

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Preface

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Introduction

Consider a stationary k–flat process Φ_k^d in $\mathbb{R}^d$, i.e. Φ_k^d is a random point process on the phase space of all k–dimensional flats in d–dimensional space, each realization of which is an at most countable "locally finite" collection of k–planes (cf. section 1 of chapter I for exact definitions). Stationarity means stability of its distribution with respect to translations in $\mathbb{R}^d$. The probability distribution θ of the direction of a "typical" flat of Φ_k^d is called the directional distribution of Φ_k^d.

Intersections of all pairs of k-planes of Φ_k^d induce the new stationary $(2k-d)$–flat process whose intensity, i.e. the $(2k-d)$–volume content in a test window, is called the intersection density of Φ_k^d. A number of authors (R. Davidson (1974), J. Janson and O. Kallenberg (1981), J. Mecke and C. Thomas (1984, 1988), J. Keutel (1992)) dealt with the following variational problem concerning Φ_k^d:find all extremal directional distributions θ of Φ_k^d that maximize its intersection density.

In the case of hyperplanes ($k = d-1$) the solution is unique and corresponds to the Haar measure on the appropriate Grassmann manifold. For other particular dimensions k the whole class of maximal measures θ was described, but nevertheless some cases are still open there, e. g. when d is not divisible by $d-k$. It is worth mentioning that the form of these extremal measures as well as the methods of treating this problem for different dimensions k depend heavily on k.

The motivation for this research was to try to describe this extremal class of directional distributions and solve the problem completely. The idea was to understand the nature of these extremal measures deeper and find their common properties.

This common approach is developed in chapter II by means of the appropriate variational calculus (section 3). The necessary conditions of maximum for arbitrary dimensions d and k are given there in terms of the roses of intersections of Φ_k^d. To be more precise, suppose one intersects Φ_k^d with an r–flat η, $r = d-k+j$. Then $\Phi_k^d \cap \eta$ is a j-flat stationary process in η with intensity $f(\eta)$ which is called the rose of intersections of Φ_k^d. Theorem II.4.1 states that the directional distribution θ is extremal if and only if the rose of intersections of Φ_k^d with all k–flats is θ – almost everywhere constant. The fact that this simple criterion does not depend on k is at the same time an advantage and a shortcoming: it systematizes the results obtained before (cf. section 2 of chapter II), but its conditions are too weak to be sufficient, i.e. to yield the solution.

Thus the new mathematical setting was born: suppose we know the rose of intersections f of Φ_k^d with all r–flats exactly. Is this information sufficient to determine the distribution of Φ_k^d completely? If it is so, how can it be done? If Φ_k^d is Poisson (cf. [59], [82]) then it is completely determined by its intensity measure $\Lambda(\cdot)$, i.e. by its intensity λ and directional distribution θ (see equation (I.1.2)). For arbitrary stationary processes Φ_k^d this is evidently false, but nevertheless the knowledge of the intensity measure allows us to make some general conclusions about the behavior of the process.

Suppose the intensity λ is fixed and the rose of intersections f of Φ_k^d with r–flats is given. In chapter IV the following two questions are considered:

1. Does there exist a one–to–one correspondence between f and θ?
2. How can θ be restored from f (exact formulae)?

The complete answer to the first question was obtained by G. Matheron (1975), P. Goodey, R. Howard and M. Reeder (1990, 1996) (cf. [51], [14] – [16], respectively). It appears that uniqueness of retrieval holds only for particular k and r (see section 1 of chapter IV).

The partial answer to the second question (fiber processes in dimensions 2 and 3) could be found in the papers by J. Mecke and W. Nagel (1980, 1981) (cf. [52], [58] and others).

The main results of the present thesis (see sections 2 and 3 of chapter IV) yield the retrieval formulae for the directional distribution θ of any stationary process of hyperplanes in $\mathbb{R}^d$ from its rose of intersections when the intersecting plane η has dimension r, $1 \leqslant r \leqslant d-1$. These results are generalized to hold for stationary manifold processes in $\mathbb{R}^d$. The case $d = 4, k = r = 2$ is considered separately in section 5 of chapter IV. The whole class of directional distributions θ corresponding to the same rose of intersections f is described there. The proofs involve inversions of various integral transforms and expansions in spherical harmonics.

The required calculus of Radon and generalized cosine transforms on Grassmann manifolds is discussed in chapter III. The action of Radon transforms R_{ij} on the functions that are the positive powers of the volumes of certain parallelepipeds is studied in §3.1 and §3.2. It is shown that the Radon transform and its dual preserve the "structure" of these functions: they map them into the same powers of some other volumes up to a constant factor (see theorem III.3.1 and proposition III.3.1).

The important corollaries that yield integral relations between the generalized cosine transforms and Radon transforms (Cauchy–Kubota – type formulae) are given in §3.3. They are used later in chapter IV to invert the generalized cosine transforms. In detail, in §1.2 of chapter III the generalized cosine transforms T_{ij} are embedded in the family of operators T_{ij}^α where α is a positive parameter. The Cauchy–Kubota – type formulae for operators T_{ij}^α state that by integration of T_{ki}^α (applying R_{ij}) the new member of the same family T_{kj}^α appears (proposition III.3.2). The name "Cauchy–Kubota – type" is due to the fact that in case $\alpha = 1$ they can be seen for particular dimensions j and measures θ as a consequence of the well known Cauchy – Kubota formula (III.3.1) (see also Ch. 13, §1, 2 of [72], [73], p. 295 and [45], p. 126) applied to projection functions of zonoids.

Some interesting corollaries of the double fibration relation for T_{ij}^α are considered in §3.4. Upper bounds for the weighted images of Radon transforms are given in §3.5.

In chapter V we consider two related problems for the roses of intersections. Section 1 is devoted to their characteristic properties: there the answer to the question "is a given function the rose of intersections of some stationary hyperplane process with lines?" is found. It is known from [51] that this problem is deeply connected with the geometry of centrally symmetric convex bodies. Namely, a rose of intersections is at the same time the generalized cosine transform of its directional distribution measure θ, and for some particular dimensions k and r it could be regarded as a support function of zonoids or its kth projection function (cf. [22], [17], [89]). Due to this fact we can apply the characterization results pertinent to this class of convex bodies to get the answer to the question posed above.

In section 2 we generalize the notion of the rose of intersections, i.e. the intensity of the process $\Phi_k^d \cap \eta$ (where η is an arbitrary r–flat), $k + r \geqslant d$, to the so-called rose of neighborhood for the case $k + r < d$. Thus the inversion formulae of chapter IV that give the directional distribution of the process Φ_k^d from its rose of intersections can be easily applied to the roses of neighborhood.

The thesis is organized as follows: almost all of its chapters begin with preliminaries and an overview of the literature on the subject, continue with the description of the involved mathematical apparatus and the proofs of main results, and end with a section of remarks and open problems.

Chapter I in of introductory character: here the main spaces and structures are defined and the basic facts from stochastic geometry are mentioned. Chapter II gives the reader the motivation of the research (variational problems for Φ_k^d), the main results are proved in chapter IV and the principal tools for that are introduced and developed in chapter III. The final chapter V deals with two problems for the roses of intersections that stay apart from the main context. Then the bibliography and the list of notations are given.

Most of the results of this research have been published or accepted for publication: see [78] for the results of chapters II and V, [79], [81] for those of chapter IV, [80] for chapter III.

Chapter I

Stationary processes of k–flats in d–dimensional space

1 Basic facts

In this section we shall follow the guidelines of [60] in introducing the basic notions of k–flat processes (cf. [82], [59] for other constructions).

Let $F(k,d)$ be the set of all k–flats in $\mathbb{R}^d$, $d \geqslant 2$, $1 \leqslant k \leqslant d-1$. Let $G(k,d)$ be the *Grassmann manifold* of all non–oriented k–dimensional linear subspaces of $\mathbb{R}^d$ (for more detailed information see section 2). Say $B \subset F(k,d)$ is *bounded* if $\sup\limits_{\xi \in B} \rho(0,\xi) < \infty$ where $\rho(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R}^d$. Let $\mathfrak{G}$, $\mathfrak{F}$ be the σ–algebras of Borel subsets of $G(k,d)$, $F(k,d)$ in their usual topologies: topology in $F(k,d)$ is generated by the following class of open sets $\mathcal{A}_K = \{\xi \in F(k,d) : \xi \bigcap K \neq \emptyset\}$, K — any compact set in $\mathbb{R}^d$, and $\mathfrak{G} = \mathfrak{F} \cap G(k,d)$. One calls $\varphi \subset F(k,d)$ a *flat field* if any bounded set $B \subset \mathbb{R}^d$ is intersected by a finite number of k–flats of φ. Let $\mathcal{M}$ be the set of all flat fields and $\mathfrak{M}$ — the σ–algebra on $\mathcal{M}$ generated by all functions $z(B,\cdot) : \mathcal{M} \to \mathbb{N}$, $z(B,\varphi) = Card\,(\{\xi \in F(k,d) : \xi \in \varphi,\ \xi \cap B \neq \emptyset\})$ for any ball B in $\mathbb{R}^d$ where $Card(A)$ denotes the cardinal number of the set A.

Definition I.1.1. *Φ_k^d is called a k–flat process if Φ_k^d which maps the probability space into the measurable space $(\mathcal{M},\mathfrak{M})$ is a random element. Its distribution is a measure $\kappa(\cdot) = P\{\Phi_k^d \in \cdot\}$ on $\mathfrak{M}$. It is an ordinary point process for $k=0$ and a hyperplane process for $k=d-1$ in $\mathbb{R}^d$.*

A k–flat process Φ_k^d is called *stationary* if its distribution is invariant with respect to all translations in $\mathbb{R}^d$. Denote by $\nu_d(\cdot)$ the d–dimensional Lebesgue measure in $\mathbb{R}^d$. We shall call λ the *intensity* of the stationary process Φ_k^d if $\lambda = \frac{E\,\nu_k(\Phi_k^d \cap B)}{\nu_d(B)}$ for every bounded subset B of $\mathbb{R}^d$ with $\nu_d(B) > 0$. The definition of λ does not depend on the choice of B. Suppose $0 < \lambda < \infty$. The *rose of directions (directional distribution)* of Φ_k^d is a probability measure on $G(k,d)$:

$$\theta(\mathcal{C}) = \frac{E\,Card\left(\{\xi \in \Phi_k^d : \xi \cap \mathbf{S}^{d-1} \neq \emptyset,\ r(\xi) \in \mathcal{C}\}\right)}{\lambda k_{d-k}}, \quad \mathcal{C} \in \mathfrak{G} \qquad \text{(I.1.1)}$$

where $r(\xi)$ is the direction of the k–flat ξ, i.e. the unique $\bar{\xi} \in G(k,d)$ that is parallel to ξ, $k_d = \frac{2\pi^{\frac{d}{2}}}{d\,\Gamma(\frac{d}{2})}$ is the volume of the unit ball, and $\mathbf{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$.

Let $\Phi_k^d(\mathcal{B})$ denote the number of k–flats of Φ_k^d that belong to a set $\mathcal{B} \in \mathfrak{F}$ (it can also get infinite values if $\mathcal{B}$ is not bounded). The measure

$$\Lambda(\mathcal{B}) = E\left(\Phi_k^d(\mathcal{B})\right), \quad \mathcal{B} \in \mathfrak{F}$$

is called *the intensity measure* of Φ_k^d.

If Φ_k^d is stationary the following factorization of its intensity measure takes place (cf. e.g. [82], [59]):

$$\Lambda(\mathcal{B}) = \lambda \int\limits_{G(k,d)} \int\limits_{\xi^\perp} I_{\mathcal{B}}(y+\xi)\nu_{d-k}^{\xi^\perp}(dy)\theta(d\xi), \quad \mathcal{B} \in \mathfrak{F} \qquad \text{(I.1.2)}$$

where $\nu_{d-k}^{\xi^\perp}(\cdot)$ is the Lebesgue measure on the orthogonal linear subspace $\xi^\perp$ and $I_{\mathcal{B}}(\cdot)$ is the indicator function of the set $\mathcal{B}$.

Introduce the following notation:

$< a_1, \ldots, a_k >$	the k–flat spanned by vectors $a_1, \ldots, a_k$;
$Vol(a_1, \ldots, a_k)$	the non–oriented k–dimensional volume of the parallelepiped spanned by vectors $a_1, \ldots, a_k$;
$\binom{d}{i}$	$= \frac{d!}{(d-i)!\,i!}$;
A^t	the transpose of the matrix A;
$\mathbb{Z}_+$	the set of all non–negative integer numbers.

For almost all $\eta \in F(d-k+j,d)$, $j \leqslant k-1$, the j–flat process

$$\Phi_k^d \cap \eta = \{\xi \cap \eta : \xi \in \Phi_k^d\}$$

is again stationary on η. Let $\lambda_{\Phi_k^d \cap \eta}$ be the intensity of $\Phi_k^d \cap \eta$. Due to the stationarity of Φ_k^d, it is sufficient to consider only those affine flats η that contain the origin, i.e., $\eta \in G(d-k+j,d)$. Then

$$\left(T_{k,d-k+j}\theta\right)(\eta) \stackrel{def}{=} \lambda_{\Phi_k^d \cap \eta} = \lambda \int\limits_{G(k,d)} [\xi,\eta]\,\theta(d\xi) \qquad \text{(I.1.3)}$$

where $[\xi,\eta]$ is the $(d-j)$ –volume of the unit parallelepiped spanned by orthonormal bases in $\xi^\perp$ and $\eta^\perp$ (cf. [51], [14]): if $\xi^\perp = < a_1,\dots,a_{d-k} >$, $\eta^\perp = < b_1,\dots,b_{k-j} >$ then

$$[\xi,\eta] = Vol(a_1,\dots,a_{d-k},b_1,\dots,b_{k-j}).$$

It is independent of the choice of the orthonormal bases in $\xi^\perp$ and $\eta^\perp$. The function $(T_{k,d-k+j}\theta)(\eta)$, $\eta \in G(d-k+j,d)$ is called *the rose of intersections of* Φ_k^d. We use the above notation to emphasize that for fixed λ the rose of intersections of Φ_k^d is the integral transform of its directional distribution θ. The subscripts denote the dimensions of Φ_k^d and of the intersecting plane η.

For integers k and d with $2k \geqslant d$ introduce the new process

$$X_2(\Phi_k^d) = \{\xi_1 \cap \xi_2 : \xi_1,\xi_2 \in \Phi_k^d, \quad \xi_1 \neq \xi_2, \quad \xi_1 \cap \xi_2 \neq \emptyset\}.$$

It is generated by the intersections of all pairs of k–flats of the original stationary process Φ_k^d. Condition $2k \geqslant d$ guarantees that any two k-flats in general position have at least one common point. This ensures that $X_2(\Phi_k^d)$ is not empty; moreover, it is a $(2k-d)$–flat stationary process in $\mathbb{R}^d$ that could be easily seen from relation

$$dim(\xi_1 \cap \xi_2) \geqslant dim(\xi_1) + dim(\xi_2) - d. \tag{I.1.4}$$

This process is sometimes called the *intersection process of* Φ_k^d *of order* 2. Its intensity $\lambda_{X_2(\Phi_k^d)}$ is known as the *intersection density of* Φ_k^d *of order* 2 (see [82], p. 253-255). One can prove by means of the Campbell – Mecke theorem that

$$\lambda_{X_2(\Phi_k^d)} = \frac{\lambda^2}{2} \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi_1,\xi_2]\, \theta(d\xi_1)\theta(d\xi_2) \tag{I.1.5}$$

where $[\xi_1,\xi_2]$ is the $2(d-k)$–volume of the unit parallelepiped spanned by the bases in $\xi_1^\perp$, $\xi_2^\perp$ (see the proof in [7] for the case of hyperplanes). Later on we make use of the notation $\mathcal{C}(\lambda,\theta)$ for $\lambda_{X_2(\Phi_k^d)}$ to stress the fact that the intersection density (I.1.5) is a non–linear functional of λ and θ.

2 Grassmannians

Assume $d \geqslant 3$, $1 \leqslant k \leqslant d-1$. Let us investigate the structure of manifolds $G(k,d)$ on which most of the measures and functions in our considerations are defined.

For any d and $k < d$ the Grassmannian $G(k,d)$, that is, the set of all *non–oriented* k–dimensional flats in $\mathbb{R}^d$ containing the origin, is a compact analytic manifold of dimension $k(d-k)$ (cf. e. g. [46], [47]). Moreover, it is a separable symmetric space. The following isomorphic representation is known:

$$G(k,d) \cong O(d)/O(k) \times O(d-k)$$

where $O(d)$ is the orthogonal group in $\mathbb{R}^d$ (cf. [32], chapter I, §6). If one considers the Grassmann manifold $\mathcal{L}(k,d)$ of oriented k–flats then

$$\mathcal{L}(k,d) \cong SO(d)/SO(k) \times SO(d-k) \tag{I.2.1}$$

where $SO(d)$ is the special orthogonal group in $\mathbb{R}^d$ (cf. [85]). For other isomorphisms of $\mathcal{L}(k,d)$ see [47], [34] p. 43-46.

The structure of $G(k,d)$, $\mathcal{L}(k,d)$ as a quotient space allows the introduction of the unique left and right invariant normalized Haar measure on $G(k,d)$, $\mathcal{L}(k,d)$ (cf. [32], chapter I, §1). It plays the role of the uniform distribution on the space of the directions of flats. Hence it is desirable to find its explicit form. It will be done in the cases $k=1$, $d-1$ for arbitrary $d \geqslant 2$ as well as $d=4$, $k=2$ (that are of particular interest to us) in chapter I, §2.1 and chapter IV, section 5.

2.1 Hyperplanes and lines

It is clear that the Grassmannian of all non–oriented lines through the origin $G(1,d)$ is isomorphic to $\mathbf{S}_+^{d-1} \stackrel{def}{=} \{u \in \mathbf{S}^{d-1} : u \equiv -u\}$ where $\equiv$ denotes the relation of identification and $\mathbf{S}_+^{d-1}$ is a sphere in $\mathbb{R}^d$ with each diameter having its end points "glued together" ($\mathbf{S}_+^{d-1}$ is obviously topologically equivalent to projective space $\mathbb{RP}^{d-1}$). Mapping any hyperplane ξ to its orthogonal complement $\xi^\perp \in G(1,d)$ we get that for the Grassmann manifold of hyperplanes the same representation is valid: $G(d-1,d) \cong \mathbf{S}_+^{d-1}$.

In the cases of hyperplanes and lines the Haar measure on the appropriate Grassmann manifolds normalized by unity is equal to the normalized surface area measure $\omega_d(\cdot)/\omega_d$ on $\mathbf{S}^{d-1}$ ($\omega_d = \omega_d(\mathbf{S}^{d-1}) = d\,\kappa_d$).

In view of that any function or measure on $G(1,d)$ or $G(d-1,d)$ can be represented as an even function or measure on $\mathbf{S}^{d-1}$. We shall use this obvious fact later on without any further references.

2.2 Grassmannian $G(2,4)$ and its isomorphisms

Consider the set of all 2-flats in $\mathbb{R}^4$ through the origin. Our aim is to give the exact proof to the following result stated by J. Mecke in [55]:

Theorem I.2.1. *The Grassmannian $\mathcal{L}(2,4)$ is homeomorphic to $\mathbf{S}^2 \times \mathbf{S}^2$. The Grassmannian $G(2,4)$ is homeomorphic to*

$$\{(u,v) \in \mathbf{S}^2 \times \mathbf{S}^2 : (u,v) \equiv (-u,-v)\} \tag{I.2.2}$$

where $\equiv$ denotes the relation of identification.

The proof will be given at the end of this paragraph. We shall use in it the special type of Grassmann coordinates introduced in [55], although other approaches are also possible (cf. [11]).

Due to theorem I.2.1 any function (measure) on $G(2,4)$ corresponds to an "even" function (measure) on $\mathbf{S}^2 \times \mathbf{S}^2$. Their "evenness" should be understood in the following way: they do not change under the mapping $(u,v) \mapsto (-u,-v)$, $(u,v) \in \mathbf{S}^2 \times \mathbf{S}^2$.

The structure of the Haar measure on $G(2,4)$ is also known (see [14], although the proof is not given there): it is proportional to the product measure $\omega_3(du)\omega_3(dv)$ on $\mathbf{S}^2 \times \mathbf{S}^2$ (cf. theorem IV.5.1). We shall provide two different proofs for it in section 5 of chapter IV.

Grassmann coordinates

Definition I.2.1. *The Grassmann coordinates of the first order of an oriented k–flat in $\mathcal{L}(k,d)$ are all minors of order k of the $k \times d$ — matrix composed of the coordinates belonging to an orthonormal basis of this k–flat in $\mathbb{R}^d$ (see [66] for details).*

In case of $\mathcal{L}(2,4)$ these are 12 coordinates, 6 of which are independent. Let a 2–flat $\xi \in \mathcal{L}(2,4)$ be spanned over the orthonormal basis vectors $a = (a_1, \ldots, a_4)$, $b = (b_1, \ldots, b_4)$. Then

$$q_{kl} = \begin{vmatrix} a_k & a_l \\ b_k & b_l \end{vmatrix}.$$

Clearly $q_{kl} = -q_{lk}$, $q_{kk} = 0$ for all k and l. There exist the following relations between the Grassmann coordinates (cf. [66]):

$$q_{kl}q_{mn} = q_{ml}q_{kn} + q_{nl}q_{mk} \qquad \text{(I.2.3)}$$

for all k, l, m, n. It means that they built a quadric in some Euclidean space which is identified with the Grassmann manifold itself.

The six independent coordinates are q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}. They do not depend on the choice of the orthonormal basis a, b. One can show it also directly. Introduce as in the paper [55] the following functions

$$\begin{array}{lllll} u_1 &=& q_{12} - q_{34} &=& a_1b_2 - a_2b_1 - a_3b_4 + a_4b_3 \\ u_2 &=& q_{13} + q_{24} &=& a_1b_3 - a_3b_1 + a_2b_4 - a_4b_2 \\ u_3 &=& q_{14} - q_{23} &=& a_1b_4 - a_4b_1 - a_2b_3 + a_3b_2 \\ v_1 &=& q_{12} + q_{34} &=& a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3 \\ v_2 &=& q_{13} - q_{24} &=& a_1b_3 - a_3b_1 - a_2b_4 + a_4b_2 \\ v_3 &=& q_{14} + q_{23} &=& a_1b_4 - a_4b_1 + a_2b_3 - a_3b_2. \end{array} \qquad \text{(I.2.4)}$$

Evidently, $-1 \leqslant u_i, v_i \leqslant 1$. Rewrite now formulae (I.2.4) in the matrix form:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_{12} \\ q_{13} \\ q_{14} \\ q_{23} \\ q_{24} \\ q_{34} \end{pmatrix},$$

or briefly,

$$\begin{pmatrix} u \\ v \end{pmatrix} = M(\vec{q}). \tag{I.2.5}$$

Proof of theorem I.2.1

We established by means of (I.2.5) the mapping of $\mathcal{L}(2,4)$ to $\mathbf{S}^2 \times \mathbf{S}^2$. Indeed, the representation u, v does not depend on the choice of the basis in ξ, because the same holds for the Grassmann coordinates. Then one should prove that $|u| = |v| = 1$ where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$: one can easily check it simply by writing $u_1^2 + u_2^2 + u_3^2$ in terms of Grassmann coordinates and then applying relations (I.2.3). It is also the mapping onto $\mathbf{S}^2 \times \mathbf{S}^2$, it is bijective and continuous: because of its linearity and since matrix M is non–degenerate one can construct the operator M^{-1} and invert this change of coordinates. The rest of the proof for the first case follows from the fact that the Grassmann coordinates of an oriented flat define it uniquely.

If we proceed now to consider the non–oriented flats, one should first mention that if we change the orientation of a flat then due to the definition of Grassmann coordinates they change the sign. It acts on the vectors u, v in the same way: they are reflected to $-u$, $-v$. Consequently, if we would like to work with non–oriented flats, we should glue together (u,v) and $(-u,-v)$. The proof is complete.

3 Representations of the integral kernel $[\xi,\eta]$

According to (I.1.3) and (I.1.5), the rose of intersections of Φ_k^d and its intersection density are integral operators of the directional distribution θ of Φ_k^d with the integral kernel $[\xi,\eta]$. The matter of this section is to find nice representations of it that will be used in chapters III and IV. First let us consider the most simple situations of lines, hyperplanes and 2–flats in dimension 4.

3.1 Particular cases

If ξ is a line from $G(1,d)$ and η is a hyperplane from $G(d-1,d)$ then the following observation is obvious: $[\xi,\eta] = |<u,v>|$ where $<\cdot,\cdot>$ stands for the usual Euclidean scalar product in $\mathbb{R}^d$ and unit vectors $\pm u$, $\pm v$ are direction vectors of lines ξ and $\eta^{\perp}$, respectively. Here $|<u, v>|$ is the cosine of the minimal angle between $\pm u$ and $\pm v$ or, equivalently, the sine of the angle between $\xi^{\perp}$ and $\eta^{\perp}$.

Analogously, if $\xi, \eta \in G(d-1, d)$ are two hyperplanes with normal vectors $\pm u$, $\pm v \in \mathbf{S}^{d-1}$ then $[\xi, \eta]$ is equal to the sine of the angle between $\pm u$ and $\pm v$: $[\xi, \eta] = \sqrt{1- <u, v>^2}$.

In fact, a more general statement holds that generalizes the above trivial cases. It will be proved in proposition I.3.1.

If ξ and η are now 2–flats from $G(2, 4)$ with coordinates (u, v) and $(\tilde{u}, \tilde{v})$ from $\mathbf{S}^2 \times \mathbf{S}^2$ (see theorem I.2.1) then the following result can be found in [55]:

$$[\xi, \eta] = \frac{1}{2} \,|<u, \tilde{u}> - <v, \tilde{v}>| \tag{I.3.1}$$

(see also [14], p. 98).

3.2 Critical angles

For integers k, r: $r \leqslant k$, $k \geqslant d/2$, $k + r \geqslant d$ consider subspaces $\xi \in G(k, d)$ and $\eta \in G(r, d)$. Let their orthogonal complements $\xi^{\perp}$ and $\eta^{\perp}$ be spanned by the orthonormal bases $a_1, \dots, a_{d-k}$ and $b_1, \dots, b_{d-r}$, respectively. If A (B) is the $(d-k) \times d$ – matrix ($(d-r) \times d$ – matrix) with the coordinates of a_i (b_i) as lines then ξ and η are defined by the following systems of equations: $x \in \xi \Leftrightarrow Ax = 0$, $x \in \eta \Leftrightarrow Bx = 0$ where x is a column vector. By [43], p. 22–25, there exist unit vectors $l_1, \dots, l_{d-k} \in \xi$ and $t_1, \dots, t_{d-k} \in \eta$ such that $<l_m, l_n>=<t_m, t_n>= 0$ for $m \neq n$ and the angles $\beta_n \in [0, \pi/2]$ between l_n and t_n are stationary among all possible angles between the vectors in ξ and η in the following sense: the values $\cos \beta_n$, $n = 1, \dots, d-k$ are critical for the minimization problem

$$<l, t> \to \min$$

with constraints

$$\begin{cases} Al = 0, \\ Bt = 0, \\ |l| = |t| = 1. \end{cases}$$

These β_n, $n = 1, \dots, d-k$ are called the *critical angles* of ξ and η. The volume $[\xi, \eta]$ as an invariant of the pair of subspaces ξ, η with respect to all rotations must have a representation in terms of β_n. We shall formulate this in proposition I.3.1. But before that we need to prove the following lemma which is the stronger version of the theorem 5.1 of [39]:

Lemma I.3.1. *There exist orthonormal bases $a_1, \dots, a_{d-k}$ and $b_1, \dots, b_{d-r}$ in $\xi^{\perp}$ and $\eta^{\perp}$, respectively such that*

$$<a_i, b_j>= \delta_{ij} \cos \beta_i \tag{I.3.2}$$

for all $i = 1, \ldots, d-k$ *and* $j = 1, \ldots, d-r$ *where* δ_{ij} *stands for the Kronecker delta.*

Proof. Let $\tilde{a}_1, \ldots, \tilde{a}_{d-k}$ and $\tilde{b}_1, \ldots, \tilde{b}_{d-r}$ be some orthonormal bases in $\xi^\perp$ and $\eta^\perp$, respectively. Then the systems of linear equations giving ξ and η are $\tilde{A}x = 0$ and $\tilde{B}x = 0$ where $\tilde{A}$ ($\tilde{B}$) is the appropriate $(d-k) \times d$ – matrix ($(d-r) \times d$ – matrix) of the coordinates of $\tilde{a}_i$ ($\tilde{b}_i$). By formula (72) of [43] the squares of the cosines of the critical angles $\cos^2 \beta_n$, $n = 1, \ldots, d-k$ are the latent roots of the symmetrical $(d-k) \times (d-k)$ – matrix $J = \tilde{A}\tilde{B}^t\tilde{B}\tilde{A}^t$. It is well–known that any symmetrical matrix has the canonical diagonal form; hence there exists an orthogonal $(d-k) \times (d-k)$ – matrix T such that

$$J = T \cdot diag(\cos^2 \beta_1, \ldots, \cos^2 \beta_{d-k}) \cdot T^t = \tilde{A}\tilde{B}^t\tilde{B}\tilde{A}^t$$

where $diag(d_1, \ldots, d_m)$ denotes the diagonal $m \times m$ – matrix with diagonal elements $d_1, \ldots, d_m$. Multiplying this equality by T^t on the left and T on the right one gets

$$diag(\cos^2 \beta_1, \ldots, \cos^2 \beta_{d-k}) = T^t\tilde{A}\tilde{B}^t\tilde{B}\tilde{A}^tT = C^tC \qquad \text{(I.3.3)}$$

where $C = \tilde{B}\tilde{A}^tT$ is the $(d-r) \times (d-k)$ – matrix. Due to (I.3.3) its columns $c_1, \ldots, c_{d-k}$ can be regarded as orthogonal vectors in $\mathbb{R}^{d-r}$ such that $< c_i, c_j >= \delta_{ij} \cos \beta_i$, $i, j = 1, \ldots, d-k$. Then there exists a rotation $D \in SO(d-r)$ such that

$$C = DE_\beta \qquad \text{(I.3.4)}$$

where E_β is the $(d-r) \times (d-k)$ – matrix of the form

$$\begin{pmatrix} \cos \beta_1 & 0 & 0 & \ldots & 0 \\ 0 & \cos \beta_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & \cos \beta_{d-k} \\ 0 & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}.$$

We get from (I.3.4) that

$$E_\beta = D^tC = D^t\tilde{B}\tilde{A}^tT = D^t\tilde{B}\left(T^t\tilde{A}\right)^t = BA^t$$

where $B = D^t\tilde{B}$, $A = T^t\tilde{A}$, and $D \in SO(d-r)$, $T \in SO(d-k)$. Let $a_1, \ldots, a_{d-k}$ and $b_1, \ldots, b_{d-r}$ be the vectors with lines of A and B as coordinates, respectively. By construction these vectors form the orthonormal bases in $\xi^\perp$ and $\eta^\perp$ with the required property (I.3.2) (cf. the equality $E_\beta = BA^t$). □

Proposition I.3.1. *For any two subspaces $\xi \in G(k,d)$, $\eta \in G(r,d)$ with dimensions k and r such that $r \leqslant k$, $k \geqslant d/2$, $k+r \geqslant d$ and critical angles $\beta_1, \dots, \beta_{d-k}$ the following identity is valid:*

$$[\xi, \eta] = \sin\beta_1 \dots \sin\beta_{d-k} = \sqrt{(1-y_1)\dots(1-y_{d-k})} \tag{I.3.5}$$

where $y_n = \cos^2\beta_n$, $n = 1, \dots, d-k$.

Proof. By lemma I.3.1 there exist orthonormal bases $a_1, \dots, a_{d-k}$ in $\xi^\perp$ and $b_1, \dots, b_{d-r}$ in $\eta^\perp$ such that $< a_n, b_m >= \delta_{nm}\cos\beta_n$ for all n and m. By definition $[\xi, \eta]$ is equal to $Vol(a_1, \dots, a_{d-k}, b_1, \dots, b_{d-r})$. Let us prove the statement of the proposition for fixed d and any r such that $r \leqslant k$, $k+r \geqslant d$ by induction on k. The base of induction ($k = d-1$) follows immediately from the definition of the volume of the parallelepiped. Let now the conjecture be true for $k = j$. One has to show that it holds also for $k = j-1$.

Let $Pr_\zeta(\cdot)$ denote the operator of orthogonal projection onto the plane ζ. We have

$$Vol(a_1, \dots, a_{d-j+1}, b_1, \dots, b_{d-r}) = |Pr_{\zeta^\perp}(a_{d-j+1})| \times \\ \times Vol(a_1, \dots, a_{d-j}, b_1, \dots, b_{d-r})$$

by the properties of the volume of the parallelepiped where

$$\zeta =< a_1, \dots, a_{d-j}, b_1, \dots, b_{d-r} > .$$

By Pythagorean theorem

$$|Pr_{\zeta^\perp}(a_{d-j+1})| = \sqrt{1 - |Pr_\zeta(a_{d-j+1})|^2}.$$

Let us prove that $|Pr_\zeta(a_{d-j+1})| = \cos\beta_{d-j+1}$. Orthogonalize the basis $a_1, \dots, a_{d-j}, b_1, \dots, b_{d-r}$ in ζ: we get the orthonormal system of vectors

$$a_1, \dots, a_{d-j}, b_{d-j+1}, \dots, b_{d-r}, \frac{b_1 - \cos\beta_1 a_1}{|b_1 - \cos\beta_1 a_1|}, \dots, \frac{b_{d-j} - \cos\beta_{d-j} a_{d-j}}{|b_{d-j} - \cos\beta_{d-j} a_{d-j}|}$$

by the properties of a_i and b_i. Then by definition of the orthogonal projection we get

$$Pr_\zeta(a_{d-j+1}) = \sum_{i=1}^{d-j} < a_i, a_{d-j+1} > a_i +$$

$$\sum_{i=1}^{d-j} \frac{< b_i - \cos\beta_i a_i, a_{d-j+1} >}{|b_i - \cos\beta_i a_i|^2}(b_i - \cos\beta_i a_i) +$$

$$\sum_{i=d-j+1}^{d-r} < b_i, a_{d-j+1} > b_i =< b_{d-j+1}, a_{d-j+1} > b_{d-j+1} = \cos\beta_{d-j+1} b_{d-j+1}.$$

Thus $|Pr_{\zeta^\perp}(a_{d-j+1})| = \sqrt{1-\cos^2\beta_{d-j+1}} = \sin\beta_{d-j+1}$, and the application of the induction step to $Vol(a_1, \dots, a_{d-j}, b_1, \dots, b_{d-r})$ completes the proof. □

3.3 Mixed volumes

Let us mention here one more interesting relation for $[\xi, \eta]$ to be found in [18]:

$$[\xi, \eta] = \frac{\binom{d}{r}}{k_r\, k_{d-r}} V(\underbrace{B_\xi, \dots, B_\xi}_{d-r}, \underbrace{B_\eta, \dots, B_\eta}_{r}) \tag{I.3.6}$$

for $\xi \in G(k,d)$, $\eta \in G(r,d)$, $k + r \geqslant d$ where B_ζ is the i–dimensional unit ball in $\zeta \in G(i,d)$ and $V(\cdot, \dots, \cdot)$ is the *mixed volume* of its arguments (cf. [73]). See remark II.5.3 for further discussion.

Chapter II

Variational problems for stationary flat processes

1 Intersection density and isoperimetric problems

One of the so–called *isoperimetric problems* that could be stated for the intersection processes of order 2 is to maximize their intersection density, i.e. for given λ find the set $\mathbf{L_0}$ of such extremal directional distributions θ_0 that $\lambda_{X_2(\Phi^d_k)}$ attains its maximal value c_{max}:

$$\mathcal{C}(\lambda,\theta)=\frac{\lambda^2}{2}\int\limits_{G(k,d)}\int\limits_{G(k,d)}[\xi_1,\xi_2]\,\theta(d\xi_1)\theta(d\xi_2)\longrightarrow\max. \qquad \text{(II.1.1)}$$

The maximum of the functional $\mathcal{C}$ is attained on the set $\mathbf{L}$ of all probability measures on $G(k,d)$ because this set is compact in the topology of weak convergence (cf. remark 3.3 of [62], it follows also from Prokhorov's theorem in [3], since all probability measures on the compact space are dense) and the functional is continuous on it: since $\mathcal{C}$ is Fréchet differentiable (cf. section 4), it is also continuous in the topology of convergence in total variation. But this convergence implies the weak convergence for the measures on the compact space $G(k,d)$. Thus the functional $\mathcal{C}$ is also continuous on the compact $\mathbf{L}$ in the sense of weak convergence, and hence it attains its extremal values.

The exact value of c_{max} is also of interest to us:

$$c_{max}=\max_{\theta\in\mathbf{L}}\mathcal{C}(\lambda,\theta). \qquad \text{(II.1.2)}$$

In section 4, $\mathcal{C}(\lambda,\theta)$ will be extended to a non–linear functional on the Banach space $\tilde{\mathbf{M}}\,(G(k,d))$ of all signed measures with finite total variation on $G(k,d)$. Then we shall use variational methods of section 3 to describe the extremal class $\mathbf{L_0}$. The appropriate necessary conditions of extremum will be found. It will be clear that the rose of intersections plays an important role in the given characterization. The obtained conditions are very weak and do not lead to the general solution of the variational problem (II.1.1) (see also the discussion in section 5). Nevertheless, they systemize the earlier results listed in section 2. Thus the common properties of the known solutions of (II.1.1) can be understood better.

Problem (II.1.1) owes its name "isoperimetric" to the deep connection between the above setting and classical isoperimetric problems for centrally symmetric convex bodies. To outline them we need some facts from convex geometry.

The *support function* of a convex body $K \subset \mathbb{R}^d$ is by definition

$$h_K(x) = \sup_{u \in K} <u, x>, \quad x \in \mathbb{R}^d.$$

The following property of zonoids that are limits of zonotopes in the Hausdorff metric can be taken as definition (cf. [22], [73], [89]):

Definition II.1.1. *A centrally symmetric convex body K with the center at $0 \in \mathbb{R}^d$ is called a (generalized) zonoid if its support function h_K has the representation*

$$h_K(x) = \int\limits_{\mathbf{S}^{d-1}} |<x, u>| \, \theta_K(du), \quad x \in \mathbb{R}^d$$

for some finite even (signed) measure θ_K on $\mathbf{S}^{d-1}$ that is called generating for K.

For some integer $k \geqslant 1$ introduce the so–called *projection generating measure* $\rho_k(K, \cdot)$ of the zonoid K with generating measure θ_K: it is the measure on the Borel subsets $\mathcal{A}$ of $G(k,d)$ defined by the relation

$$\rho_k(K, \mathcal{A}) = \int\limits_{\mathbf{S}^{d-1}} \cdots \int\limits_{\mathbf{S}^{d-1}} Vol(u_1, \ldots, u_k) I_{\{<u_1, \ldots, u_k> \in \mathcal{A}\}}(u_1, \ldots, u_k) \, \theta_K(du_1) \ldots \theta_K(du_k)$$

(cf. [22]). If $d = 2k$, $k \geqslant 1$ one has for a zonoid K the following relations between its mixed volume $V(K, \ldots, K)$, *mixed functional* $\Phi^{(0)}_{k,k}(K, K)$ and integral representation (II.1.1) (cf. [21]):

$$\binom{d}{k} \nu_d(K) = \binom{d}{k} V(K, \ldots, K) = \Phi^{(0)}_{k,k}(K, K) = \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi_1, \xi_2] \, \rho_k(K, d\xi_1) \rho_k(K, d\xi_2).$$

We preserve here the notation of [21] for the mixed functional, which, as we hope, will not confuse the reader by its closeness to our notation Φ_k^d of the k–flat processes. If we allow measures θ in (II.1.1) to be chosen from the smaller class of projection generating measures of zonoids in $\mathbb{R}^d$ for $d = 2k$ then problem (II.1.1) rewrites (up to a constant factor) as follows: find zonoids K of maximal volume $\nu_d(K)$ provided that the total mass of their projection generating measure is λ, i.e. $\rho_k\,(K, G(k,d)) = \lambda$.

If $k = 1$ then $\rho_k(K, \cdot\,)$ is obviously equal to $\theta_K(\cdot)$. Recalling the fact that in two–dimensional space the class of zonoids coincides with the class of all centrally symmetric convex bodies, one gets that problem (II.1.1) in $\mathbb{R}^2$ has the following isoperimetric meaning without any further constraints on probability measure θ: find a centrally symmetric convex body $K \subset \mathbb{R}^2$ with maximal volume $\nu_2(K)$ provided that its generating measure θ_K has total mass λ. It is not difficult to show that the perimeter $p(K)$ of the zonoid $K \subset \mathbb{R}^2$ is equal to $4\,\theta_K(\mathbf{S}^1)$: by [27] $p(K) = \pi \bar{w}(K)$ where $\bar{w}(L)$ is the *mean width functional* of the convex body $L \subset \mathbb{R}^d$:

$$\bar{w}(L) \stackrel{def}{=} \frac{1}{\omega_d} \int\limits_{\mathbf{S}^{d-1}} (h_L(u) + h_L(-u))\, \omega_d(du).$$

Then by [59], p. 108 (cf. also [84], remark 1) $\bar{w}(K) = \frac{4}{\pi}\,\theta_K(\mathbf{S}^1)$. Hence $p(K) = 4\lambda$. The zonoid K with generating measure proportional to the rose of directions of a stationary line process Φ_1^2 is called the *Steiner compact* of Φ_1^2 (cf. [51]). Thus setting (II.1.1) is a classical isoperimetric problem for Steiner compacts of Φ_1^2, see also [84] for generalizations of these ideas to hyperplane processes Φ_{d-1}^d in arbitrary dimensions d.

The case $d = 2k$, $k > 1$ will be considered more detailed in section 5 of this chapter.

2 Some historical notes

The following is an outline of the results known for the problem (II.1.1) – (II.1.2). The involved mathematical apparatus as well as the solutions themselves depend to a great extent on dimensions d and k:

- $d \geqslant 2$, $k = d - 1$: $\mathbf{L_0} = \{\omega_d(\cdot)/\omega_d\}$, $c_{max} = \frac{\lambda^2\Gamma^2(\frac{d}{2})}{2\Gamma(\frac{d+1}{2})\Gamma(\frac{d-1}{2})}$ where $\omega_d(\cdot)/\omega_d$ is the uniform distribution of directions on $\mathbf{S}^{d-1}$ coinciding with the Haar probability measure on $G(d-1,d)$. The case of a line process on the plane ($d = 2$, $k = 1$) was considered in the pioneering paper of R. Davidson (1974) [4]. J. Janson and O. Kallenberg (1981) [40] investigated the general case using spherical harmonics, while C. Thomas (1984) [84] employed the isoperimetric inequalities for Minkowski functionals.

- $k < d-1$: J. Mecke and C. Thomas (1986) [60] proved that the Haar measure is not extremal. Further developments can be found in [54], [57] and [74].
  - $d = 2k$, $k > 2$: J. Mecke (1988) [56] showed that $c_{max} = \frac{\lambda^2}{4}$ and

$$\mathbf{L_0} = \left\{ \theta_0(\cdot) = \frac{1}{2}\big(\delta_\xi(\cdot) + \delta_{\xi^\perp}(\cdot)\big) \,:\, \xi \in G(k,d) \right\}$$

    where $\delta_\xi(\cdot)$ is the Dirac measure concentrated in ξ.
  - $d = 4$, $k = 2$: the value c_{max} is the same as above the but the class $\mathbf{L_0}$ is essentially larger as in the previous case (cf. J. Mecke (1988) [55]).
  - $d-k \mid d$, i.e. $d-k$ divides d, $k < d-2$: J. Keutel (1992) [44] proved that $c_{max} = \frac{\lambda^2}{2}\frac{k}{d}$ and the class $\mathbf{L_0}$ consists of measures

$$\theta_0(\cdot) = \frac{d-k}{d}\big(\delta_{\xi_1}(\cdot) + \ldots + \delta_{\xi_{\frac{d}{d-k}}}(\cdot)\big)$$

    for $\xi_i \in G(k,d)$, $\xi_i^\perp \perp \xi_j^\perp$, $i \neq j$.
  - $d-k \mid d$, $k = d-2$: c_{max} remains the same as in the previous case, the class $\mathbf{L_0}$ is larger but not yet completely known (cf. [44]).
  - $d-k$ does not divide d, $k < d-1$: the problem is still open. In [44] some bounds for c_{max} are given.

The necessary conditions of maximum (cf. section 4) bring unification in this variety of results whose form depends heavily on the dimension k: it appears that for any directional distribution θ_0 from $\mathbf{L_0}$ its rose of intersections with k–flats $T_{kk}\theta_0$ should be θ_0–almost surely constant.

3 Variational calculus on the space of signed measures

In what follows we make use of the papers [64], [62], [61] in order to state results that will be instrumental for us in obtaining the necessary conditions of extremum in section 4.

Let X be a measurable Polish locally compact space, and let $\tilde{\mathbf{M}}(X)$ be the Banach space of all signed measures on X with finite total variation norm: for $\mu \in \tilde{\mathbf{M}}(X)$

$$\|\mu\| = \sup_{|\phi(\omega)|\leqslant 1} \left| \int_X \phi(\omega)\mu(d\omega) \right| < \infty$$

where the supremum is taken over all measurable functions $\phi : X \to \mathbb{R}$ with the absolute value bounded from above by unity. Let $\mathbf{M}(X) \subset \tilde{\mathbf{M}}(X)$ be the cone of all non–negative finite measures equipped with convergence in total variation. Let functionals $F : \tilde{\mathbf{M}}(X) \to \mathbb{R}$, $H : \tilde{\mathbf{M}}(X) \to \mathbb{R}$ be continuous and Fréchet differentiable (cf. [48]) on a closed convex subset A of $\mathbf{M}(X)$. We shall tackle the following optimization problem with equality constraints:

$$\begin{cases} F(\mu) \longrightarrow \inf, \\ \mu \in A, \\ H(\mu) = 0 \end{cases} \tag{II.3.1}$$

where the first Fréchet derivative of the constraint function H at μ does not depend on μ and has the form

$$H'(\mu)[\eta] = \int\limits_X h\, d\eta \tag{II.3.2}$$

for some measurable function $h : X \to \mathbb{R}$.

Now let us cite the necessary conditions of infimum in the problem (II.3.1) with one equality constraint $H(\mu) = 0$ (cf. theorem 3.5 and remark 3.3 [64]):

Theorem II.3.1. *Let functionals F and H be twice Fréchet differentiable at any measure μ satisfying constraints in (II.3.1) and (II.3.2). Assume that there exists a measurable function $f : X \to \mathbb{R}$ such that*

$$F'(\mu)[\eta] = \int\limits_X f\, d\eta, \quad \eta \in \tilde{\mathbf{M}}(X). \tag{II.3.3}$$

Let μ_0 be a local minimum point in the optimization problem (II.3.1). Let there exist positive ε such that

(i) $(1 \pm \varepsilon)\mu_0 \in A$,

(ii) $\mu_0 + t\delta_x \in A$ *for all* $x \in X$ *and* $0 < t \leqslant \varepsilon$.

Then there exists a real u such that $f(x) \geqslant u \cdot h(x)$ for all $x \in X$ and $f(x) = u \cdot h(x)$ μ_0–a.e.

Necessary conditions of supremum can be deduced from theorem II.3.1 by replacing F by $-F$.

4 Necessary conditions of maximum

Express now the isoperimetric problem (II.1.1) in terms of variational calculus. In this case $X = G(k, d)$ (it is a compact Polish space) and

$A = \mathbf{M}(G(k,d))$. According to (II.3.1), we shall write $F(\theta) = \mathcal{C}(1,\theta)$ (the intensity λ is supposed to be fixed, we put $\lambda = 1$ without loss of generality), $H(\theta) = \theta(G(k,d)) - 1$. Thus the problem (II.1.1) rewrites in the following optimization setting:

$$\begin{cases} F(\theta) = \frac{1}{2} \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi_1, \xi_2]\, \theta(d\xi_1)\theta(d\xi_2) \longrightarrow \max, \\ \theta \in \mathbf{M}(G(k,d)), \\ H(\theta) = \theta(G(k,d)) - 1 = 0. \end{cases} \tag{II.4.1}$$

Now we are ready to prove the following result:

Theorem II.4.1. *Let θ_0 be a directional distribution on $G(k,d)$ that maximizes the intersection density of order 2 of the stationary k–flat process Φ_k^d. Let c_{max} be its maximal value. Then its rose of intersections $T_{kk}\theta_0$ satisfies the following necessary conditions:*

(i) $(T_{kk}\theta_0)(\eta) = \int\limits_{G(k,d)} [\xi,\eta]\, \theta_0(d\xi) = 2c_{max}$ *θ_0–a.e.;*

(ii) $(T_{kk}\theta_0)(\eta) \leqslant 2c_{max}$ *for all $\eta \in G(k,d)$.*

Proof. First we check the conditions of theorem II.3.1. Due to the fact that $A = \mathbf{M}(G(k,d))$, any finite positive measure $\theta_0 \in A$ satisfies assumptions **(i)** and **(ii)** of theorem II.3.1. We shall prove now that functionals F and H are twice Fréchet differentiable at any $\mu \in \mathbf{M}(G(k,d))$ and

$$F'(\mu)[\nu] = \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\, \mu(d\xi)\nu(d\eta), \tag{II.4.2}$$

$$F''(\mu)[\nu,\nu] = 2F(\nu),$$

$$H'(\mu)[\nu] = \nu(G(k,d)),$$

$$H''(\mu)[\nu,\nu] \equiv 0.$$

Consider the difference

$$F(\mu+\nu) - F(\mu) = \tag{II.4.3}$$
$$\int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\, \mu(d\xi)\nu(d\eta) + \frac{1}{2} \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\, \nu(d\xi)\nu(d\eta)$$

for an arbitrary $\nu \in \tilde{\mathbf{M}}(G(k,d))$. It can be easily seen that the second term in the right–hand side of (II.4.3) is $o(\|\nu\|)$ as $\|\nu\| \to 0$: due to the inequality $[\xi,\eta] \leqslant 1$ one can prove that

$$\left| \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\, \nu(d\xi)\nu(d\eta) \right| \leqslant \|\nu\|^2.$$

The first term in the right–hand side of (II.4.3) is a linear functional on ν, it is also bounded: its operator norm is not greater than $|\mu\,(G(k,d))| < \infty$. Then it is continuous, and the first Fréchet derivative of F exists and is equal to (II.4.2).

Analogously to the considerations above

$$F'(\mu+\nu)[\tau] - F'(\mu)[\tau] = \int\limits_{G(k,d)}\int\limits_{G(k,d)} [\xi,\eta]\,\nu(d\xi)\tau(d\eta) = F''(\mu)[\nu,\tau]$$

for all $\tau \in \tilde{\mathbf{M}}\,(G(k,d))$. This bilinear form does not depend on μ.

Then we find the derivative of H: it is

$$H(\mu+\nu) - H(\mu) = \nu\,(G(k,d)) = H'(\mu)[\nu],$$

this difference does not depend on μ, which yields $H''(\mu)[\cdot,\cdot] = 0$.

Furthermore, $F'(\mu)$ has representation (II.3.3) with

$$f(\eta) = \int\limits_{G(k,d)} [\xi,\eta]\,\mu(d\xi) = (T_{kk}\theta_0)\,(\eta)$$

(see (II.4.2)). The functional H satisfies (II.3.2) with $h(\cdot) \equiv 1$. Take a probability measure θ_0 on $G(k,d)$ to be the local maximum point of (II.4.1). Then by theorem II.3.1 there exists a constant u such that

$$\begin{aligned}(T_{kk}\theta_0)\,(\eta) &= \int\limits_{G(k,d)} [\xi,\eta]\,\theta_0(d\xi) = u \quad \theta_0\text{–a.e.},\\ (T_{kk}\theta_0)\,(\eta) &\leqslant u \text{ for all } \eta \in G(k,d),\end{aligned}$$

and since $c_{max} = \frac{1}{2}\int\limits_{G(k,d)} T_{kk}\theta_0(\eta)\,\theta_0(d\eta)$ we conclude that $u = 2c_{max}$. □

Let us illustrate this theorem by the following

Example II.4.1. *The measures that are proved to be maximal in (II.1.1) can be verified to satisfy the above necessary conditions. We consider for convenience just the simplest cases:*

1) Hyperplane case $k = d-1$: it can be shown (cf. corollary III.3.3) that

$$(T_{d-1,d-1}1)\,(\eta) = \frac{\Gamma^2\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}$$

for all η.

2) Suppose that the maximal directional distribution θ_0 is discrete:

$$\theta_0 = p_1\delta_{\xi_1} + \ldots + p_n\delta_{\xi_n},$$

$p_1 + \ldots + p_n = 1$, for some $\{\xi_i\}_{i=1}^n \subset G(k,d)$. Then due to condition **(i)** *of theorem II.4.1*

$$\sum_{j=1}^{n} p_j[\xi_i, \xi_j] = 2c_{max}$$

for each i. In case $d-k \mid d$ we have $\xi_i^\perp \perp \xi_j^\perp$ for $i \neq j$, $p_i = 1/n$ with $n = d/(d-k)$, which yields $c_{max} = k/(2d)$.

Now consider the class A of all absolutely continuous measures with respect to the Haar measure on $G(k,d)$.

Proposition II.4.1. *Let θ_0 be a directional distribution on $G(k,d)$ that maximizes the intersection density of order 2 of the stationary k–flat process Φ_k^d. Let c_{max} be its maximal value. If θ_0 is absolutely continuous with respect to the Haar measure on $G(k,d)$ then the following necessary conditions hold for its rose of intersections $T_{kk}\theta_0$:*

(i) $(T_{kk}\theta_0)(\eta) = 2c_{max}$ *a.e.;*

(ii) $\inf_{E\in\mathfrak{N}} \sup_{G(k,d)\setminus E} (T_{kk}\theta_0)(\eta) \leqslant 2c_{max}$ *where $\mathfrak{N}$ is the ensemble of all sets from $\mathfrak{G}$ with zero Haar measure.*

Proof. Here we have again an optimization problem of the type (II.4.1). The proof is conducted analogously to theorem II.4.1 using theorem 4.1 of [64]. □

5 Remarks and open problems

The necessary conditions of extremum for the isoperimetric problem (II.1.1) proven above can not be essentially improved by means of variational calculus: they do not depend on dimensions d and k, while the class $\mathbf{L_0}$ of extremal directional distributions depends heavily on them. Although these conditions do not solve the problem in general due to the reasons below, they enable reproducing of the result of [84] in the case $k = d-1$. That is based, in part, on the fact that the directional distribution of Φ_k^d can be uniquely restored from its rose of intersections. For all k different from $d-1$ there is no uniqueness (cf. section 1 of chapter IV). So even if we knew that the rose of intersections was constant (not only θ_0–a.e.) we could not make profit from this knowledge.

Nevertheless, the description of the class of directional distributions that correspond to the same rose of intersections is sometimes possible, e.g. for $d = 4$, $k = 2$ (cf. section 5 of chapter IV).

Now we outline some other approaches that might be helpful in solving (II.1.1) and mention the difficulties that one faces there.

Remark II.5.1 (Fourier expansions). *R. Davidson [4] proved that the isotropic stationary process Φ_1^2 has the maximal intersection density by using the Fourier decomposition of the integral kernel $|\sin(\alpha-\beta)|$ of T_{11}. The first Fourier coefficient corresponding to the Haar measure was positive and all other coefficients were negative, which immediately yields the solution.*

One can think of doing the same in higher dimensions. Indeed, the kernel $[\xi,\eta]$ as a simple function of the squares of critical angles y_i (cf. proposition I.3.1) can be expanded into the complete orthonormal system of the so-called generalized Jacobi polynomials, although the calculations are not trivial there since the exact form of these polynomials is not known (see also the discussion in section 6 of chapter IV). But even in the solved case $d=4$, $k=2$ there is an infinite number of positive and negative multipliers of $[\xi,\eta]$ in its expansion in spherical harmonics (see §5.6 of chapter IV), so that the structure of extremal measures is not clear.

We believe that in the general case the same difficulties arise. So the advantages of this approach are vague.

Remark II.5.2 (Potential theory). *The problem (II.1.1) can be set in terms of the potential theory. Namely, the double integral in (II.1.1) is the energy for the potential $T_{kk}\theta$ (see [49]). Unfortunately, the general potential theory expects the integral kernel $k(x,y)$ of the potential $\int_X k(x,y)\,\mu(dx)$ to have a singularity at $x=y$, which is not true in our case: $[\xi,\eta]=0$ for $\xi=\eta$. So the direct application of the results of the potential theory is not possible.*

Remark II.5.3 (Mixed volumes). *According to the representation (I.3.6), the kernel $[\xi,\eta]$ is a mixed volume of some lower dimensional balls lying in ξ and η. This can lead us to the idea to use the Brunn–Minkowski theory to investigate the open cases of (II.1.1). The theorem 4.1.6 of [73] can help us to express the mixed volume in (I.3.6) through the usual volume of the Minkowski sum $\alpha B_\xi+(1-\alpha)B_\eta$, $0<\alpha<1$. Then some inequalities that would give sharp upper bounds for this volume are required. The interesting open problem here is to construct such inequalities.*

The last remark of this section gives the simple stochastic meaning to the projection generating measures introduced in section 1 of this chapter.

Remark II.5.4. *We mentioned on page 16 that the problem (II.1.1) can have an isoperimetric meaning if the directional distributions of the processes Φ_k^d for $d=2k$, $k>1$ are chosen from the class of projection generating measures $\rho_k(K,\cdot)$ of zonoids K with generating measure $\theta_K(\cdot)$. Now we shall give a simple description of this class of measures. Let the process*

$$X_k(\Phi_{d-1}^d)=\{\xi_1\cap\ldots\cap\xi_k:\xi_i\in\Phi_{d-1}^d,\quad\bigcap_{i=1}^k\xi_i\neq\emptyset\}$$

be generated by all intersections of any k *hyperplanes of some stationary process* Φ^d_{d-1} *with directional distribution* θ_K*. Then one can show by means of lemmas 2.1, 2.2 of [7] that the directional distribution of* $X_k(\Phi^d_{d-1})$ *is equal to* $\rho_k(K,\cdot)/\rho_k(K,G(k,d))$*. Then the intersection density of order 2 of the original process* Φ^d_k *is proportional to the intensity of* $X_{2k}(\Phi^d_{d-1})$*, that is, to the intersection density of order* $2k$ *of* Φ^d_{d-1}*. Thomas [84] showed that it is maximal for* θ_K *being the Haar measure on* $G(d-1,d)$*. It means that the measure* θ *that maximizes the intersection density of order 2 of* Φ^d_k *in the class of directional distributions of the form* $\rho_k(K,\cdot)$ *is equal to the measure* $\rho_k(B,\cdot)/\rho_k(B,G(k,d))$ *where* B *is the ball in* $\mathbb{R}^d$*, i.e.* $\theta_B(\cdot)$ *is proportional to* $\omega_d(\cdot)$*.*

Chapter III

Integral transforms

In this chapter we introduce generalized cosine and Radon transforms (sections 1 and 2) that are necessary to prove certain integral formulae of Cauchy–Kubota type (section 3). They will enable us to solve the inverse problem of retrieving the directional distribution of a stationary k–flat process from its rose of intersections in chapter IV.

1 Generalized cosine transforms

Denote by $C(X)$ the space of all continuous functions on X. Introduce for $i + j \geqslant d$ the *generalized cosine transform*

$$T_{ij} : \tilde{\mathbf{M}}\,(G(i,d)) \to C\,(G(j,d))\,,$$
$$(T_{ij}\theta)\,(\xi) = \int\limits_{G(i,d)} [\xi,\eta]\,\theta(d\eta), \qquad \text{(III.1.1)}$$

where $\theta \in \tilde{\mathbf{M}}\,(G(i,d))$, $\xi \in G(j,d)$. We use here the same notation T_{ij} as for the rose of intersections of Φ_i^d because these objects are identical (up to a constant factor λ in (I.1.3) which we suppose later on to be equal to one without loss of generality). We shall consider also transforms T_{ij} of integrable functions f on $G(i,d)$ meaning that the transform (III.1.1) is applied to the measure $f(\eta)d\eta$ where $d\eta$ denotes the integration with respect to the Haar measure on the corresponding Grassmannian normalized by unity. See section 1 of chapter IV about the injectivity of T_{ij}.

1.1 Spherical cosine and sine transforms

Transforms (III.1.1) generalize the well–known notion of the *spherical cosine transform* (cf. [26] for other generalizations):

$$T\theta(u) = \int\limits_{\mathbf{S}^{d-1}} |{<}u,v{>}|\,\theta(dv) = \left(T_{d-1,1}\theta^{\perp}\right)(u), \quad u \in \mathbf{S}^{d-1}$$

for $\theta \in \tilde{\mathbf{M}}(\mathbf{S}^{d-1})$ where $\theta^\perp$ denotes the measure $\theta^\perp(d\xi) = \theta(d\xi^\perp)$ on the Grassmannian $G(d-1,d)$ (see §2.1 of chapter I). There exists exhaustive literature on spherical cosine transforms and their use in geometry (see e.g. [76], [77], [6], [8], [20]).

The *sine transform* of a signed measure $\theta \in \tilde{\mathbf{M}}(\mathbf{S}^{d-1})$ is by definition

$$K\theta(u) = \int\limits_{\mathbf{S}^{d-1}} \sqrt{1- <u,v>^2}\,\theta(dv) = \left(T_{d-1,d-1}\theta^\perp\right)(u^\perp), \quad u \in \mathbf{S}^{d-1}.$$

The cosine and sine transforms can be also defined on integrable functions g on the sphere:

$$Tg(u) = \int\limits_{\mathbf{S}^{d-1}} |<u,v>|\, g(v)\,\omega_d(dv) = \omega_d\left(T_{d-1,1}g^\perp\right)(u), \tag{III.1.2}$$

$$Kg(u) = \int\limits_{\mathbf{S}^{d-1}} \sqrt{1- <u,v>^2}\, g(v)\,\omega_d(dv) = \omega_d\left(T_{d-1,d-1}g^\perp\right)(u^\perp) \tag{III.1.3}$$

where $g^\perp$ stands for the function $g^\perp(\xi) = g(\xi^\perp)$, $\xi \in G(d-1,d)$.

Denote by $C_e^p(\mathbf{S}^{d-1})$, $p \in \mathbb{N} \cup \{\infty\}$ the space of all p times continuously differentiable even functions on $\mathbf{S}^{d-1}$; let $C_e(\mathbf{S}^{d-1})$ be the space of all continuous even functions on $\mathbf{S}^{d-1}$. It is known that cosine and sine transforms are injective on $C_e^\infty(\mathbf{S}^{d-1})$.

1.2 Operator families $\{T_{ij}^\alpha\}$ and $\{\tilde{T}_{ij}^\alpha\}$

Introduce the new families of operators $\{T_{ij}^\alpha\}$, $\{\tilde{T}_{ij}^\alpha\}$ parameterized by $\alpha > 0$:

$$T_{ij}^\alpha,\, \tilde{T}_{ij}^\alpha : \tilde{\mathbf{M}}\left(G(i,d)\right) \to C\left(G(j,d)\right),$$

$$\left(T_{ij}^\alpha\theta\right)(\xi) = \int\limits_{G(i,d)} [\xi,\eta]^\alpha\,\theta(d\eta) \quad \text{for } i+j \geqslant d,$$

$$\left(\tilde{T}_{ij}^\alpha\theta\right)(\xi) = \int\limits_{G(i,d)} [\xi,\eta^\perp]^\alpha\,\theta(d\eta) \quad \text{for } j \geqslant i.$$

The above transforms on integrable functions can be introduced as before. Evidently, these families comprise generalized cosine (T_{ij}^1) transforms. It is also clear that

$$\left(\tilde{T}_{ij}^\alpha\theta\right)(\xi) = \left(T_{d-i,j}^\alpha\theta^\perp\right)(\xi). \tag{III.1.4}$$

The consideration of such operator families will enable us to obtain some integral formulae for them which will be used in chapter IV for $\alpha = 1$. Although these formulae do not find practical applications for $\alpha \neq 1$, they give interesting relations between the moments of order α of the typical angles in intersection of a stationary process Φ_k^d with r–flats (see the next paragraph).

1.3 Connection to convex and stochastic geometry

The generalized cosine transforms find important applications in convex and stochastic geometry.

Thus, $(T_{ij}\theta)(\eta)$ evidently coincides with the rose of intersections of the stationary i–flat process Φ_i^d in $\mathbb{R}^d$ with unit intensity and directional distribution θ with an j–dimensional flat η through the origin.

Then

$$\left(T_{ij}^{\alpha}\theta\right)(\xi) = \int\limits_{G(i,d)} [\xi,\eta]^{\alpha}\,\theta(d\eta) = E\left([\xi,\eta]^{\alpha}|\eta \in \Phi_i^d\right)$$

is the moment of order α of the "sine of the angle" between the typical (chosen at random) i–flat of the process Φ_i^d and a fixed *test flat* $\xi \in G(j,d)$ with respect to the directional distribution θ. Indeed, $[\xi,\eta]$ is the natural generalization of the notion of the sine of the angle between two hyperplanes to the cases of flats of lower dimensions (cf. §3.2 of chapter I).

One can also find another interpretation for T_{ij}^{α} at least in $\mathbb{R}^2$ and $\mathbb{R}^3$ which can be extended analogously to arbitrary dimensions d. It can be shown (see [82], p. 286–303) that the measure

$$\mathcal{A} \mapsto \frac{\int\limits_{\mathcal{A}}[\xi,\eta]\,\theta(d\eta)}{\int\limits_{G(i,d)}[\xi,\eta]\,\theta(d\eta)}, \quad \mathcal{A} \in \mathfrak{G}$$

is the distribution of the typical angle in intersection between the test hyperplane (or line) ξ and Φ_i^d ($d = 2$ or $d = 3$). Then if we rewrite T_{ij}^{α} in the form

$$c \cdot \frac{\int\limits_{G(i,d)} [\xi,\eta]^{\alpha-1}[\xi,\eta]\,\theta(d\eta)}{(T_{ij}\theta)(\xi)} = c \cdot E\left([\xi,\eta]^{\alpha-1}|\eta \in \Phi_i^d,\ \eta \cap \xi \neq \emptyset\right)$$

where $c = (T_{ij}\theta)(\xi)$ we get that $\left(T_{ij}^{\alpha}\theta\right)(\xi)$ is proportional to the moment of order $\alpha - 1$ of the absolute value of the sine of the difference between the typical angle in intersection of Φ_i^d with a fixed test flat (line) ξ and the angle attributed to this test plane (line) by appropriate parameterization.

The meaning of $\left(\tilde{T}_{ij}^{1}\theta\right)(\xi)$ becomes to be transparent due to relation (III.1.4): it is the rose of intersections of a stationary $d-i$–flat process with directional distribution $\theta^{\perp}$ and unit intensity with a j–flat ξ.

In convex geometry the *ith projection function* $v_i(K;\eta)$ of a zonoid K, i.e. the i–dimensional volume of $Pr_{\eta}(K)$ for $\eta \in G(i,d)$, is the generalized cosine transform of the projection generating measure $\rho_i(K,\cdot)$:

$$v_i(K,\eta) = (T_{i,d-i}\rho_i(K,\cdot))(\eta^{\perp}), \quad \eta \in G(i,d)$$

(cf. [22]).

2 Radon transforms

In this section we outline some facts about Radon integral transforms on Grassmannians and on the sphere (cf. [32], [34], [10], [8]).

2.1 Radon transforms on Grassmannians

Let $L^p(X)$ be the space of all functions f such that $|f|^p$ is integrable on X. Introduce the *Radon transform on Grassmannians* and its *dual* following the paper of Grinberg [25]: for $1 \leqslant i < j \leqslant d-1$

$$(R_{ij}f)(\xi) = \int\limits_{\eta \in G(i,d):\, \eta \subset \xi} f(\eta)\, \sigma(d\eta),$$

$$(R_{ji}g)(\eta) = \int\limits_{\xi \in G(j,d):\, \xi \supset \eta} g(\xi)\, \sigma(d\xi)$$

where $f \in L^1(G(i,d))$, $g \in L^1(G(j,d))$, $\xi \in G(j,d)$, $\eta \in G(i,d)$, and $\sigma(\cdot)$ is the unique rotation invariant measure on the appropriate integration space with total mass 1. Such Radon transforms find numerous applications in convex geometry, see e.g. [19], [22], [12], [13], [23].

It is known that R_{ij} is injective on the space $C^\infty(G(i,d))$ of all infinitely differentiable functions on $G(i,d)$ if and only if $i+j \leqslant d$, $i<j$ or $i+j \geqslant d$, $i>j$ (cf. [9], [25]).

2.2 Spherical Radon transforms and their inversion

An important particular case of R_{ij} form the so–called spherical Radon transforms. For $\eta \in G(r,d)$, $1 \leqslant r \leqslant d-1$ denote by $\mathbf{S}^{r-1}_\eta$ the totally geodesic submanifold $\mathbf{S}^{d-1} \cap \eta$ of $\mathbf{S}^{d-1}$, $dim(\mathbf{S}^{r-1}_\eta) = r-1$. For any integrable function g on $\mathbf{S}^{d-1}$ introduce *the spherical Radon transform of order* r:

$$(R_r g)(\eta) = \frac{1}{\omega_r} \int\limits_{\mathbf{S}^{r-1}_\eta} g(u)\, \omega^\eta_r(du) = (R_{1r}\, g)(\eta) \qquad \text{(III.2.1)}$$

where $\omega^\eta_r(\cdot)$ is the surface area measure on the subsphere $\mathbf{S}^{r-1}_\eta$. If $r = d-1$ we shall write $R = R_{d-1}$ and call it simply *the spherical Radon transform:* identifying $\eta \in G(d-1,d)$ with the direction unit vector v of the line $\eta^\perp$ we have

$$Rg(v) = \frac{1}{\omega_{d-1}} \int\limits_{<u,v>=0} g(u)\, \omega^{v^\perp}_{d-1}(du) = (R_{1,d-1}\, g)(v^\perp). \qquad \text{(III.2.2)}$$

Radon transforms R_r are injective on $C_e^\infty(\mathbf{S}^{d-1})$. The following inversion formulae being true for all $d \geqslant 3$ follow from the general results of S. Helgason in [33] and [34], p. 54: for any $g \in C_e(\mathbf{S}^{d-1})$, $\eta \in G(r,d)$

$$g(v) = c_R \times \qquad \text{(III.2.3)}$$
$$\times \left(\frac{d}{d(x^2)}\right)^{r-1} \left[\int_0^x y^{r-1}(x^2-y^2)^{\frac{r-3}{2}} \int\limits_{d(\mathbf{S}_\eta^{r-1},v)=\arccos y} (R_r g)(\eta)\,\sigma(d\eta)\,dy \right]\Bigg|_{x=1}$$

where $d(\cdot,\cdot)$ is the geodesic distance on $\mathbf{S}^{d-1}$ and

$$c_R = \frac{2^r}{(r-2)!}; \qquad \text{(III.2.4)}$$

$$g(v) = c_R^1 \left(\frac{d}{d(x^2)}\right)^{d-2} \left[\int_0^x (x^2-y^2)^{\frac{d}{2}-2} \int\limits_{<u,v>^2=1-y^2} Rg(u)\,\omega_{d-1}^{v^\perp,y}(du)\,dy \right]\Bigg|_{x=1} \qquad \text{(III.2.5)}$$

where $\omega_{d-1}^{v^\perp,y}(\cdot)$ is the surface area measure on the pair of subspheres $\{u \in \mathbf{S}^{d-1} : <u,v>^2 = 1-y^2\}$ of $\mathbf{S}^{d-1}$ and

$$c_R^1 = \frac{2^{d-3}}{(d-3)!\,\omega_{d-1}}. \qquad \text{(III.2.6)}$$

We shall find now a more compact form of (III.2.5) and will use it in the proofs later on; nevertheless, (III.2.5) can be also used instead. In our opinion, formula (III.2.7) below with its integration over the part of a sphere gives a reader more geometrical clearness and agrees with the earlier results of Pogorelov (see remark III.2.1).

Lemma III.2.1 (Modified inversion formula for the Radon transform).
For any $g \in C_e(\mathbf{S}^{d-1})$ and $d \geqslant 3$ the following inversion formula holds:

$$g(v) = c_R^2 \left(\frac{d}{d(\mu^2)}\right)^{d-2} \left[\int\limits_{<u,v>^2>\mu^2} \frac{Rg(u)\,|<u,v>|}{(<u,v>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(du) \right]\Bigg|_{\mu=0} \qquad \text{(III.2.7)}$$

where

$$c_R^2 = \frac{(-1)^{d-2}2^{d-3}}{(d-3)!\,\omega_{d-1}}. \qquad \text{(III.2.8)}$$

Proof. Applying to (III.2.5) subsequently the following changes of variables:

$$y^2 = 1 - t^2, \quad \mu^2 = 1 - x^2$$

we get

$$g(v) = c_R^1(-1)^{d-2}\left(\frac{d}{d(\mu^2)}\right)^{d-2}\left[\int_\mu^1 \frac{t(t^2-\mu^2)^{\frac{d}{2}-2}}{\sqrt{1-t^2}} \int_{<u,v>^2=t^2} Rg(u)\,\omega_{d-1}^{v^\perp,t}(du)\,dt\right]\Bigg|_{\mu=0}.$$

Then using $<u,v>= t$ the result follows from Fubini's theorem. □

Remark III.2.1. *A. V. Pogorelov gives another proof of (III.2.7) for the case $d = 3$ (cf. [67]) that does not depend on inversion formula (III.2.5). See the detailed discussion in [79].*

2.3 Useful integral and differential relations

Introduce the differential operator

$$\Box = \frac{1}{2\omega_{d-1}}(\triangle_0 + d - 1) \tag{III.2.9}$$

where $\triangle_0$ is the Beltrami – Laplace operator on $\mathbf{S}^{d-1}$:

$$\triangle_0 g(x) = \left(\triangle g\left(\frac{x}{|x|}\right)\right)\Bigg|_{\mathbf{S}^{d-1}}.$$

Here $\triangle$ denotes the standard Laplace operator. It can be shown (cf. [22]) that

$$\Box T = R, \tag{III.2.10}$$

$$K = \frac{\omega_{d-1}}{2k_{d-2}}RT \tag{III.2.11}$$

on $C_e(\mathbf{S}^{d-1})$, or equivalently to (III.2.11),

$$T = \frac{2k_{d-2}}{\omega_{d-1}}R^{-1}K, \tag{III.2.12}$$

as R is invertible. As for relation (III.2.11) we shall prove it in more general form for the Radon transforms on Grassmannians and operators T_{ij}^α in proposition III.3.2 (see also corollary III.3.1).

3 Integral formulae

In this section some integral relations for T_{ij}^α, $\tilde{T}_{ij}^\alpha$ and R_{ij} will be obtained that enable the inversion of T_{ij} in cases of its injectivity (see chapter IV). The main technical result is proved in §3.1. Its important corollaries are given in §3.3 and §3.4. Finally, §3.5 yields the bounds for the weighted images of Radon transform R_{ij} and its dual.

3.1 Radon transforms of the power of the volume

Let $e_1, \ldots, e_d$ be the Cartesian unit basis vectors in $\mathbb{R}^d$ and $\zeta_0 =< e_1, \ldots, e_k >$ for some $k < d$. The aim of this paragraph is to investigate the action of Radon transforms R_{ij} on functions $g(\eta) = [\eta, \zeta_0]^\alpha$ for $\alpha > 0$. One partial result of that kind for the dual Radon transform with $\alpha = 1$ and dimension $k = j$ can be found in lemma 4.1 of [21]. The argument there uses connection between volumes $[\cdot, \cdot]$, mixed volumes and projection functions. Then one applies the following Cauchy–Kubota–type formula (equation (2.3) of [21]) to the latter:

$$R_{ij}\left(v_i(K;\cdot)\right)(\xi) = \frac{i!k_i(j-i)!k_{j-i}}{j!k_j} V_i\left(Pr_\xi(K)\right) \tag{III.3.1}$$

for a convex body K and all $\xi \in G(j,d)$ where $i < j < d$ and $V_i(K)$ is the *ith intrinsic volume* (cf. [73]) of K.

Our approach differs from the described above that allows us to gain more generality in dimensions of involved linear subspaces and real positive powers of the volume.

Let $Vol(\xi)$ be the non–oriented volume of the parallelepiped spanned by the basis vectors of a k–flat ξ (the choice of the basis will be clear from the context). Denote by $b(\xi)$ a set of the orthonormal basis vectors $a_1, \ldots, a_k$ such that $\xi =< a_1, \ldots, a_k >$ for $\xi \in G(k,d)$. We shall prove the following

Theorem III.3.1. *Let $\xi \in G(j,d)$, $i < j$, $i + k \geqslant d$, $d \geqslant 3$, $\alpha > 0$. Then*

$$\left(R_{ij}[\,\cdot, \zeta_0]^\alpha\right)(\xi) = \int\limits_{\eta \in G(i,d):\, \eta \subset \xi} [\eta, \zeta_0]^\alpha \, \sigma(d\eta) = c(\alpha)[\xi, \zeta_0]^\alpha \tag{III.3.2}$$

where

$$c(\alpha) = \prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{j-l}{2}\right)\Gamma\left(\frac{i-l+\alpha}{2}\right)}{\Gamma\left(\frac{i-l}{2}\right)\Gamma\left(\frac{j-l+\alpha}{2}\right)}. \tag{III.3.3}$$

Proof. Let us fix an orthonormal basis $\xi_1, \ldots, \xi_j$ of $\xi \in G(j,d)$ so that $\xi_1, \ldots, \xi_i$ is an orthonormal basis of $\eta \in G(i,d)$, $i < j$. Let

$$(\eta)^\perp_\xi =< \xi_{i+1}, \ldots, \xi_j > \tag{III.3.4}$$

be the orthogonal complement of η in ξ. We denote by $\xi_{j+1}, \ldots, \xi_d$ a certain orthonormal basis of $\xi^\perp$. Then vectors $\xi_{i+1}, \ldots, \xi_d$ form an orthonormal basis in $\eta^\perp$. If $[\xi, \zeta_0] = 0$, i.e. $dim\left(\xi^\perp \cap \zeta_0^\perp\right) > 0$, then $[\eta, \zeta_0] = 0$ for all $\eta \subset \xi$ since $\xi^\perp \subset \eta^\perp$ and $dim\left(\eta^\perp \cap \zeta_0^\perp\right) > 0$. Hence formula (III.3.2) holds automatically. It means that in the following it suffices to prove (III.3.2) for the case $[\xi, \zeta_0] \neq 0$, i.e.

$$\xi^\perp \cap < e_{k+1}, \ldots, e_d >= \{0\}. \tag{III.3.5}$$

The following relation holds:

$$[\eta, \zeta_0] \equiv Vol(\xi_{i+1}, \dots, \xi_d, e_{k+1}, \dots, e_d) =$$
$$Vol(\xi_{j+1}, \dots, \xi_d, e_{k+1}, \dots, e_d) \cdot Vol\left(Pr_{<\xi_{j+1},\dots,\xi_d,e_{k+1},\dots,e_d>^{\perp}}\{\xi_{i+1}, \dots, \xi_j\}\right),$$

or briefly,

$$[\eta, \zeta_0] = [\xi, \zeta_0] Q(\xi, \eta) \tag{III.3.6}$$

where $Q(\xi, \eta)$ denotes the $(j-i)$–dimensional volume of the parallelepiped spanned by projections of $\xi_{i+1}, \dots, \xi_j$ (cf. (III.3.4)) onto the plane

$$< \xi_{j+1}, \dots, \xi_d, e_{k+1}, \dots, e_d >^{\perp} = < \xi^{\perp}, \zeta_0^{\perp} >^{\perp}. \tag{III.3.7}$$

Thus by (III.3.6)

$$\int\limits_{\eta \in G(i,d):\, \eta \subset \xi} [\eta, \zeta_0]^{\alpha} \sigma(d\eta) = c_{\alpha}(\xi)[\xi, \zeta_0]^{\alpha},$$

$$c_{\alpha}(\xi) = \int\limits_{\eta \in G(i,d):\, \eta \subset \xi} Q(\xi, \eta)^{\alpha} \sigma(d\eta). \tag{III.3.8}$$

Let us write $Q(\xi, \eta)$ in a different form. First we give another representation to the plane (III.3.7):

$$< \xi^{\perp}, \zeta_0^{\perp} >^{\perp} = \xi \cap \zeta_0. \tag{III.3.9}$$

Indeed, if $\tau \in < \xi^{\perp}, \zeta_0^{\perp} >^{\perp}$ then $\tau \perp \{\xi_{j+1}, \dots, \xi_d\}$ and $\tau \perp \{e_{k+1}, \dots, e_d\}$. Hence $\tau \in < \xi_{j+1}, \dots, \xi_d >^{\perp} = \xi$, $\tau \in < e_{k+1}, \dots, e_d >^{\perp} = \zeta_0$, or $\tau \in \xi \cap \zeta_0$. Thus, $< \xi^{\perp}, \zeta_0^{\perp} >^{\perp} \subseteq \xi \cap \zeta_0$. Since $dim\left(< \xi^{\perp}, \zeta_0^{\perp} >^{\perp}\right) = dim(\xi \cap \zeta_0)$ by (I.1.4) the relation (III.3.9) is proved.

Let us now show that

$$Q(\xi, \eta) = Vol\left(\xi_{j+1}, \dots, \xi_d, b\left((\xi \cap \zeta_0)_{\xi}^{\perp}\right)\right) \stackrel{def}{=} [\xi \cap \zeta_0, \eta]_{\xi}. \tag{III.3.10}$$

By definition of $Q(\xi, \eta)$, owing to (III.3.9) one can write

$$Q(\xi, \eta) = Vol\left(Pr_{\xi \cap \zeta_0}\left((\eta)_{\xi}^{\perp}\right)\right). \tag{III.3.11}$$

The following formula holds for any flats a, c in arbitrary ambient space:

$$[a^{\perp}, c] \stackrel{def}{=} Vol\left(b(a), b(c^{\perp})\right) = Vol\left(Pr_c(a)\right) = Vol\left(Pr_{a^{\perp}}\left(c^{\perp}\right)\right).$$

Hence (III.3.11) yields

$$Q(\xi, \eta) = Vol\left(Pr_{\xi \cap \zeta_0}\left((\eta)_{\xi}^{\perp}\right)\right) = Vol\left(Pr_{\eta}\left((\xi \cap \zeta_0)_{\xi}^{\perp}\right)\right) = [\xi \cap \zeta_0, \eta]_{\xi}$$

(here the ambient space is ξ). Thus, relation (III.3.10) is proved and (III.3.8) gives

$$c_\alpha(\xi) = \int\limits_{\eta\in G(i,d):\,\eta\subset\xi} [\xi\cap\zeta_0,\eta]^\alpha_\xi\,\sigma(d\eta).$$

Let us prove that $c_\alpha(\xi)$ does not depend on ξ. According to (III.3.5) it is sufficient to consider only the case

$$\xi^\perp\cap\zeta_0^\perp = \{0\}. \tag{III.3.12}$$

For any $\xi\in G(j,d)$ there exists a rotation $\gamma\in SO(d)$ such that $\xi=\gamma\xi_0$, $\xi_0 =< e_1,\dots,e_j >$. Then

$$c_\alpha(\xi) = \int\limits_{\eta\in G(i,d):\,\eta\subset\xi_0} [\gamma\xi_0\cap\zeta_0,\gamma\eta]^\alpha_{\gamma\xi_0}\,\sigma(d\eta) = \int\limits_{\eta\in G(i,d):\,\eta\subset\xi_0} [\xi_0\cap\gamma^{-1}\zeta_0,\eta]^\alpha_{\xi_0}\,\sigma(d\eta),$$

and (III.3.12) has the form

$$(\gamma\xi_0)^\perp\cap\zeta_0^\perp = \gamma\xi_0^\perp\cap\zeta_0^\perp = \xi_0^\perp\cap\gamma^{-1}\zeta_0^\perp = \{0\}.$$

Without loss of generality one can substitute γ by γ^{-1}. Thus we should show that

$$\tilde c_\alpha(\gamma) \stackrel{def}{=} \int\limits_{\eta\in G(i,d):\,\eta\subset\xi_0} [\xi_0\cap\gamma\zeta_0,\eta]^\alpha_{\xi_0}\,\sigma(d\eta) \tag{III.3.13}$$

is constant on the set

$$G_{jk} = \left\{\gamma\in SO(d) : \xi_0^\perp\cap\gamma\zeta_0^\perp = \{0\}\right\}. \tag{III.3.14}$$

First let us prove that

$$dim(\xi_0\cap\gamma\zeta_0) = j+k-d$$

for any $\gamma\in G_{jk}$. By (I.1.4) the dimension of $\xi_0\cap\gamma\zeta_0$ can not be less then $j+k-d$. Let us prove that it can not be also greater then $j+k-d$. Suppose, ex adverso, that $dim(\xi_0\cap\gamma\zeta_0) = m > j+k-d$. Let $\tau_1,\dots,\tau_m$ be the basis in $\xi_0\cap\gamma\zeta_0$. Amplify it to the bases in ξ_0 and $\gamma\zeta_0$:

$$\xi_0 =< \tau_1,\dots,\tau_m,\tau_{m+1},\dots,\tau_j >,$$

$$\gamma\zeta_0 =< \tau_1,\dots,\tau_m,\tilde\tau_{m+1},\dots,\tilde\tau_k > .$$

The number of different unit vectors in ξ_0, $\gamma\zeta_0$ is evidently equal to $j+k-m$. Then, as $m > j+k-d$, this number is less then d. It means that there exists at least one unit vector $x\in\mathbb{R}^d$ that does not belong to the linear hull of the bases in ξ_0 and $\gamma\zeta_0$. Then $x \in \xi_0^\perp \cap (\gamma\zeta_0)^\perp$. We arrived at the contradiction with (III.3.14).

Thus we proved that any transform $\gamma \in G_{jk}$ preserves the dimension of $\beta \overset{def}{=} \xi_0 \cap \gamma\zeta_0 \subset \xi_0$. Identifying ξ_0 with $\mathbb{R}^j$ we can rewrite the relation (III.3.13) as follows:

$$\bar{c}_\alpha(\beta) = \int\limits_{\eta\in G(i,d):\,\eta\subset\mathbb{R}^j} [\beta,\eta]^\alpha_{\mathbb{R}^j}\,\sigma(d\eta) = \int\limits_{G(i,j)} [\beta,\eta]^\alpha_{\mathbb{R}^j}\,d\eta$$

for $\beta \subset \xi_0$, $dim(\beta) = j+k-d$. Let us prove that $\bar{c}_\alpha(\beta)$ does not depend on $\beta \in G(j+k-d,j)$. Indeed, we have by rotation invariance (since $dim(\beta)$ does not depend on $\gamma \subset G_{jk}$) that

$$\bar{c}_\alpha(\beta) = \bar{c}_\alpha(\beta_0), \quad \beta_0 =< e_1,\dots,e_{j+k-d} > .$$

Thus we have proved that

$$c(\alpha) = \int\limits_{G(i,j)} [\beta_0,\eta]^\alpha_{\mathbb{R}^j}\,d\eta$$

is a constant, $\beta_0 =< e_1,\dots,e_{j+k-d} >\in G(j+k-d,j)$. Now our aim is to calculate $c(\alpha)$. Introduce the notation

$$b_\alpha(n,m,r) = \int\limits_{G(n,m)} [\beta_0,\eta]^\alpha_{\mathbb{R}^m}\,d\eta$$

for $n+r \geqslant m$, $\beta_0 =< e_1,\dots,e_r >$. Then $c(\alpha) = b_\alpha(i,j,j+k-d)$. Let us calculate $b_\alpha(n,m,r)$ for all n, m, r such that $n+r \geqslant m \geqslant 2$. At the first step, let us prove that

$$b_\alpha(k,d,i) = b_\alpha(i,j,i+j-d)\cdot b_\alpha(k,d,j) \qquad \text{(III.3.15)}$$

for $i+k \geqslant d$, $i<j$, $i \geqslant d/2$, $d \geqslant 2$. Integrate with respect to ζ the equality

$$\int\limits_{\eta\in G(i,d):\,\eta\subset\xi} [\eta,\zeta]^\alpha\,\sigma(d\eta) = c(\alpha)[\xi,\zeta]^\alpha \qquad \text{(III.3.16)}$$

where $\zeta \in G(k,d)$ and $\xi \in G(j,d)$. Relation (III.3.16) follows from (III.3.2) by rotation invariance. One gets by Fubini's theorem that

$$\int\limits_{\eta\in G(i,d):\,\eta\subset\xi} \left(\int\limits_{G(k,d)} [\eta,\zeta]^\alpha\,d\zeta\right)\sigma(d\eta) = c(\alpha)\int\limits_{G(k,d)} [\xi,\zeta]^\alpha\,d\zeta.$$

Since the integrand in parentheses in the left–hand side of the above relation does not depend on η because of the rotation invariance and is equal to $b_\alpha(k,d,i)$ we can write

$$b_\alpha(k,d,i) \int\limits_{\eta\in G(i,d):\, \eta\subset\xi} \sigma(d\eta) = c(\alpha) b_\alpha(k,d,j).$$

Then, as the total mass of the measure σ is one, the above relation completes the proof of (III.3.15).

By lemma 2.2 (a) of [70] (one should apply operators A and A^* to a constant function there)

$$b_\alpha(d-1,d,d-k) = b_\alpha(d-k,d,d-1) = \frac{\omega_{d-k}\omega_k}{\omega_d} \int\limits_0^1 (1-t^2)^{(k-2)/2} t^{d-k-1+\alpha} dt. \qquad \text{(III.3.17)}$$

The integral in the right-hand side of (III.3.17) is equal to

$$1/2 \int\limits_0^1 (1-u)^{k/2-1} u^{\frac{d-k+\alpha}{2}-1} du = \frac{1}{2} \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)}.$$

Then by (III.3.17)

$$b_\alpha(d-1,d,d-k) = \frac{1}{2}\frac{\omega_{d-k}\omega_k}{\omega_d} \frac{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)} = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d-k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}.$$

Thus we have proved that

$$b_\alpha(d-1,d,d-k) = b_\alpha(d-k,d,d-1) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+\alpha-k}{2}\right)}{\Gamma\left(\frac{d-k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}. \qquad \text{(III.3.18)}$$

Let us prove that for all k

$$b_\alpha(d-k,d,k) = b_\alpha(k,d,d-k).$$

By definition, one can write for some $\eta \in G(k,d)$

$$b_\alpha(d-k,d,k) = \int\limits_{G(d-k,d)} [\beta_0,\eta]^\alpha \, d\eta = \int\limits_{G(d-k,d)} [\beta_0^\perp,\eta^\perp]^\alpha \, d\eta =$$

$$\int\limits_{G(k,d)} [\beta_0^\perp,\eta^\perp]^\alpha \, d\eta^\perp = \int\limits_{G(k,d)} [\beta_0^\perp,\nu]^\alpha \, d\nu = b_\alpha(k,d,d-k).$$

The said above is true since $[\beta,\eta] = [\beta^\perp,\eta^\perp]$ for all $\beta \in G(d-k,d)$, $\eta \in G(k,d)$.

Furthermore, by (III.3.18) we have for all $1 \leqslant r \leqslant d-1$

$$b_\alpha(d-1,d,r) = b_\alpha(r,d,d-1) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{r+\alpha}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}. \tag{III.3.19}$$

By (III.3.15) and (III.3.19) the following equality is true:

$$b_\alpha(k,d,d-2) = b_\alpha(d-2,d-1,k-1)\cdot b_\alpha(k,d,d-1) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{k+\alpha}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{d+\alpha}{2}\right)}\cdot\frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{k-1+\alpha}{2}\right)}{\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{d-1+\alpha}{2}\right)}.$$

One can prove in the same way by induction on r that for $\alpha > 0$

$$b_\alpha(k,d,d-r) = \prod_{l=0}^{r-1}\frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{k-l+\alpha}{2}\right)}{\Gamma\left(\frac{k-l}{2}\right)\Gamma\left(\frac{d-l+\alpha}{2}\right)},$$

or, more generally,

$$b_\alpha(k,d,r) = \prod_{l=0}^{d-r-1}\frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{k-l+\alpha}{2}\right)}{\Gamma\left(\frac{k-l}{2}\right)\Gamma\left(\frac{d-l+\alpha}{2}\right)}, \quad k+r \geqslant d. \tag{III.3.20}$$

Then as $c(\alpha) = b_\alpha(i,j,j+k-d)$ and by (III.3.20) we get that relation (III.3.3) holds, and the theorem is proved. □

3.2 Dual Radon transforms of the power of the volume

Now it will be of interest to us to consider the action of the dual Radon transform on functions $g(\eta) = [\eta^\perp, \zeta_0]^\alpha$ for $\alpha > 0$.

Proposition III.3.1. *Let $i < j \leqslant k < d$, $d \geqslant 3$. Then for $\alpha > 0$ the following relation holds:*

$$(R_{ji}[\cdot^\perp, \zeta_0]^\alpha)(\eta) = c^*(\alpha)[\eta^\perp, \zeta_0]^\alpha \tag{III.3.21}$$

where $\eta \in G(i,d)$,

$$c^*(\alpha) = \prod_{l=0}^{d-k-1}\frac{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-j-l+\alpha}{2}\right)}{\Gamma\left(\frac{d-j-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha}{2}\right)}. \tag{III.3.22}$$

Proof. Let $\eta \subset \xi \in G(j,d)$. Then $\xi^\perp \subset \eta^\perp$, and using the duality relation (2.3) of [19] for the Radon transform one can write

$$(R_{ji}[\cdot^\perp, \zeta_0]^\alpha)(\eta) = \left(R_{d-j,d-i}[\cdot, \zeta_0]^\alpha\right)(\eta^\perp).$$

Here $d-j < d-i$, $\eta^\perp \in G(d-i,d)$, so we can use the result of theorem III.3.1:

$$\left(R_{d-j,d-i}[\cdot, \zeta_0]^\alpha\right)(\eta^\perp) = c^*(\alpha)[\eta^\perp, \zeta_0]^\alpha$$

where $c^*(\alpha) = b_\alpha(d-j, d-i, d-i+k-d) = b_\alpha(d-j, d-i, k-i)$ (cf. proof of theorem III.3.1). Then by (III.3.20)

$$c^*(\alpha) = \prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-j-l+\alpha}{2}\right)}{\Gamma\left(\frac{d-j-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha}{2}\right)},$$

and the assertion is proved. □

3.3 Cauchy – Kubota – type formulae for the generalized cosine transforms

First let us prove the following

Proposition III.3.2. *For any $\alpha > 0$ and dimensions i, j, k with $i+k \geqslant d$, $i < j$ the following integral relation is valid on the space $\tilde{\mathbf{M}}\,(G(k,d))$:*

$$R_{ij}T^\alpha_{ki} = c(\alpha)T^\alpha_{kj}$$

where the constant $c(\alpha)$ is defined in (III.3.3).

Proof. Integrate both sides of equality (III.3.16) with respect to some measure $\theta \in \tilde{\mathbf{M}}\,(G(k,d))$ and use Fubini's theorem:

$$\int\limits_{\eta\in G(i,d):\,\eta\subset\xi}\ \int\limits_{G(k,d)} [\eta,\zeta]^\alpha\,\theta(d\zeta)\sigma(d\eta) = c(\alpha)\int\limits_{G(k,d)} [\xi,\zeta]^\alpha\,\theta(d\zeta).$$

The comparison of both sides of the above equation with the expression for T^α_{ij} completes the proof. □

The following corollary is an easy consequence of proposition III.3.2 for $\alpha = 1$:

Corollary III.3.1. *For any i, j, k with $i + k \geqslant d$, $i < j$ the following relation holds:*

$$R_{ij}T_{ki} = \frac{\omega_{i+1-d+k}\omega_{j+1}}{\omega_{j+1-d+k}\omega_{i+1}}T_{kj}.$$

The case $i = 1$, $j = r$, $k = d-1$ will be of importance to us:

Corollary III.3.2. *For any $d \geqslant 3$, $1 < r \leqslant d-1$, $\eta \in G(d,r)$, and signed measure $\theta \in \tilde{\mathbf{M}}(\mathbf{S}^{d-1})$ the following integral relation holds:*

$$\left(T_{d-1,r}\theta^\perp\right)(\eta) = \frac{\omega_r}{2k_{r-1}}\left(R_rT\theta\right)(\eta). \tag{III.3.23}$$

Remark III.3.1. *If $r = d-1$ then equation (III.3.23) coincides with (III.2.11). Thus proposition III.3.2 is a generalization of the classical relation (III.2.11).*

The above connection between generalized cosine transforms of different orders can be used in practice. For example, in tomography and stereology there is often a need to estimate the statistical parameters of the shape of some geometric structure under consideration (e.g. porous media, microscopic shots of tissues, fiber collections, etc.) from the experimental data gained by sections of examined patterns, or in our terms, by intersections of patterns with flats of different dimensions. The above mentioned patterns are sometimes modeled as k–dimensional manifold processes in $\mathbb{R}^d$ which can be seen locally as k–flat processes with directional distribution θ if one considers all k–flats tangent to the pieces of k–manifolds in the corresponding neighborhood. The characteristics known from their intersections with i–flats will be in this case often their roses of intersections $T_{ki}\theta$ (or the corresponding moments $T^{\alpha}_{ki}\theta$). Proposition III.3.2 states that from the knowledge of the characteristics of the lower–dimensional sections of the pattern ($T^{\alpha}_{ki}\theta$) conclusions about the same characteristics in higher–dimensional sections ($T^{\alpha}_{kj}\theta$, $j > i$) can be obtained by simple integration (R_{ij}).

The proof of the theorem III.3.1 yields the well–known result below (cf. formula (III.3.20) with $\alpha = 1$) that we would like to emphasize: it is the value of the rose of intersections of the stationary isotropic (i.e. with the uniform distribution of directions) Poisson k–flat process (cf. [82]) with arbitrary r–flats ($T_{kr}1$). It coincides with the rose of intersections of the stationary isotropic Poisson r–flat process with k–planes:

Corollary III.3.3. *For any k, r such that $k + r \geqslant d$*

$$T_{kr}1 = T_{rk}1 = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{k+r-d+1}{2}\right)} = \frac{\omega_{d+1}\omega_{k+r-d+1}}{\omega_{k+1}\omega_{r+1}}.$$

3.4 Double fibration for $\{T^{\alpha}_{ij}\}$ and $\{\tilde{T}^{\alpha}_{ij}\}$

The following double fibration relation (cf. [32], p. 168) for R_{ij} takes place for all absolute integrable $f \in L^1(G(i,d))$, $\varphi \in L^1(G(j,d))$:

$$\int\limits_{G(j,d)} (R_{ij}f)(\xi)\varphi(\xi)\,d\xi = \int\limits_{G(i,d)} f(\eta)\,(R_{ji}\varphi)(\eta)\,d\eta. \tag{III.3.24}$$

Here R_{ji} is dual to the transform R_{ij}. Let us investigate now the behaviour of this relation on functions $f(\eta) = [\eta, \zeta_0]^{\alpha}$, $\eta \in G(i,d)$, $i < j$. By theorem III.3.1 one gets from (III.3.24) that

$$c(\alpha) \int\limits_{G(j,d)} [\xi, \zeta_0]^{\alpha}\varphi(\xi)\,d\xi = \int\limits_{G(i,d)} [\eta, \zeta_0]^{\alpha}\,(R_{ji}\varphi)(\eta)\,d\eta, \tag{III.3.25}$$

and using the rotation invariance of the above equation we obtain the following result:

Proposition III.3.3. *For all $i < j$, $i + k \geqslant d$, $\alpha > 0$ and all absolute integrable functions $\varphi \in L^1(G(j,d))$*

$$\left(T^{\alpha}_{jk}\varphi\right)(\zeta) = c^{-1}(\alpha) T^{\alpha}_{ik}\left(R_{ji}\varphi\right)(\zeta), \quad \zeta \in G(k,d) \tag{III.3.26}$$

where $c(\alpha)$ is given by (III.3.3).

It would be of some interest to illustrate the usage of the above proposition in stochastic geometry. Namely, for the values $\alpha = 1$, $d = 3$, $i = 1$, $j = 2$ and $k = 2$ we have the equality

$$\left(T_{22}\varphi\right)(\zeta) = \frac{\pi}{2} T_{12}\left(R_{21}\varphi\right)(\zeta), \quad \zeta \in G(2,3). \tag{III.3.27}$$

The transform $\left(T_{22}\varphi\right)(\zeta)$ is the rose of intersections of the stationary process of planes Φ^3_2 in three dimensions with a test plane ζ. The process Φ^3_2 has the unit intensity and the directional distribution with density φ. This rose of intersections is equal by (III.3.27) to the rose of intersections T_{12} of the process Φ^3_1 of lines with the same test plane ζ where this new process Φ^3_1 has the unit intensity and directional distribution density $R_{21}\varphi$ obtained from φ by integration.

Proposition III.3.4. *The adjoint operator of T^{α}_{ij} on $L^1(G(i,d))$ is operator T^{α}_{ji} on $L^1(G(j,d))$:*

$$\left(T^{\alpha}_{ij}\right)^* = T^{\alpha}_{ji}.$$

Proof. The desired relation

$$\int\limits_{G(i,d)} f(\eta)\left(T^{\alpha}_{ji}\varphi\right)(\eta)\, d\eta = \int\limits_{G(j,d)} \left(T^{\alpha}_{ij}f\right)(\xi)\varphi(\xi)\, d\xi$$

for $f \in L^1(G(i,d))$ and $\varphi \in L^1(G(j,d))$ follows easily from the more general relation

$$\int\limits_{G(i,d)} \left(T^{\alpha}_{ji}\theta\right)(\eta)\, \mu(d\eta) = \int\limits_{G(j,d)} \left(T^{\alpha}_{ij}\mu\right)(\xi)\, \theta(d\xi)$$

for any $\theta \in \tilde{\mathbf{M}}(G(j,d))$, $\mu \in \tilde{\mathbf{M}}(G(i,d))$ which can be seen directly by Fubini's theorem. □

Let us state now the result for the dual Radon transform similar to the proposition III.3.3:

Proposition III.3.5. *For all $i < j \leqslant k < d$, $\alpha > 0$, and all absolute integrable functions $g \in L^1(G(i,d))$*

$$\left(\tilde{T}^{\alpha}_{ik}g\right)(\zeta) = \left(c^*(\alpha)\right)^{-1} \tilde{T}^{\alpha}_{jk}\left(R_{ij}g\right)(\zeta), \quad \zeta \in G(k,d) \tag{III.3.28}$$

where $c^*(\alpha)$ is given by (III.3.22).

Proof. First it is worth mentioning that the assertion of proposition III.3.1 is also true for any $\zeta \in G(k,d)$ instead of ζ_0 by rotation invariance. One can write by (III.3.24) and (III.3.21) that

$$c^*(\alpha) \int\limits_{G(i,d)} [\eta^\perp, \zeta]^\alpha g(\eta)\, d\eta = \int\limits_{G(j,d)} [\xi^\perp, \zeta]^\alpha \left(R_{ij} g\right)(\xi)\, d\xi, \tag{III.3.29}$$

which together with the definition of $\tilde{T}^\alpha_{ij}$ completes the proof. □

3.5 Bounds for the weighted images of Radon transforms

For $\alpha > 0$ introduce the following functionals on the space of absolute integrable functions g on $G(i,d)$ and φ on $G(j,d)$:

$$\|g\|_{(\alpha)} \stackrel{def}{=} \left| \int\limits_{G(i,d)} g(\eta)[\eta, \zeta_0]^\alpha \, d\eta \right|,$$

$$\|\varphi\|^\perp_{(\alpha)} \stackrel{def}{=} \left| \int\limits_{G(j,d)} \varphi(\xi)[\xi^\perp, \zeta_0]^\alpha \, d\xi \right|.$$

Let $\|\cdot\|_p$ denote the usual norm in L^p–spaces. This paragraph will be devoted to the construction of the inequalities that would bound the weighted images of Radon transforms and their duals from above.

Proposition III.3.6. *Choose the numbers $p, q > 1$ such that $1/p + 1/q = 1$.*

1) Let $i < j$, $i + k \geqslant d$, $\alpha > 0$, and $\varphi \in L^p\left(G(j,d)\right)$. Then

$$\|R_{ji}\varphi\|_{(\alpha)} \leqslant d(\alpha, q)\|\varphi\|_p \tag{III.3.30}$$

where

$$d(\alpha, q) = c(\alpha) \cdot \left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{j-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{j-l}{2}\right)\Gamma\left(\frac{d-l+\alpha+q}{2}\right)} \right)^{1/q} \tag{III.3.31}$$

and $c(\alpha)$ is defined by (III.3.3);

2) Let $i < j \leqslant k < d$, $\alpha > 0$, and $g \in L^p\left(G(i,d)\right)$. Then

$$\|R_{ij} g\|^\perp_{(\alpha)} \leqslant d^*(\alpha, q)\|g\|_p \tag{III.3.32}$$

where

$$d^*(\alpha, q) = c^*(\alpha) \cdot \left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right)\Gamma\left(\frac{d-i-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{d-i-l}{2}\right)\Gamma\left(\frac{d-l+\alpha+q}{2}\right)} \right)^{1/q} \tag{III.3.33}$$

and $c^*(\alpha)$ is defined by (III.3.22).

Proof. First let us prove the upper bound for the image of the dual Radon transform: we have by (III.3.25) that

$$\|R_{ji}\varphi\|_{(\alpha)} \leqslant c(\alpha) \cdot |||\varphi|||_{(\alpha)}.$$

Then applying Hölder's inequality one gets

$$|||\varphi|||_{(\alpha)} \leqslant \left(\int\limits_{G(j,d)} |\varphi(\xi)|^p \, d\xi \right)^{1/p} \left(\int\limits_{G(j,d)} [\xi, \zeta_0]^{\alpha+q} \, d\xi \right)^{1/q} = \|\varphi\|_p \cdot b^{1/q}_{\alpha+q}(j,d,k),$$

while by (III.3.20)

$$b^{1/q}_{\alpha+q}(j,d,k) = \left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right) \Gamma\left(\frac{j-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{j-l}{2}\right) \Gamma\left(\frac{d-l+\alpha+q}{2}\right)} \right)^{1/q}.$$

The proof of the second statement of the proposition is conducted analogously: by relation (III.3.29)

$$\|R_{ij} g\|^{\perp}_{(\alpha)} \leqslant c^*(\alpha) \cdot |||g|||^{\perp}_{(\alpha)}.$$

By Hölder's inequality

$$|||g|||^{\perp}_{(\alpha)} \leqslant \left(\int\limits_{G(i,d)} |g(\eta)|^p \, d\eta \right)^{1/p} \left(\int\limits_{G(i,d)} [\eta^{\perp}, \zeta_0]^{\alpha+q} \, d\eta \right)^{1/q} =$$

$$\|g\|_p \left(\int\limits_{G(d-i,d)} [\nu, \zeta_0]^{\alpha+q} \, d\nu \right)^{1/q} = \|g\|_p \cdot b^{1/q}_{\alpha+q}(d-i,d,k),$$

where by (III.3.20)

$$b^{1/q}_{\alpha+q}(d-i,d,k) = \left(\prod_{l=0}^{d-k-1} \frac{\Gamma\left(\frac{d-l}{2}\right) \Gamma\left(\frac{d-i-l+\alpha+q}{2}\right)}{\Gamma\left(\frac{d-i-l}{2}\right) \Gamma\left(\frac{d-l+\alpha+q}{2}\right)} \right)^{1/q}.$$

□

Remark III.3.2. *All results of section 3 can be reformulated for complex α. But several technical difficulties arise here because of the singularities of the kernel $[\xi, \eta]^{\alpha}$ for $Re\alpha < 0$. Indeed, one needs to prove even the existence of operators T^{α}_{ij}, which seems to be not trivial. It is very likely that for complex α the integral formulae of this section could be proved only for measures θ that are absolutely continuous with respect to the Haar measure on the appropriate Grassmannian. A proper normalization of operators T^{α}_{ij} by some complex coefficients depending on the gamma functions of α will be necessary to make this family of operators analytic on α. To avoid these complications we consider only real positive α here.*

Chapter IV

An inverse problem for the roses of intersections

Suppose Φ_k^d is a stationary k–flat process with unit intensity intersected with any r–flat η, $r = d - k + j$. In this chapter we prove the formulae that yield its directional distribution θ from the rose of intersections $T_{kr}\theta$.

1 The rose of intersections and the directional distribution

First of all, the following natural question is to be answered: does the rose of intersections $T_{k,d-k+j}\theta$ of Φ_k^d determine the directional distribution θ uniquely? In other words, for which k and j the generalized cosine transform $T_{k,d-k+j}$ is injective on $\mathbf{L}$? Introduce the set $V_f(d,k,j)$ of all probability measures θ_0 on $\mathfrak{G}$ such that $(T_{k,d-k+j}\theta_0)(\eta) = f(\eta)$ for all η. Then the uniqueness would imply $\mid V_f(d,k,j) \mid = 1$. One can distinguish the following particular cases for any dimension d:

- $j = 0$ ($\Phi_k^d \cap \eta$ is an ordinary point process):
  - $k = d - 1$ or 1: $\mid V_f(d,k,j) \mid = 1$ (G. Matheron, 1975, cf. [51]).
  - $2 \leqslant k \leqslant d - 2$: $V_f(d,k,j)$ is infinite dimensional (P. Goodey, R. Howard, 1990, cf. [14], [15]).
- $1 \leqslant j < k \leqslant d - 1$ (P. Goodey, R. Howard, 1990):
  - $k < d - 1$: $V_f(d,k,j)$ is infinite dimensional.
  - $k = d - 1$: $\mid V_f(d,k,j) \mid = 1$.

See also [23] for the description of the subspaces of injectivity of T_{kr} for $k < d - 1$.

As $\theta(\mathcal{A}) = \int\limits_{G(k,d)} I_{\mathcal{A}}(\xi)\,\theta(d\xi)$ for any $\mathcal{A} \in \mathfrak{G}$ and $I_{\mathcal{A}}(\cdot)$ can be approximated by smooth functions one can easily see that any measure θ is uniquely determined by all integrals

$$\int\limits_{G(k,d)} g(\xi)\,\theta(d\xi) \qquad \text{(IV.1.1)}$$

where g belongs to some sufficiently large class of functions, say $C^p(G(k,d))$, $p \in \mathbb{N} \cup \{\infty\}$ or $C(G(k,d))$. J. Mecke [52] provides an easy integral retrieval formula for fiber processes on the plane ($d = 2$, $k = r = 1$) that expresses (IV.1.1) through the integrals of $T_{11}\theta$, while J. Mecke and W. Nagel [58] obtain a sort of expansion formula in spherical harmonics for the case $d = 3$, $k = 1$, $r = 2$. In both cases $T_{kr}\theta$ is injective.

In what follows we shall generalize these results ($k = d-1$) for arbitrary dimensions d and r and also consider one case of non–uniqueness $d = 4$, $k = r = 2$. Here the whole set $V_f(4,2,0)$ will be described. The choice of parameters k and r could be explained by the fact that only in these cases the appropriate Grassmannian $G(k,d)$ is isomorphic to a sphere (or a product of spheres) (see section 2 of chapter I), and by this means the standard methods of harmonic analysis on the sphere are applicable. All other cases of non–uniqueness remain still open.

2 Inversion formulae for $k = d - 1$, $r = 1$

Let Φ^d_{d-1} be a stationary hyperplane process with unit intensity, directional distribution θ and rose of intersections with lines $T_{d-1,1}\theta = f$.

Proposition IV.2.1. *If $\theta^{\perp}(\cdot)$ is absolutely continuous with respect to $\omega_d(\cdot)$ with density $\gamma \in C_e(\mathbf{S}^{d-1})$ then*

$$\gamma(v) = c_R^2 \left(\frac{d}{d(\mu^2)}\right)^{d-2} \left[\int\limits_{<u,v>^2>\mu^2} \frac{\Box f(u)\;|<u,v>|}{(<u,v>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(du)\right]\Bigg|_{\mu=0} \qquad \text{(IV.2.1)}$$

where c_R^2 is given by (III.2.8).

Proof. We apply formula (III.2.7) for $g = \gamma$ and then note that in view of (III.2.10) and provided that $f = T\gamma$ we get the above result. □

Relation (IV.2.1) in three dimensions can be found in the book of A. V. Pogorelov [67].

Theorem IV.2.1. *Let Φ^d_{d-1} be a stationary hyperplane process with arbitrary directional distribution measure θ and rose of intersections with lines*

f. Then for any $g \in C_e^m\left(\mathbf{S}^{d-1}\right)$, $m \geqslant (d+5)/2$, and dimension $d \geqslant 3$ the following formula holds:

$$\int\limits_{\mathbf{S}^{d-1}} g(v)\,\theta^{\perp}(dv) = c_R^2 \times$$

$$\times \int\limits_{\mathbf{S}^{d-1}} f(u)\,\Box\left(\left(\frac{d}{d(\mu^2)}\right)^{d-2}\left[\int\limits_{<u,v>^2>\mu^2} \frac{g(v)\,|<u,v>|}{(<u,v>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(dv)\right]\Bigg|_{\mu=0}\right)\omega_d(du)$$

where c_R^2 is given by (III.2.8).

Proof. By theorem 4.1 in [20], for any $g \in C_e^m\left(\mathbf{S}^{d-1}\right)$ there exists an integrable function h on $\mathbf{S}^{d-1}$ such that

$$g = Th. \tag{IV.2.2}$$

Then by Fubini's theorem

$$\int\limits_{\mathbf{S}^{d-1}} g(v)\,\theta^{\perp}(dv) = \int\limits_{\mathbf{S}^{d-1}} (Th)\,(v)\,\theta^{\perp}(dv) = \int\limits_{\mathbf{S}^{d-1}}\int\limits_{\mathbf{S}^{d-1}} |<u,v>|h(u)\,\theta^{\perp}(dv)\omega_d(du) =$$

$$\int\limits_{\mathbf{S}^{d-1}} h(u)\left(T\theta^{\perp}\right)(u)\,\omega_d(du) = \int\limits_{\mathbf{S}^{d-1}} h(u)f(u)\,\omega_d(du).$$

Then as $\Box$ and R^{-1} commute and by (III.2.10) we have from (IV.2.2) that

$$h = T^{-1}g = R^{-1}\Box g = \Box R^{-1} g.$$

Then using lemma III.2.1 one gets

$$\int\limits_{\mathbf{S}^{d-1}} g(v)\,\theta^{\perp}(dv) = \int\limits_{\mathbf{S}^{d-1}} f(u)\Box R^{-1} g(u)\,\omega_d(du) =$$

$$c_R^2 \int\limits_{\mathbf{S}^{d-1}} f(u)\,\Box\left(\left(\frac{d}{d(\mu^2)}\right)^{d-2}\left[\int\limits_{<u,v>^2>\mu^2} \frac{g(v)\,|<u,v>|}{(<u,v>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(dv)\right]\Bigg|_{\mu=0}\right)\omega_d(du),$$

and we are done. □

Remark IV.2.1. *A result similar to theorem IV.2.1 was obtained by W. Weil (cf. [88], [22]) in the setting of distributions. Namely, the support function of a generalized zonoid is the cosine transform of a finite signed measure. Weil generalized this idea and introduced the generating distribution for any centrally symmetric compact convex body in $\mathbb{R}^d$. It was shown in [88] how this distribution can be restored from its cosine transform.*

Remark IV.2.2 (Case $k=1$, $r=d-1$). *All results of this section can be applied directly to the dual case of a stationary line process Φ_1^d intersected with hyperplanes: $r=d-1$.*

Remark IV.2.3. *One can also use other inversion formulae for the spherical Radon transform (cf. [31], [32] p. 186–187) in order to prove relations similar to those of proposition IV.2.1 and theorem IV.2.1 (see [79]). These inversion formulae involve certain polynomials of the Beltrami–Laplace operator, if d is even, and for odd dimensions they can be written in terms of fractional integrals and wavelets (cf. [69]–[71]).*

3 Inversion formulae for $k=d-1$, $1<r\leqslant d-1$

Consider a stationary hyperplane process Φ_{d-1}^d with unit intensity, directional distribution θ and rose of intersections with r–planes $T_{d-1,r}\theta=f$. The results of the previous section allow us to get θ (or the density γ of $\theta^\perp$ on $\mathbf{S}^{d-1}$ with respect to $\omega_d(\cdot)$) if we know $T\theta^\perp$ ($T\gamma$, respectively). Now let us rewrite these expressions in terms of f. It follows from (III.3.23) that

$$T\theta^\perp=\frac{2k_{r-1}}{\omega_r}R_r^{-1}f.$$

Then applying inversion formula (III.2.3) for the spherical Radon transform of order r one gets

$$\left(T\theta^\perp\right)(v)=\frac{2k_{r-1}c_R}{\omega_r}\times$$
$$\times\left(\frac{d}{d(x^2)}\right)^{r-1}\left[\int_0^x y^{r-1}(x^2-y^2)^{\frac{r-3}{2}}\int_{d(\mathbf{S}_\eta^{r-1},\,v)=\arccos y} f(\eta)\,\sigma(d\eta)\,dy\right]\Bigg|_{x=1}. \quad \text{(IV.3.1)}$$

Now substituting $T\theta^\perp$ from the above relation for f in the results of section 2 we get the same scope of retrieval formulae for the case $1<r\leqslant d-1$:

Theorem IV.3.1. *Let Φ_{d-1}^d be a stationary hyperplane process with directional distribution θ and rose of intersections with r–planes f, $1<r\leqslant d-1$, $d\geqslant 3$. Then $T\theta^\perp$ can be determined using (IV.3.1) and we have*

1) *If $\theta^\perp$ is absolutely continuous with respect to $\omega_d(\cdot)$ with density $\gamma\in C_e(\mathbf{S}^{d-1})$ then*

$$\gamma(v)=c_R^2\left(\frac{d}{d(\mu^2)}\right)^{d-2}\left[\int_{<u,v>^2>\mu^2}\frac{\square\left(T\theta^\perp\right)(u)\,|<u,v>|}{(<u,v>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(du)\right]\Bigg|_{\mu=0};$$

2) For arbitrary probability measure θ on $G(d-1,d)$ and any $g \in C_e^m\left(\mathbf{S}^{d-1}\right)$, $m \geqslant (d+5)/2$

$$\int\limits_{\mathbf{S}^{d-1}} g(v)\,\theta^{\perp}(dv) = c_R^2 \times$$

$$\times \int\limits_{\mathbf{S}^{d-1}} \left(T\theta^{\perp}\right)(u)\,\square \left(\frac{d}{d(\mu^2)}\right)^{d-2} \left[\int\limits_{<u,v>^2>\mu^2} \frac{g(v)\,|<u,v>|}{(<u,v>^2 - \mu^2)^{2-\frac{d}{2}}}\,\omega_d(dv)\right]\Bigg|_{\mu=0} \omega_d(du)$$

where c_R^2 is constant (III.2.8).

Remark IV.3.1. *In the case $k = r = d-1$ the fact that $G(d-1,d) \cong \mathbf{S}_+^{d-1}$ and the Haar measure is just the normalized surface area measure on the sphere makes relation (IV.3.1) simpler (cf. [79]).*

4 Inversion via expansions in spherical harmonics

The following section will be devoted to obtaining the inversion formulae for $T_{d-1,1}$ and $T_{d-1,d-1}$ by means of expansions in spherical harmonics. To this end already known results about the eigenvalues of cosine and sine transforms (cf. [28]) will be used.

Introduce the scalar product $<g,h>_{\mathbf{S}^{d-1}} = \int\limits_{\mathbf{S}^{d-1}} g(u)h(u)\,\omega_d(du)$ of any real functions g and h from $L^2(\mathbf{S}^{d-1})$. Let

$$\{S_{n,j} : n \in \mathbb{Z}_+, \quad j = 1, \ldots, N(d,n)\}$$

be an orthonormal basis of spherical harmonics on $\mathbf{S}^{d-1}$ in the usual norm $\|\cdot\|_{\mathbf{S}^{d-1}}$ in $L^2(\mathbf{S}^{d-1})$ (see [65], [5], [27], [28]),

$$N(d,n) = \begin{cases} \frac{(2n+d-2)\Gamma(n+d-2)}{\Gamma(n+1)\Gamma(d-1)}, & n \geqslant 1 \\ 1, & n = 0. \end{cases} \tag{IV.4.1}$$

For any integrable function g: $\int\limits_{\mathbf{S}^{d-1}} |g(u)|\,\omega_d(du) < \infty$ there exists its expansion in spherical harmonics:

$$g(u) \sim \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} c_{nj}\,S_{n,j}(u) \tag{IV.4.2}$$

where

$$c_{nj} = <g, S_{n,j}>_{\mathbf{S}^{d-1}} \tag{IV.4.3}$$

We know that any measure μ on $\mathbf{S}^{d-1}$ is defined by the values of its integrals $\int_{\mathbf{S}^{d-1}} g(u)\,\mu(du)$ for all $g \in C_e^\infty(\mathbf{S}^{d-1})$. Let g have expansion (IV.4.2). Then because of the uniform convergence of this series to g (cf. [28]) we have

$$\int_{\mathbf{S}^{d-1}} g(u)\,\mu(du) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} c_{nj}\,B_{nj}$$

where

$$B_{nj} = \int_{\mathbf{S}^{d-1}} S_{n,j}(u)\,\mu(du). \tag{IV.4.4}$$

Therefore is it sufficient to know all B_{nj} to get a complete description of μ. If $\mu = \theta^\perp$ is the directional distribution measure of a hyperplane process and f its rose of intersections with lines or hyperplanes then B_{nj} can be determined from f, i.e. from its expansion coefficients $b_{nj} = <f, S_{nj}>_{\mathbf{S}^{d-1}}$.

The following result is a direct corollary from lemmas 3.4.5, 3.4.7 [28]:

Proposition IV.4.1 (Intersections with lines and hyperplanes). *Let Φ_{d-1}^d be a stationary hyperplane process with directional distribution θ and rose of intersections with r–planes $T_{d-1,r}\theta = f$ ($r \in \{1, d-1\}$). Then for any $g \in C_e^\infty(\mathbf{S}^{d-1})$ with expansion in spherical harmonics (IV.4.2)*

$$\int_{\mathbf{S}^{d-1}} g(u)\,\theta^\perp(du) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} c_{nj}\,B_{nj}$$

where

$$B_{nj} = \left\{ \begin{array}{ll} 0, & n \text{ odd} \\ \frac{<f,S_{nj}>_{\mathbf{S}^{d-1}}}{\omega_{d-1}\,a_n}, & n \text{ even} \end{array} \right| , \quad j = 1, \ldots, N(d,n).$$

The value of a_n is

1) in case of lines ($r = 1$):

$$a_n = \left\{ \begin{array}{ll} \frac{2}{d-1}, & n = 0 \\ \frac{2}{(d-1)(d+1)}, & n = 2 \\ 2(-1)^{\frac{n-2}{2}} \frac{1\cdot 3\cdot \ldots \cdot (n-3)}{(d-1)(d+1)\ldots(d+n-1)}, & \text{for even } n \geqslant 4 \end{array} \right. ,$$

2) in case of hyperplanes ($r = d-1$):

$$a_n = -\frac{(d-2)!}{2} \frac{\Gamma\left(\frac{n+d}{2}\right)\Gamma\left(\frac{n-1}{2}\right)n!}{\Gamma\left(\frac{n+d+1}{2}\right)(n+d-2)!\,(n/2)!} \quad \text{for even } n \geqslant 0.$$

5 Case $G(2,4)$

Let us proceed now to the case $G(2,4)$. Suppose the stationary process Φ_2^4 is intersected by a 2–plane $\eta \in G(2,4)$. By theorem I.2.1, the isomorphism M (cf. (I.2.5)) maps $G(2,4)$ onto

$$\{(u,v) \in \mathbf{S}^2 \times \mathbf{S}^2 : (u,v) \equiv (-u,-v)\}.$$

Then for all ξ, $\eta \in G(2,4)$ $\xi \mapsto (u,v)$, $\eta \mapsto (\tilde{u},\tilde{v})$, where u, v, $\tilde{u}$, $\tilde{v} \in \mathbf{S}^2$. If θ is the directional distribution of Φ_2^4 (i.e. a probability measure on $G(2,4)$) then one can prove that its image under the isomorphism M is again a probability measure $\tilde{\theta}$ on $\mathbf{S}^2 \times \mathbf{S}^2$. By (I.3.1), the rose of intersections of Φ_2^4 is

$$(T_{22}\theta)(\eta) = f(\tilde{u},\tilde{v}) = \frac{1}{2} \int\limits_{\mathbf{S}^2 \times \mathbf{S}^2} |<u,\tilde{u}> - <v,\tilde{v}>| \; \tilde{\theta}\big(d(u,v)\big). \qquad \text{(IV.5.1)}$$

It will be shown in the following paragraphs that for absolute continuous measure θ with density g, i.e. when $\theta(d\xi) = g(\xi)\, d\xi$, its image $\tilde{\theta}$ has the following form: $\tilde{\theta}\big(d(u,v)\big) = \frac{1}{16\pi^2} g\left(M^{-1}(u,v)\right) \omega_3(du)\omega_3(dv)$. The key point here is to prove that the Haar measure on $\mathcal{L}(2,4)$ is mapped into the product of surface area measures on $\mathbf{S}^2 \times \mathbf{S}^2$. Two independent proofs of this fact (see theorem IV.5.1) will be given in §5.2 and §5.5. The idea of the first one belongs to Prof. B. Rubin. The second proof is due to Prof. J. Mecke. Then the inversion formula for T_{22} in its parametric representation (IV.5.1) on $\mathbf{S}^2 \times \mathbf{S}^2$ will be given in §5.6.

In the sequel we shall use θ instead of $\tilde{\theta}$ without abuse of notation.

5.1 The Haar measure on $\mathcal{L}(2,4)$

In what follows we shall study the structure of the Haar measure on the oriented Grassmannian $\mathcal{L}(2,4)$. Namely, we shall give two proofs of the following result:

Theorem IV.5.1. *The image of the normalized Haar measure on* $\mathcal{L}(2,4)$ *under the homeomorphism* $\mathcal{L}(2,4) \to \mathbf{S}^2 \times \mathbf{S}^2$ *is the measure*

$$\frac{1}{16\pi^2}\,\omega_3(du)\omega_3(dv),$$

i.e. the normalized product of the surface area measures on each sphere.

The first proof involves the exact form of the Haar measure on $SO(3)$ given by integration with respect to the Euler angles (cf. [87], p. 11, 23).

5.2 The first proof of the theorem IV.5.1

In this proof we shall find another homeomorphism $\mathcal{L}(2,4) \to \mathbf{S}^2 \times \mathbf{S}^2$ (different to M) such that the assertion of theorem IV.5.1 is true.

Let $Stab(\xi)$ be the *stabilizer* of subspace $\xi \subset \mathbb{R}^d$, i.e. the subgroup of $SO(d)$ such that for any rotation $\kappa \in Stab(\xi)$ $\kappa\xi = \xi$. Denote by $e_1, \ldots, e_4$ the Cartesian orthonormal basis in $\mathbb{R}^4$. By lemma 2.1 of [70] we have for any smooth function φ on $\mathcal{L}(2,4)$ that

$$\int\limits_{\mathcal{L}(2,4)} \varphi(\xi)\, d\xi = 2 \int\limits_0^{\pi/2} \sin\tau \cos\tau \int\limits_{Stab(e_4)} \varphi(\kappa g_\tau^{-1} \xi_0)\, d\kappa d\tau \qquad \text{(IV.5.2)}$$

where $\xi_0 =< e_1, e_2 >$, $Stab(e_4) \cong SO(3)$, $d\kappa$ is the Haar measure on $SO(3)$ normalized by unity, and g_τ^{-1} is the following rotation in the plane $< e_2, e_4 >$:

$$g_\tau^{-1} = \begin{pmatrix} \sin\tau & -\cos\tau \\ \cos\tau & \sin\tau \end{pmatrix}. \qquad \text{(IV.5.3)}$$

Without loss of generality we can identify φ with $\varphi^\perp$ and ξ_0 with $\xi_0^\perp =< e_3, e_4 >$. Then by (IV.5.3)

$$\kappa g_\tau^{-1} e_3 = \kappa e_3, \quad \kappa g_\tau^{-1} e_4 = -\kappa e_2 \cos\tau + e_4 \sin\tau,$$

and

$$\varphi(\kappa g_\tau^{-1} \xi_0) = \varphi(< \kappa e_3, -\kappa e_2 \cos\tau + e_4 \sin\tau >).$$

By [87], p. 23 the Haar measure on $SO(3)$ has the following representation:

$$d\kappa = \frac{1}{8\pi^2} \sin\gamma\, d\gamma d\beta d\alpha$$

in the parameterization by the *Euler angles* $0 \leqslant \alpha, \beta < 2\pi$, $0 \leqslant \gamma < \pi$. Then relation (IV.5.2) rewrites

$$\int\limits_{\mathcal{L}(2,4)} \varphi(\xi)\, d\xi = \frac{1}{4\pi^2} \int\limits_0^{\pi/2} \int\limits_0^{2\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi} \tilde\varphi(\alpha,\beta,\gamma,\tau) \sin\gamma \sin\tau \cos\tau\, d\gamma d\beta d\alpha d\tau =$$

$$\frac{1}{16\pi^2} \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi} \tilde\varphi(\alpha,\beta,\gamma,\tau_1/2) \sin\gamma \sin\tau_1\, d\gamma d\alpha d\tau_1 d\beta =$$

$$\frac{1}{16\pi^2} \int\limits_{\mathbf{S}^2 \times \mathbf{S}^2} \bar\varphi(u,v)\, \omega_3(du) \omega_3(dv)$$

where

$$\tilde\varphi(\alpha,\beta,\gamma,\tau) = \varphi(< \kappa(\alpha,\beta,\gamma) e_3, -\kappa(\alpha,\beta,\gamma) e_2 \cos\tau + e_4 \sin\tau >)$$

and (α,β,γ) are the Euler angles of the rotation κ, $\tau_1 = 2\tau$, and u, v are the vectors from $\mathbf{S}^2$ with spherical coordinates (α,γ) and (β,τ_1), respectively. Here $\bar\varphi$ is equal to $\tilde\varphi$ as a function of u, v. The proof is complete.

5.3 Quaternions and rotations in $\mathbb{R}^4$

In order to give the second proof of theorem IV.5.1 we need some facts from the theory of quaternions (see [2], [68], [30]). Let

$$\mathbb{K}^4 = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_l \in \mathbb{R}, \quad l = 1, \dots, 4\}$$

be the skew field of all quaternions in $\mathbb{R}^4$. We shall denote the real (imaginary) part of a $q \in \mathbb{K}^4$ by $\Re q \overset{def}{=} q_0$ ($\Im q \equiv \hat{q} \overset{def}{=} q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$) and the set of all purely imaginary quaternions (with $\Re q = 0$) by $\Im\left(\mathbb{K}^4\right)$. There is an obvious isomorphism between $\mathbb{K}^4$ and the vector space $\mathbb{R}^4$:

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \longleftrightarrow \vec{q} = (q_0, q_1, q_2, q_3).$$

Later on we shall identify $\vec{q}$ and $q \in \mathbb{K}^4$ without abuse of notation. It will be clear from the context which interpretation of it we use. Introduce the *quaternion product* of $x, y \in \mathbb{K}^4$:

$$x \cdot y = x_0 y_0 + x_0\hat{y} + y_0\hat{x} - <\hat{x}, \hat{y}> + [\hat{x}, \hat{y}]_0 \tag{IV.5.4}$$

where $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{R}^d$ (here $d = 3$), $[\cdot, \cdot]_0$ is the vector product in $\mathbb{R}^3$, and $\hat{x}$, $\hat{y}$ are understood as vectors in $\mathbb{R}^3$ with basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Introduce the *conjugate* to a quaternion q: $\bar{q} \overset{def}{=} q_0 - \hat{q}$, the *absolute value* $|q| = \sqrt{<q, q>}$, the *inverse* $q^{-1} = \bar{q}/|q|^2$ and the subset

$$\mathbb{H} = \{q \in \mathbb{K}^4 : |q| = 1\}.$$

Then one can prove that

$$<x, y> = \frac{1}{2}(x \cdot \bar{y} + y \cdot \bar{x}), \quad x, y \in \mathbb{K}^4.$$

It is also known that every rotation $A \in SO(4)$ considered to act on $\mathbb{K}^4$ can be presented in the form

$$Ax = a \cdot x \cdot b, \quad x \in \mathbb{K}^4 \tag{IV.5.5}$$

for some $a, b \in \mathbb{H}$. The above representation is unique up to the change of sign of a, b: $\{a, b\}$ and $\{-a, -b\}$ form the same rotation. Every rotation $B \in SO(3)$ understood to act on the three–dimensional space of imaginary quaternions $\Im\left(\mathbb{K}^4\right)$ can be expressed in the form

$$Bx = b \cdot x \cdot b^{-1}, \quad x \in \mathbb{K}^4 \tag{IV.5.6}$$

for some $b \in \mathbb{H}$. The above representation is unique up to the change of sign of b.

5.4 Quaternionic coordinates for $\mathcal{L}(2,4)$

Let us find a good description of any 2–flat from $\mathcal{L}(2,4)$ by means of quaternions. Let a 2–flat $\xi \in \mathcal{L}(2,4)$ be spanned over the oriented orthonormal basis $\{x,y\}$ in $\mathbb{R}^4$: $|x| = |y| = 1$, $<x,y>= 0$. We shall consider x and y as quaternions and write later on $\xi = L\{x,y\}$, $x, y \in \mathbb{H}$. Let $\xi_0 = L\{1,\mathbf{i}\}$ be the oriented coordinate 2–plane. And as each $\xi \in \mathcal{L}(2,4)$ can be obtained from ξ_0 by rotation one can write by (IV.5.5) that $\xi = L\{ab, a\mathbf{i}b\}$ for some $a, b \in \mathbb{H}$ and thus

$$\mathcal{L}(2,4) = \{L\{ab, a\mathbf{i}b\} : a, b \in \mathbb{H}\}\,. \tag{IV.5.7}$$

Note that the representation of $\xi \in \mathcal{L}(2,4)$ by $\xi = L\{ab, a\mathbf{i}b\}$ is not unique. Let us prove the following result:

Proposition IV.5.1. *For every 2–plane $\xi = L\{ab, a\mathbf{i}b\} \in \mathcal{L}(2,4)$, $a, b \in \mathbb{H}$, the pair $\{\bar{b}\mathbf{i}b, a\mathbf{i}\bar{a}\}$ does not depend on the special choice of a, b (quaternionic coordinates).*

Proof. It is clear that for each 2–flat ξ such coordinates exist, and vice versa, for each pair $\{\bar{b}\mathbf{i}b, a\mathbf{i}\bar{a}\}$ there exists a plane $\xi \in \mathcal{L}(2,4)$ with these coordinates. Now it should be shown that the above pair does not depend on the choice of rotation that maps ξ_0 to ξ, i.e., on quaternions a and b. Namely, we shall prove that if $\xi = L\{ab, a\mathbf{i}b\} = L\{cd, c\mathbf{i}d\}$ for some $a, b, c, d \in \mathbb{H}$ then $a\mathbf{i}\bar{a} = c\mathbf{i}\bar{c}$, $\bar{b}\mathbf{i}b = \bar{d}\mathbf{i}d$.

First let us describe all rotations $x \mapsto cxd$, $c, d \in \mathbb{H}$ that map $L\{1,\mathbf{i}\}$

elf preserving orientation. It means that $L\{1,\mathbf{i}\} = L\{cd, c\mathbf{i}d\}$, or that
d lie in $L\{1,\mathbf{i}\}$. Since $cd \perp c\mathbf{i}d$ and they are the linear combination
nd $\mathbf{i}$, then they differ from each other by the factor $\mathbf{i}$: $\mathbf{i}cd = c\mathbf{i}d$ (the
$\{cd, c\mathbf{i}d\}$ should have the same orientation as $\{1,\mathbf{i}\}$). Dividing it by
have that c and $\mathbf{i}$ commute. This means that c_2 and c_3 vanish in
$+ c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, or equivalently, $c \in L\{1,\mathbf{i}\}$. And as $cd, c \in L\{1,\mathbf{i}\}$
ve (again by division) that d also belongs to ξ_0.
ow let us return to the initial problem. Namely, let two rotations
$\mapsto axb$ and $B : x \mapsto cxd$ map the same 2–flat ξ_0 in ξ. It means that
$\mapsto \bar{c}axb\bar{d}$ maps ξ_0 to itself. In detail, $B^{-1}A\xi_0 = L\{\bar{c}ab\bar{d}, \bar{c}a\mathbf{i}b\bar{d}\}$
s that $\bar{c}a$ and $b\bar{d}$ commute with $\mathbf{i}$:

$$\bar{c}a\mathbf{i} = \mathbf{i}\bar{c}a,$$

$$b\bar{d}\mathbf{i} = \mathbf{i}b\bar{d}.$$

plies the first of the above equalities by c on the left and by
ht and the second equality by $\bar{b}$ on the left and by d on the
ts the desired uniqueness of coordinates, and the proposition is

□

Let us denote the quaternionic coordinates of a flat $\xi = L\{ab, aib\}$ by u and v:

$$\begin{cases} u &= \bar{b}ib, \\ v &= ai\bar{a}. \end{cases} \tag{IV.5.8}$$

Evidently $\bar{u} = -u$, $\bar{v} = -v$, which gives us $\Re u = \Re v = 0$, or

$$u, v \in \Im\left(\mathbb{K}^4\right) \cong \mathbb{R}^3.$$

And as $u, v \in \mathbb{H}$ then considered as vectors in $\mathbb{R}^3$ they belong to $\mathbf{S}^2$. Thus we proved once again (cf. theorem I.2.1) that there exists an isomorphism between $\mathcal{L}(2,4)$ and $\mathbf{S}^2 \times \mathbf{S}^2$.

Proposition IV.5.2. *Let ξ be a 2–plane with quaternionic coordinates u, v spanned by the orthogonal unit vectors x and y. Then the following formulas connecting quaternions u, v and vectors x, y are true:*

$$\begin{cases} u &= \frac{1}{2}(\bar{x}y - \bar{y}x), \\ v &= \frac{1}{2}(y\bar{x} - x\bar{y}). \end{cases} \tag{IV.5.9}$$

These coordinate relations coincide with (I.2.4) if one supposes $\{a_i\}$ and $\{b_i\}$ to be the Cartesian coordinates of the vectors x, y.

Proof. One can write by (IV.5.7) that $x = ab$, $y = aib$ for some $a, b \in \mathbb{H}$. Then $u = \bar{x}y = -\bar{y}x$, $v = y\bar{x} = -x\bar{y}$ and (IV.5.9) is proved. In order to verify the second statement of the proposition one should just write down formulas (IV.5.9) in Cartesian coordinates of x and y and then using (IV.5.4) check the stated coincidence. □

Now we are ready to give the second proof of theorem IV.5.1.

5.5 Mecke's proof of theorem IV.5.1

The Haar measure on $\mathcal{L}(2,4)$ is by definition the unique (up to a constant factor) measure on this Grassmannian invariant under the action of the group $SO(4)$. The transformation group $SO(4)$ acting on $\mathcal{L}(2,4)$ generates by means of the mapping of $\mathcal{L}(2,4)$ onto $\mathbf{S}^2 \times \mathbf{S}^2$ the appropriate transformation group on $\mathbf{S}^2 \times \mathbf{S}^2$. Let us explore its properties.

Take an arbitrary rotation $A \in SO(4)$: $x \mapsto qxp$ for some $p, q \in \mathbb{H}$. Let ξ be a 2–flat with quaternionic coordinates u, v. There exist quaternions $a, b \in \mathbb{H}$ such that by (IV.5.8) $u = \bar{b}ib$, $v = ai\bar{a}$. Then $\xi = L\{ab, aib\}$. The image $A\xi$ is equal to $L\{qabp, qaibp\}$, thus the quaternionic coordinates of $A\xi$ are $\tilde{u} = \bar{p}up$, $\tilde{v} = qv\bar{q}$. By (IV.5.6) they are images under two rotations on two spheres $\mathbf{S}^2$. Thus we have proved that any rotation A acts on $\mathbf{S}^2 \times \mathbf{S}^2$ as a product of two rotations from $SO(3)$. Then the image of the rotation invariant measure on $\mathbf{S}^2 \times \mathbf{S}^2$ should not change under all rotations on the factor spheres. The measure $\omega_3(du)\omega_3(dv)$ obviously satisfies this condition.

Then due to the uniqueness (up to a constant factor) of the rotation invariant measure $d\xi$ on $\mathcal{L}(2,4)$ the measure

$$\frac{\omega_3(du)\omega_3(dv)}{(\omega_3)^2}$$

is its image on $\mathbf{S}^2 \times \mathbf{S}^2$. Theorem IV.5.1 is proved.

Remark IV.5.1. *Since by proposition IV.5.2 the quaternionic approach described above coincides coordinatewise with that of §2.2 of chapter I, it preserves the view of the integral kernel* $[\xi,\eta]$*:*

$$[\xi,\eta] = 1/2| < u,\tilde{u} > - < v,\tilde{v} > |$$

where $\{u,v\}$ *and* $\{\tilde{u},\tilde{v}\}$ *are the quaternionic coordinates of* ξ *and* η*.*

5.6 The structure of $V_f(2,4,0)$

Consider $\{S_{n,j} : n \in \mathbb{Z}_+, \quad j = 1,\dots,N(3,n)\}$ — an orthonormal basis of spherical harmonics on $\mathbf{S}^2$, $N(3,n) = 2n+1$. The generalized cosine transform of the directional distribution θ of Φ_2^4 (its rose of intersections with 2–flats) $(T_{22}\theta)(\eta) = f(\tilde{u},\tilde{v})$ can be expanded in this system of $S_{n,j}$ as a function of two independent variables (cf. (IV.5.1)):

$$f(\tilde{u},\tilde{v}) \sim \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} b_{njki}\, S_{n,j}(\tilde{u}) S_{k,i}(\tilde{v}) \tag{IV.5.10}$$

where

$$b_{njki} = \int_{\mathbf{S}^2} \int_{\mathbf{S}^2} f(\tilde{u},\tilde{v}) S_{n,j}(\tilde{u}) S_{k,i}(\tilde{v})\, \omega_3(d\tilde{u})\omega_3(d\tilde{v}). \tag{IV.5.11}$$

One can show that $\{S_{n,j} \cdot S_{k,i}\}_{n,k,j,i}$ constitute a basis in $L^2(\mathbf{S}^2 \times \mathbf{S}^2)$. Again, we are looking for integrals

$$\int_{\mathbf{S}^2 \times \mathbf{S}^2} g(u,v)\, \theta\big(d(u,v)\big) \quad \text{for all } g \in C_e^\infty(\mathbf{S}^2 \times \mathbf{S}^2).$$

If

$$g(u,v) \sim \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} c_{njki}\, S_{n,j}(u) S_{k,i}(v) \tag{IV.5.12}$$

then

$$\int_{\mathbf{S}^2 \times \mathbf{S}^2} g(u,v)\, \theta\big(d(u,v)\big) = \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} c_{njki}\, B_{njki}$$

where

$$B_{njki} = \int\limits_{\mathbf{S}^2\times\mathbf{S}^2} S_{n,j}(u)S_{k,i}(v)\,\theta\big(d(u,v)\big). \tag{IV.5.13}$$

Hence the coefficients B_{njki} define θ completely. By Funk – Hecke's theorem (cf. [65])

$$|{<}u,\tilde{u}{>} - {<}v,\tilde{v}{>}| \sim \tag{IV.5.14}$$

$$\sum_{n,k=0}^{\infty} 4\pi^2 a_{nk} \sum_{j=1}^{2n+1}\sum_{i=1}^{2k+1} S_{n,j}(u)S_{n,j}(\tilde{u})S_{k,i}(v)S_{k,i}(\tilde{v}),$$

$$a_{nk} = \int\limits_{-1}^{1}\int\limits_{-1}^{1} |\, t - x \,|\, P_n(x)P_k(t)\,dt\,dx \tag{IV.5.15}$$

where

$$P_n(t) = \frac{1}{2^n\, n!}\left(\frac{d}{dt}\right)^n (t^2-1)^n \tag{IV.5.16}$$

are Legendre polynomials in three dimensions (cf. Rodrigues formula in [65]). Namely, $P_0(t) = 1$, $P_1(t) = t$, $P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$, etc. Function

$$|{<}u,\tilde{u}{>} - {<}v,\tilde{v}{>}|$$

is continuous on (u,v). Therefore its expansion (IV.5.14) converges to it in the sense of Abel summation (cf. [65]). Moreover, taking into account the explicit expressions for a_{nk} (cf. theorem IV.5.2) one can show that uniform convergence in (IV.5.14) takes place. Then integrating (IV.5.14) with respect to θ and interchanging integration and summation one gets

$$f(\tilde{u},\tilde{v}) \sim \sum_{n,k=0}^{\infty}\sum_{j=1}^{2n+1}\sum_{i=1}^{2k+1} 2\pi^2 a_{nk}\, B_{njki}\, S_{n,j}(\tilde{u})S_{k,i}(\tilde{v}).$$

Ergo if $a_{nk} \neq 0$ then for all $i = 1,\dots,2k+1$, $j = 1,\dots,2n+1$

$$B_{njki} = \frac{b_{njki}}{2\pi^2 a_{nk}}. \tag{IV.5.17}$$

As T_{22} is not injective (see section 1) we shall not be able to find all B_{njki}. As proved in [14], p. 102–103, all a_{nk}, $|n-k| > 2$ are equal to zero that explains the situation.

Theorem IV.5.2. *Assume that* $f : G(2,4) \to \mathbb{R}$, $f = T_{22}\theta$ *is the rose of intersections of a stationary* 2*–flat process in* $\mathbb{R}^4$. *Let* $V_f(2,4,0)$ *be the set of all directional distributions of* 2*–flat processes with rose of intersections* f.

Then, the probability measure θ is an element of $V_f(2,4,0)$ iff its coefficients (IV.5.13) satisfy the following conditions:

$$B_{njki} = B_{kinj} \text{ for all } n,\, k, \tag{IV.5.18}$$

$$B_{njki} = \begin{cases} \dfrac{\int\limits_{\mathbf{S}^2}\int\limits_{\mathbf{S}^2} f(\tilde{u},\tilde{v}) S_{n,j}(\tilde{u}) S_{k,i}(\tilde{v})\, \omega_3(d\tilde{u})\omega_3(d\tilde{v})}{2\pi^2 a_{nk}}, & |n-k| \in \{0,2\}, \\ 0, & |n-k| \text{ is odd} \end{cases}$$

where coefficients a_{nk} defined in (IV.5.15) have the following properties:

$$a_{nk} = a_{kn} \text{ for all } n,\, k,$$

$$a_{nk} = \begin{cases} \frac{8}{3}, & n = k = 0, \\ -\frac{2\,(2n)!}{2^{2n}\, n!\,(n+1+1/2)!\,(n-1/2)}, & n = k \geqslant 1, \\ -\frac{1}{2} a_{mm}, & n = m-1,\ k = m+1, \\ -\frac{1}{2} a_{mm}, & n = m+1,\ k = m-1, \\ 0, & \text{otherwise}, \end{cases}$$

$(m+\frac{1}{2})! \overset{def}{=} (1+\frac{1}{2})(2+\frac{1}{2}) \cdot \ldots \cdot (m+\frac{1}{2})$. *Then for any $g \in C_e^\infty(\mathbf{S}^2 \times \mathbf{S}^2)$ with expansion in spherical harmonics (IV.5.12)*

$$\int\limits_{\mathbf{S}^2 \times \mathbf{S}^2} g(u,v)\, \theta\big(d(u,v)\big) = \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} c_{njki}\, B_{njki}. \tag{IV.5.19}$$

Proof. The symmetry of B_{njki} on k, i and n, j is evident from relations (IV.5.13) and (I.2.2): $\theta(A,B) = \theta(B,A)$ for all spherical Borel sets A and B. Let $n+k$ be odd; making the change of variables $u \mapsto -u$, $v \mapsto -v$ in (IV.5.13) we get

$$B_{njki} = (-1)^{n+k} B_{njki} = -B_{njki}$$

by the homogeneity of spherical harmonics. Consequently $B_{njki} = 0$. Now calculate a_{nk} for even $n+k$. Clearly $a_{nk} = a_{kn}$ because of the symmetry of (IV.5.15). One can show that $a_{nk} \neq 0$ iff $|n-k| \in \{0,2\}$ (cf. [16], p. 267). It is also a consequence of the lemmas below: due to (IV.5.15), (IV.5.20), the symmetry of a_{nk} on n and k and lemma IV.5.2 $a_{nk} \neq 0$ iff $n+k$ is even and

$$\begin{cases} n+k-2 \leqslant 2k \\ n+k-2 \leqslant 2n \end{cases}$$

which yields $n-2 \leqslant k \leqslant n+2$, and in view of the fact that $n+k$ is even one gets $k = n-2, n, n+2$. One should note in this regard that the values of B_{njki} are undetermined for those n, k ($|n-k|$ even) that $a_{nk} = 0$ (cf. (IV.5.17)), i.e. if $|n-k|$ is even and not in $\{0,2\}$. These B_{njki} can be chosen freely as long as their symmetry condition (IV.5.18) holds and by (IV.5.19) a probability measure is obtained. Furthermore, calculating directly we have by (IV.5.15) and (IV.5.16)

$$a_{00} = \int_{-1}^{1}\int_{-1}^{1} |t-x| \; dt\, dx = \frac{8}{3}, \quad a_{11} = \int_{-1}^{1}\int_{-1}^{1} |t-x| \, tx\, dt\, dx = -\frac{8}{15};$$

the rest of a_{nk} will be obtained in lemmas IV.5.1, IV.5.2.

Lemma IV.5.1. *For any* $n, k \geqslant 2$

$$a_{nk} = \frac{(-1)^n}{2^{n+k-1}\, n!\, k!} \int_{-1}^{1} (x^2-1)^n \left(\frac{d}{dx}\right)^{n+k-2} (x^2-1)^k dx. \tag{IV.5.20}$$

Proof. First one should prove that for $n, k \geqslant 2$

$$a_{nk} = \frac{1}{2^{k-1}\, k!} \int_{-1}^{1} P_n(x) \left(\frac{d}{dx}\right)^{k-2} (x^2-1)^k dx. \tag{IV.5.21}$$

Then we use lemma 11 on p. 17 of [65]: for all $f \in C^n[-1,1]$

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{(-1)^n}{2^n\, n!} \int_{-1}^{1} (x^2-1)^n \left(\frac{d}{dx}\right)^{n} f(x) dx$$

to get the desired result. Let us now prove (IV.5.21):

$$a_{nk} = \int_{-1}^{1} \left[\int_{-1}^{x} (x-t) P_k(t)\, dt - \int_{x}^{1} (x-t) P_k(t)\, dt \right] P_n(x)\, dx = \tag{IV.5.22}$$

$$\int_{-1}^{1} h(x) P_n(x)\, dx$$

where $h(x)$ can be rewritten as

$$h(x) = x \left(\int_{-1}^{x} P_k(t)\, dt + (-1)^{k+1} \int_{-1}^{-x} P_k(t)\, dt \right) -$$

$$\left(\int_{-1}^{x} t P_k(t)\, dt + (-1)^{k} \int_{-1}^{-x} t P_k(t)\, dt \right).$$

One can easily see that $h(x)$ is the solution of the following Cauchy problem on $[-1,1]$ with initial conditions due to the orthogonality of $P_k(t)$, $k \geqslant 2$ to $P_0(t)=1$, $P_1(t)=t$:

$$\begin{cases} h''(x) &= 2P_k(x), \quad x \in [-1,1], \\ h'(1) &= \int\limits_{-1}^{1} P_k(t)\,dt = 0, \\ h(1) &= \int\limits_{-1}^{1} (1-t)P_k(t)\,dt = 0. \end{cases} \tag{IV.5.23}$$

Then solving (IV.5.23) we get

$$h(x) = \frac{1}{2^{k-1}\,k!}\left(\frac{d}{dx}\right)^{k-2}(x^2-1)^k, \quad k \geqslant 2.$$

Substituting this representation of h into (IV.5.22) we prove (IV.5.21). □

The following lemma enables us to get all a_{nk} other than zero for all n, k:

Lemma IV.5.2. *For all $n \geqslant 1$*

$$a_{nn} = -\frac{2\,(2n)!}{2^{2n}\,n!\,(n+1+1/2)!\,(n-1/2)},$$

$$a_{n\,n+2} = \frac{(2n)!}{2^{2n}\,n!\,(n+2+1/2)!}.$$

Proof. Case $n=1$ can be verified by direct calculation. Suppose now $n \geqslant 2$. Then by lemma IV.5.1

$$a_{nn} = \frac{(-1)^n}{2^{2n-1}\,(n!)^2}\int\limits_{-1}^{1}(t^2-1)^n\left(\frac{d}{dt}\right)^{2n-2}(t^2-1)^n\,dt =$$

$$\frac{2(-1)^n}{2^{2n}\,(n!)^2}\int\limits_{-1}^{1}(t^2-1)^n\left(\frac{(2n)!}{2}t^2 - n(2n-2)!\right)dt =$$

$$\frac{2(-1)^n}{2^{2n}\,(n!)^2}\left(\frac{(2n)!}{2}\int\limits_{-1}^{1}t^2(t^2-1)^n\,dt - n(2n-2)!\int\limits_{-1}^{1}(t^2-1)^n\,dt\right) =$$

$$\frac{2}{2^{2n}\,(n!)^2}\left(\frac{(2n)!\,n!}{2(n+1+1/2)!} - n(2n-2)!\frac{2n!}{(n+1/2)!}\right) =$$

$$-\frac{2\,(2n)!}{2^{2n}\,n!\,(n+1+1/2)!\,(n-1/2)},$$

$$a_{n\,n+2} = a_{n+2\,n} = \frac{(-1)^n \int\limits_{-1}^{1} (t^2-1)^{n+2} \left(\frac{d}{dt}\right)^{2n} (t^2-1)^n\,dt}{2^{2n+1}\,n!\,(n+2)!} =$$

$$\frac{(-1)^n\,(2n)! \int\limits_{-1}^{1} (t^2-1)^{n+2}\,dt}{2^{2n+1}\,n!\,(n+2)!} = \frac{(2n)!}{2^{2n}\,n!\,(n+2+1/2)!}.$$

□

Using the formulae of lemma IV.5.2 one can easily show that

$$a_{nn} = -2a_{n-1\,n+1}, \quad n \geqslant 2.$$

It could be verified by direct calculations that the above relation holds also for $n = 1, 2$: $a_{02} = a_{20} = \frac{4}{15}$, $a_{13} = a_{31} = \frac{4}{105}$, $a_{11} = -\frac{8}{15}$, $a_{22} = -\frac{8}{105}$. □

6 Remarks and open problems

Remark IV.6.1. *After the submission of the paper [81] containing most of the results of this chapter the preprint [70] appeared, where the main problem considered in sections 2–3 was solved via the inversion of a certain analytic family of functional operators in L^p–spaces that contains both the spherical Radon and the generalized cosine transforms.*

Operators of the form (IV.5.1) arise also in another context in the paper of Strichartz [83].

Remark IV.6.2 (Generalized Jacobi polynomials). *The problem of the inversion of T_{kr} for values k and r different from those considered in this chapter is still open. Here we outline one approach that might be possibly used to tackle it. By proposition I.3.1 the integral kernel $[\xi, \eta]$ of T_{kr} ($r \leqslant k$, $k + r \geqslant d$, $k \geqslant d/2$) is equal to a simple symmetric function of the squared cosines y_i, $i = 1, \ldots, d-k$ of the critical angles of ξ and η. By [39] (see also [86], p. 313), there exists the system of symmetric polynomials $P_{\vec{n}}(y_1, \ldots, y_{d-k})$, $\vec{n} = (n_1, \ldots, n_{d-k})$ such that*

1) *$P_{\vec{0}} = 1$;*

2) *$P_{\vec{n}}$ are orthonormal on*

$$\{y_1, \ldots, y_{d-k} \in [0,1] : 0 \leqslant y_{d-k} \leqslant \ldots \leqslant y_1 \leqslant 1\}$$

with respect to the invariant measure

$$\omega(d\vec{y}) = c \cdot \prod_{i=1}^{d-k} y_i^{-1/2} (1-y_i)^{\frac{1}{2}(2k-d-1)} \prod_{i<j} (y_i - y_j)\,dy_1 \ldots dy_{d-k}$$

with constant c chosen so that

$$\int\limits_{0\leqslant y_{d-k}\leqslant\cdots\leqslant y_1\leqslant 1} \omega(d\vec{y}) = 1$$

(cf. also [35], [37]).

These $P_{\vec{n}}$ *are called the generalized Jacobi polynomials. Since* y_i *are the squared cosines of the critical angles of* ξ, η *one can write* $P_{\vec{n}}(\vec{y}) = P_{\vec{n}}(\xi,\eta)$. *These polynomials have the following interesting property: the operator*

$$g(\xi) \mapsto \int\limits_{G(k,d)} P_{\vec{n}}(\xi,\eta) g(\xi)\, d\xi$$

maps one of the invariant subspaces of the irreducible representation of $O(d)$ *in* $C(G(k,d))$ *into the corresponding invariant subspace of* $C(G(r,d))$, *while all other invariant subspaces of* $C(G(k,d))$ *are mapped into zero. So if* $[\xi,\eta] = \sqrt{(1-y_1)\dots(1-y_{d-k})}$ *has the decomposition*

$$\sum_{\vec{n}} c_{\vec{n}} P_{\vec{n}}(\vec{y})$$

in the generalized Jacobi polynomials then the coefficients $c_{\vec{n}}$ *equal to zero will give us the description of the kernel*

$$Ker(T_{kr}) = \{f \in C(G(k,d)) : T_{kr} f = 0\}.$$

Then the proof of a result similar to theorem IV.5.2 would be possible. The main technical obstacle here is to calculate $c_{\vec{n}}$, *since the exact form of* $P_{\vec{n}}$ *is not known yet (cf. [86], p. 316), although some recurrent relations exist (see [38]) that, formally speaking, make these calculations possible.*

Another approach would be to get expansions of $T_{kr}\theta$ *in the spherical functions on* $G(k,d)$ *that are the analogs of the spherical harmonics on* $\mathbf{S}^{d-1}$ *(cf. [50], [36]).*

Remark IV.6.3 (Manifold processes). *Inversion formulae of sections 2 – 5 of this chapter could be generalized to a wider class of point processes on abstract spaces, namely, to fiber and hypersurface processes in* $\mathbb{R}^d$ *or, more generally, to processes of manifolds in* $\mathbb{R}^d$. *This kind of processes was extensively studied in a large number of papers by J. Mecke, W. Nagel, I. Molchanov, D. Stoyan, M. Zähle, etc. ([52], [58], [53], [63], see [82], chapter 9 for more references). In this case the rose of intersections* $f(\eta)$ *of a stationary process* Σ_k^d *of* k*–dimensional manifolds with an* r*–flat* η *is the mean total surface area of* $\Sigma_k^d \cap B$ *for a unit test window* $B \subset \eta$. *It can be shown that* $f(\eta)$ *is well–defined and equal to (I.1.3) for almost all* $\eta \in G(r,d)$. *Thus the inversion formulae for* T_{kr} *of this chapter can be applied directly to* f.

Chapter V

Related problems

In section 1 of this chapter the range characterization of $T_{d-1,1}$ will be discussed. Section 2 is devoted to the new notion of the rose of neighborhood of Φ_k^d with r–flats, $k+r<d$, which is the analogue of the rose of intersections for the case $k+r \geqslant d$.

1 Characteristic properties of the roses of intersections

Suppose a function $f: G(1,d) \to \mathbb{R}$ is given. The matter of this section is to answer to the following question: is f a rose of intersections? In other words, does there exist a stationary hyperplane process Φ_{d-1}^d such that its rose of intersections with lines coincides with f?

1.1 Preliminaries

In two dimensions such characterization result is well–known (cf. [1] §2.11): a function $f: \mathbf{S}^1 \to \mathbb{R}$ is the rose of intersections of a stationary line process Φ_1^2 with lines iff it is a support function of a centrally symmetric convex body in $\mathbb{R}^2$. It means that f is non–negative and even on $\mathbf{S}^1$, and its *radial extension* $\hat{f}(x) \stackrel{def}{=} \|x\| f\left(\frac{x}{\|x\|}\right)$, $x \in \mathbb{R}^2$ is *subadditive*: $\hat{f}(x+y) \leqslant \hat{f}(x)+\hat{f}(y)$ for all $x, y \in \mathbb{R}^2$ (cf. [73], p. 26, 38).

Also in the hyperplane case ($d \geqslant 3$) an elegant characterization criterion was already proved in [51], theorem 4.5.1 and corollary 3:

Theorem V.1.1 (Matheron). *A function $f \in C(\mathbf{S}^{d-1})$ is the rose of intersections of some stationary hyperplane process Φ_{d-1}^d with lines iff the function $-\hat{f}(x)$, $x \in \mathbb{R}^d$ is conditionally positive definite, i.e.*

$$\sum_{i,j=1}^{n} \omega_i \omega_j \left(-\hat{f}(x_i - x_j)\right) \geqslant 0 \qquad \text{(V.1.1)}$$

for all natural n, $x_i \in \mathbb{R}^d$ *and* $\omega_i \in \mathbb{R}$ *such that* $\sum_{i=1}^{n} \omega_i = 0$.

We shall also use Matheron's idea of the connection between support functions of zonoids and roses of intersections and prove another version of the characterization result (see theorem V.1.2). In addition, various sufficient conditions for a function f to be the rose of intersections of some Φ^d_{d-1} with particular directional distribution θ will be found. The unsolved characterization problems for other dimensions k and r will be touched upon in section 3.

1.2 Hyperplane processes intersected with lines

Let us formulate the following assumptions for functions $f : \mathbf{S}^{d-1} \to \mathbb{R}$ that will be required for the theorem below:

(i) $f(x) \geqslant 0$ for all $x \in \mathbf{S}^{d-1}$;

(ii) f is even on $\mathbf{S}^{d-1}$;

(iii) the radial extension $\hat{f}(x)$ is subadditive.

We say that a subset $\mathcal{A}$ of the dual space $\left(C_e(\mathbf{S}^{d-1})\right)^*$ of all even signed measures on $\mathbf{S}^{d-1}$ with the topology of weak convergence is *uniformly dense* on the subset of functions $\mathcal{D} \subset C_e(\mathbf{S}^{d-1})$ if for any even signed measure μ there exists a sequence $\{\mu_n\} \subset \mathcal{A}$ such that $\mu_n \to \mu$ and this convergence is uniform on $\mathcal{D}$.

Theorem V.1.2. *Let a function* $f : \mathbf{S}^{d-1} \to \mathbb{R}$, $d \geqslant 3$, *satisfy the conditions* **(i)**–**(iii)** *above.*

1) Suppose $\mathcal{A}$ *is a dense subset of* $\left(C_e(\mathbf{S}^{d-1})\right)^*$ *which is uniformly dense on the set of the support functions of all centered line segments of length 2 in* $\mathbb{R}^d$. *The function* f *is the rose of intersections of a stationary hyperplane process* Φ^d_{d-1} *with lines iff*

$$\int_{\mathbf{S}^{d-1}} f(u)\,\mu(du) \geqslant 0$$

for all μ *from* $\mathcal{A}$ *that satisfy*

$$\int_{\mathbf{S}^{d-1}} |<x,u>|\,\mu(du) \geqslant 0, \quad x \in \mathbf{S}^{d-1}.$$

2) The function $f \in C^k_e(\mathbf{S}^{d-1})$ *is the rose of intersections with lines of a stationary hyperplane process* Φ^d_{d-1} *with directional distribution* θ *such that* $\theta^\perp$ *has a density* $g \in C_e(\mathbf{S}^{d-1})$ *with respect to* $\omega_d(\cdot)$ *if*

$$k = \begin{cases} d+2, & d \text{ even}, \\ d+3, & d \text{ odd} \end{cases},$$

$$(-1)^d \left(\frac{d}{d(\mu^2)}\right)^{d-2} \left[\int\limits_{<u,v>^2>\mu^2} \frac{\Box f(u)\,|<u,v>|}{(<u,v>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(du) \right]\Bigg|_{\mu=0} \geqslant 0 \qquad \text{(V.1.2)}$$

for all $v \in \mathbf{S}^{d-1}$.

3) *The function* f *is the rose of intersections with lines of a stationary hyperplane process* Φ^d_{d-1} *with directional distribution* θ *that is not concentrated on any subset of the sphere lying in* $\mathbf{S}^{d-1} \cap \xi$ *for some* $\xi \in G(r,d)$, $r < d$ *iff* f *is strictly positive and* $-\hat{f}$ *is conditionally positive definite.*

Proof. Any rose of intersections of Φ^d_{d-1} with lines is the support function of a zonoid. Hence the necessity of the above conditions follows from the characterization results for zonoids (cf. [20] p. 683–684, [22] theorem 3.3, [17] theorem 3). Let us prove the sufficiency of the assumptions of the theorem. For any f satisfying conditions **(i)–(iii)** its radial extension $\hat{f}$ is a support function of a centrally symmetric convex body K. One needs to prove that K is a zonoid in the three cases above:

1) The required property of K follows from [17], theorem 3.

2) By [20], p. 683–684 there exists a function $g \in C_e(\mathbf{S}^{d-1})$ such that

$$f(x) = \int\limits_{\mathbf{S}^{d-1}} |<x,u>|\, g(u)\, \omega_d(du), \quad x \in \mathbf{S}^{d-1}.$$

One needs to prove that g is non–negative which follows from assumption (V.1.2): by proposition IV.2.1 g is proportional to the left–hand side expression in (V.1.2) as the inverse of the cosine transform.

3) Since f is positive the centrally symmetric convex body K has inner points. Then K is a zonoid by theorem 3.3 of [22].

Then there exists an even measure $\mu(\cdot)$ on $\mathbf{S}^{d-1}$ such that

$$f(x) = \int\limits_{\mathbf{S}^{d-1}} |<x,u>|\, \mu(du) \quad \text{for } x \in \mathbf{S}^{d-1}.$$

Let $\lambda = \mu(\mathbf{S}^{d-1})$, $\theta(\cdot) = \mu^{\perp}(\cdot)/\lambda$. Then for this λ and the probability measure θ construct the measure $\Lambda(\cdot)$ on $F(d-1,d)$ by formula (I.1.2). There exists a unique stationary Poisson hyperplane process with intensity measure Λ. By (I.1.3) its rose of intersections is equal to f. □

2 Roses of neighborhood

In the previous chapters the main object of our investigations was the rose of intersections T_{kr} of Φ_k^d with r–flats. One inevitable restriction on dimensions k and r was imposed: $k+r \geqslant d$. Any pair of flats ξ and η in general position of dimensions k and r, $k+r<d$, do not have any common points: thus a generalization of the intersection process $\Phi_k^d \cap \eta$ is desirable. In this section we introduce the so called roses of neighborhood of Φ_k^d that will remove the restriction on k and r mentioned above.

R. Schneider introduced in [75] the notion of *proximity* of Φ_k^d, $k<d/2$ that generalizes the usual intersection density of order 2 for $k \geqslant d/2$ (cf. [40] for the similar approach). We shall follow his ideas to construct the *process of neighborhood* $\Phi_k^d \odot \eta$ for $k+r<d$.

Let the relation $\not\parallel$ mean that the flats ξ and η are in general position, i.e. there does not exist a translation t of $\mathbb{R}^d$ such that $t\xi \subseteq \eta$ or $t\eta \subseteq \xi$. Let φ be a realization of a stationary Φ_k^d, let $\eta \in F(k,d)$. For any $\xi \in \varphi$ with $\xi \not\parallel \eta$ there exists a unique point $x_\xi \in \eta$ given by

$$dist(\xi,\eta) = \inf_{y\in\xi,\ x\in\eta} \rho(x,y) = \inf_{y\in\xi} \rho(x_\xi, y)$$

where $\rho(x,y)$ is the Euclidean distance in $\mathbb{R}^d$. Clearly the collection of all points

$$\{x_\xi \in \eta : dist(\xi,\eta) < a,\ \xi \in \Phi_k^d,\ \xi \not\parallel \eta\}$$

for some $a>0$ forms a stationary point process $\Phi_k^d \odot \eta$ in η that will be called the *a–process of neighborhood* (we suppress a in the notation). Its intensity $N_{kr}(a,\eta)$ will be called the *a–rose of neighborhood* of Φ_k^d, and in case $a=1$ we shall write $N_{kr}(\eta)$ instead of $N_{kr}(1,\eta)$ and call it the *rose of neighborhood* of Φ_k^d. Due to stationarity of Φ_k^d consider only those $\eta \in F(r,d)$ that contain the origin, i.e. $\eta \in G(r,d)$.

For any stationary k–flat process Φ_k^d with intensity λ and directional distribution θ introduce the family of *dual processes* $\mathfrak{D}(\lambda,\theta)$: a stationary $(d-k)$–flat process Φ_{d-k}^d belongs to $\mathfrak{D}(\lambda,\theta)$ iff its intensity is equal to λ and its directional distribution is $\theta^\perp(d\zeta) = \theta(d\zeta^\perp)$ for $\zeta \in G(d-k,d)$.

Theorem V.2.1. *In case $k+r<d$ for the rose of neighborhood N_{kr} of the stationary k–flat process Φ_k^d the following equality holds:*

$$N_{kr}(\eta) = k_{d-k-r}\left(T_{d-k,d-r}\theta^\perp\right)(\eta^\perp), \quad \eta \in G(r,d) \tag{V.2.1}$$

where $T_{d-k,d-r}\theta^\perp$ is the rose of intersections of the dual processes $\Phi_{d-k}^d \in \mathfrak{D}(\lambda,\theta)$ with $(d-r)$–flats. By (I.1.3) it is the same for all processes of that family.

Proof. For any $(d-k)$–flat ζ and $(d-r)$–flat β, $\zeta \not\parallel \beta$, their intersection is not empty since $d-k+d-r>d$. Therefore, the usual rose of intersections of $\Phi^d_{d-k} \in \mathfrak{D}(\lambda,\theta)$ with $(d-r)$–flats is well–defined. The intensity of $\Phi^d_k \odot \eta$ is given by the expression

$$N_{kr}(a,\eta) = \frac{1}{k_r} E\left(\sum_{\zeta\in\Phi^d_k} J_a(\zeta)\right) \qquad \text{(V.2.2)}$$

where $J_a(\zeta) = I_{\{\zeta \not\parallel \eta:\, dist(\zeta,\eta)<a, x_\zeta \in B_1(0)\subset\eta\}}(\zeta)$, $B_m(0)$ is the ball in the appropriate ambient subspace of $\mathbb{R}^d$ with radius m and the center in the origin. Determining the expectation in (V.2.2) by means of Campbell's theorem (cf. [82]) and using (I.1.2) for the intensity measure $\Lambda(\cdot)$ of Φ^d_k one gets

$$N_{kr}(a,\eta) = \frac{1}{k_r}\int\limits_{F(k,d)} J_a(\zeta)\,\Lambda(d\zeta) = \frac{\lambda}{k_r}\int\limits_{G(k,d)}\int\limits_{\xi^\perp} J_a(y+\xi)\,\nu^{\xi^\perp}_{d-k}(dy)\theta(d\xi) =$$
$$\frac{\lambda}{k_r}\int\limits_{G(d-k,d)}\int\limits_{\xi^\perp} J_a(y+\xi)\,\nu^{\xi^\perp}_{d-k}(dy)\theta^\perp(d\xi^\perp).$$

Now prove that

$$\int\limits_{\xi^\perp} J_a(y+\xi)\,\nu^{\xi^\perp}_{d-k}(dy) = k_{d-k-r}k_r a^{d-k-r}[\xi^\perp,\eta^\perp]$$

for any $\xi \in G(k,d)$, $\xi \not\parallel \eta$. Using the reasoning similar to that of [75], formulae (7), (8) we get

$$\int\limits_{\xi^\perp} J_a(y+\xi)\,\nu^{\xi^\perp}_{d-k}(dy) = [\xi^\perp,\eta^\perp]\int\limits_{H} J_a(z+\xi)\,\nu^{H}_{d-k}(dz)$$

where $H = (\xi+\eta)^\perp + \eta$ in the sense of Minkowski summation. We shall show that the integral

$$\int\limits_{H} J_a(z+\xi)\,\nu^{H}_{d-k}(dz) \qquad \text{(V.2.3)}$$

is equal to $k_{d-k-r}k_r a^{d-k-r}$: as $H = \eta \oplus (\eta)^\perp_H$, a direct orthogonal sum, where $(\eta)^\perp_H$ stands for the orthogonal complement of η in H, we have $z = x_{y+\xi} + l$ for any $z \in H$, where $x_{y+\xi} \in \eta$, $l \in (\eta)^\perp_H$. Then

$$J_a(z+\xi) = \begin{cases} 1, & x_{y+\xi} \in B_1(0) \subset \eta,\ l \in B_a(0) \subset (\eta)^\perp_H, \\ 0, & \text{otherwise.} \end{cases}$$

It means that integral (V.2.3) is equal to the desired expression. Thus

$$N_{kr}(a,\eta) = \frac{\lambda}{k_r} k_r k_{d-k-r} a^{d-k-r} \int\limits_{G(d-k,d)} [\xi^\perp,\eta^\perp]\,\theta^\perp(d\xi^\perp) =$$
$$k_{d-k-r}a^{d-k-r}\left(T_{d-k,d-r}\theta^\perp\right)(\eta^\perp),$$

hence $N_{kr}(\eta) = k_{d-k-r}\left(T_{d-k,d-r}\theta^{\perp}\right)(\eta^{\perp})$. □

The above theorem shows that the question of restoring the directional distribution θ of a stationary process Φ_k^d from its rose of neighborhood $N_{kr}(\eta)$ constructed for r–flats η, $k+r<d$, can be reduced to the dual problem for any $\Phi_{d-k}^d \in \mathfrak{D}(\lambda,\theta)$ intersected with $(d-r)$–planes; its partial solution was already given in chapter IV. Thus, the directional distribution θ of a line process Φ_1^d can be retrieved in this way from its rose of neighborhood with r–flats, $1 \leqslant r < d-1$. This means, from the stereological and statistical point of view, that in order to estimate θ we can use now just lines as test objects instead of planes, which can possibly find potential statistical applications.

Example V.2.1 **($d=3,\ k=r=1$).** *Consider a stationary line process Φ_1^3 with intensity λ and directional distribution θ. Let $N_{11}(v)$ be its rose of neighborhood with lines ($v \in \mathbf{S}^2$ is the direction vector of a test line). Then by theorem V.2.1*

$$N_{11}(v) = 2\left(T_{22}\theta^{\perp}\right)(v^{\perp}) = 2\lambda\int_{\mathbf{S}^2}\sqrt{1-<u,v>^2}\,\theta(du) = 2\lambda K\theta(v),$$

and applying theorem IV.3.1 to invert K one gets that for any $g \in C_e^4(\mathbf{S}^2)$

$$\int_{\mathbf{S}^2} g(v)\,\theta(dv) = \frac{1}{16\pi^4\lambda}\times$$

$$\times\int_{\mathbf{S}^2} W(u)\,(\Delta_0+2)\left(\frac{d}{d(\mu^2)}\right)\left[\int_{<u,v>^2>\mu^2}\frac{g(v)\ |<u,v>|\ \omega_3(dv)}{\sqrt{<u,v>^2-\mu^2}}\right]\Bigg|_{\mu=0}\omega_3(du)$$

where

$$W(u) = \left(\frac{d}{d(\mu^2)}\right)\left[\int_{<u,t>^2>\mu^2}\frac{N_{11}(t)\ |<u,t>|\ \omega_3(dt)}{\sqrt{<u,t>^2-\mu^2}}\right]\Bigg|_{\mu=0}.$$

3 Open problems

The problem of characterizing a rose of intersections of Φ_k^d with r–flats for $k=d-1$, $1<r\leqslant d-1$ and $k<d-1$ is still open. In other terms, one should describe the range $T_{kr}\left(\mathbf{M}(G(k,d))\right)$ of the generalized cosine transform. By formula (III.3.23) and theorem V.1.1 the above problem for $k=d-1$ can be reduced to the description of the range $R_r\left(C_e(\mathbf{S}^{d-1})\right)$ of the totally geodesic Radon transform R_r on r–circles of $\mathbf{S}^{d-1}$. Unfortunately, the

only range description for R_r (see [24], [41], [42]) was obtained for $C_e^\infty(\mathbf{S}^{d-1})$ in terms of the solutions of partial differential equations. This corresponds to the case of our setting when the directional distribution of Φ_{d-1}^d has a smooth density. Thus the general characterization can be obtained here only if $R_r\left(C_e(\mathbf{S}^{d-1})\right)$, or more precisely, $R_r\left(T\big(\mathbf{M}(\mathbf{S}^{d-1})\big)\right)$ is known.

Another possible way to state the problem of finding the range $T_{kr}\left(\mathbf{M}(G(k,d))\right)$ for $r = d-k$, $k < d-1$ is by means of kth projection functions of centrally symmetric convex bodies. In §7, theorem 7.1 of [22] the classes $\mathcal{K}(k)$ of centrally symmetric convex bodies are described whose kth projection function $v_k(K;\eta)$ on $\eta \in G(k,d)$ can be represented as

$$v_k(K;\eta) = \int\limits_{G(k,d)} [\eta^\perp, \xi]\, \mu(d\xi)$$

for some positive measure μ on $G(k,d)$. Suppose one can determine whether a given function $f : G(k,d) \to \mathbb{R}$ is the kth projection function of some centrally symmetric convex body K. Then the criterion cited above yields that $f^\perp$ is the rose of intersections of a stationary k–flat process Φ_k^d with $d-k$–flats. Thus the original problem can be reduced to the characterization of projection functions which is not obtained yet.

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List of special symbols

$[\cdot,\cdot]$, 7
$[\cdot,\cdot]_0$, 51
$\equiv$, 8
$\cong$, 8
$\square$, 30
$\triangle$, 30
$\triangle_0$, 30
$|\cdot|$, 10
$\|\cdot\|$, 18
$\|\cdot\|_{(\alpha)}$, 40
$\|\cdot\|^{\perp}_{(\alpha)}$, 40
$\|\cdot\|_p$, 40
$\|\cdot\|_{\mathbf{S}^{d-1}}$, 47
$\nparallel$, 64
$< \cdot,\cdot >$, 10
$< \cdot,\cdot >_{\mathbf{S}^{d-1}}$, 47
$< a_1,\ldots,a_k >$, 6

A^t, 6

B_ζ, 14
$b(\cdot)$, 31
β_n, 11

$Card(A)$, 5
$\mathcal{C}(\lambda,\theta)$, 7
$C(X)$, 25
$C_e^p(\mathbf{S}^{d-1})$, 26
$C_e^\infty(\mathbf{S}^2\times\mathbf{S}^2)$, 54
$\left(C_e(\mathbf{S}^{d-1})\right)^*$, 62

$\mathfrak{D}(\lambda,\theta)$, 64
$d(\cdot,\cdot)$, 29
$diag(d_1,\ldots,d_m)$, 12
$d\eta$, 25
δ_{ij}, 12
$\delta_\xi(\cdot)$, 18

e_i, 31
$(\eta)^{\perp}_{\xi}$, 31

$F(k,d)$, 5
$\mathfrak{F}$, 5
$\hat{f}$, 61

$G(k,d)$, 5, 7
$\mathfrak{G}$, 5
$g^\perp$, 26

$\mathbb{H}$, 51
$h_K(x)$, 16

$I_A(\cdot)$, 6
$\Im$, 51
$\Im\left(\mathbb{K}^4\right)$, 51

K, 26
$Ker(\cdot)$, 60
$\mathbb{K}^4$, 51
k_d, 6

$\mathbf{L}$, 15
$\mathbf{L_0}$, 15
$L\{\cdot,\cdot\}$, 52
$L^p(X)$, 28
$\mathcal{L}(k,d)$, 8
$\Lambda(\cdot)$, 6
λ, 5

$\mathbf{M}(X)$, 19
$\tilde{\mathbf{M}}(X)$, 18
$(m+\frac{1}{2})!$, 56
$\mathcal{M}$, 5
$\mathfrak{M}$, 5

$N_{kr}(\eta)$, 64
$N_{kr}(a,\eta)$, 64
$\nu_d(\cdot)$, 5
$\nu_i^\eta(\cdot)$, 6

$O(d)$, 8
ω_d, 8
$\omega_d(\cdot)$, 8
$\omega_r^\eta(\cdot)$, 28
$\omega_{d-1}^{v^\perp,y}(\cdot)$, 29

$p(K)$, 17
$P_n(t)$, 55
$P_{\vec{n}}(y_1,\ldots,y_{d-k})$, 59
$Pr_\zeta(\cdot)$, 13
Φ_k^d, 5
$\Phi_k^d(\cdot)$, 6
$\Phi_k^d \odot \eta$, 64

$|q|$, 51
$\bar{q}$, 51
$\hat{q}$, 51
q^{-1}, 51
q_{kl}, 9

R, 28
R_r, 28
R_{ij}, 28
$\Re$, 51
$\mathbb{RP}^{\,d-1}$, 8
$r(\cdot)$, 6
$\rho(\cdot,\cdot)$, 5
$\rho_k(K,\cdot)$, 16

$\mathbf{S}^{d-1}$, 6
$\mathbf{S}_+^{d-1}$, 8
$\mathbf{S}_\eta^{r-1}$, 28
$SO(d)$, 8
$Stab(\cdot)$, 50
$\sigma(\cdot)$, 28
Σ_k^d, 60

T, 25
T_{ij}, 6, 25
T_{ij}^α, 26
$\tilde{T}_{ij}^\alpha$, 26
$\theta(\cdot)$, 6
$\theta^\perp$, 26
θ_K, 16

$v_i(\cdot;\cdot)$, 27
$V(\cdot,\ldots,\cdot)$, 14
$V_i(\cdot)$, 31
$V_f(d,k,j)$, 43
$Vol(\xi)$, 31
$Vol(a_1,\ldots,a_k)$, 6

$\bar{w}(L)$, 17

X, 18
$X_2(\Phi_k^d)$, 7
$X_k(\Phi_{d-1}^d)$, 23
$\xi^\perp$, 6

$\mathbb{Z}_+$, 6