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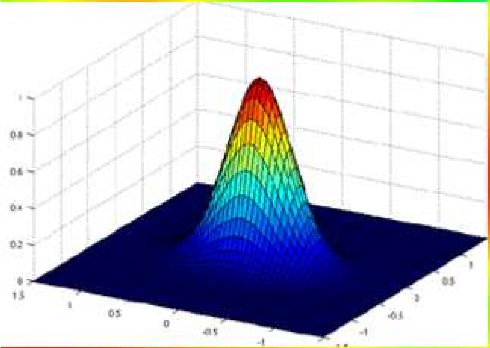

Limit Theorems in Stochastic Geometry

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Overview

- Introduction
- Specific intrinsic volumes of random closed sets
- Scan statistics for Lévy measures and point processes
- Open problems
  - Wiener sausage
  - Level (excursion) sets of stationary random fields

Introduction

In stochastic geometry:

Limit theorems for

- Random geometric graphs (Penrose, Yukich, ...)
- Random polytopes (Bárány, Buchta, Hug, Reitzner, Schneider, ...)
- Random closed sets (RACS), geometrical random fields and measures (Heinrich, Molchanov, S., Vitale, Weil, ...)

Introduction

Preliminaries

$\mathcal{K}$	family of non–empty compact convex sets (convex bodies) in $\mathbb{R}^d$
$\mathcal{R}$	$= \{\bigcup_{i=1}^n K_i : K_i \in \mathcal{K},\ i = 1, \dots, n,\ \forall n\}$ convex ring
$\mathcal{S}$	$= \{K : K \cap W \in \mathcal{R},\ \forall\, W \in \mathcal{K}\}$ extended convex ring
$B_r(a)$	ball with center in a and radius r
κ_j	volume of $B_1(o)$ in $\mathbb{R}^j$, $j = 0, \dots, d$
$K_1 \oplus K_2$	$= \bigcup_{x \in K_2} (K_1 + x)$ Minkowski addition
$K_1 \ominus K_2$	$= \bigcap_{x \in K_2} (K_1 + x)$ Minkowski subtraction
$\lvert \cdot \rvert$	Lebesgue measure (volume)

Introduction

Limit theorem for random polytopes

(Bárány (1982); Reitzner (2003, 2005); Vu (2005))

For $K \in \mathcal{K}$ with $\partial K \in C^3$, let $K_n = conv(X_1, \ldots, X_n)$, where $X_i \sim U(K)$ are *iid* random points in K. Then, as $n \to \infty$, it holds

- $\mathbb{E}\,|K_n| = |K| - c_1(d, K)n^{-1/(d+1)}$,
- $Var\,|K_n| \sim c_2(K)n^{-(d+3)/(d+1)}$,
- $(|K_n| - \mathbb{E}\,|K_n|)/\sqrt{Var\,|K_n|} \xrightarrow{d} N(0, 1)$.

Introduction

LT for RACS w. r. to Minkowski addition (Weil (1982))

- Hausdorff metric: For two nonempty compacts $A, B \subset \mathbb{R}^d$
  $d_H(A, B) = \min \{r > 0 : A \subseteq B \oplus B_r(o), B \subseteq A \oplus B_r(o)\}$.
- Norm of a set: $\|A\| = \sup\{|x| : x \in A\}$
- Support function: for $A \in \mathcal{K}$, $s_A(u) = \sup\{u \cdot v : v \in A\}$, $u \in \mathbf{S}^{d-1}$.
- Expectation of RACS (Aumann (1965)): $\mathbb{E}\,\Xi =$ convex set with support function $\mathbb{E}\,s_\Xi(\cdot)$

For a sequence of *iid* RACS $\Xi_i \stackrel{\text{d}}{=} \Xi$ with $\mathbb{E}\,\|\Xi\| < \infty$, it holds

$$\sqrt{n}\, d_H\left((\Xi_1 \oplus \ldots \oplus \Xi_n)/n, \mathbb{E}\,\Xi\right) \xrightarrow{d} \sup_{u \in \mathbf{S}^{d-1}} X(u), \quad n \to \infty,$$

where X is the centered Gaussian process on $\mathbf{S}^{d-1}$ with cov. f. $C(u, v) = \mathbb{E}\,[s_\Xi(u)s_\Xi(v)] - \mathbb{E}\,s_\Xi(u)\mathbb{E}\,s_\Xi(v)$, $u, v \in \mathbf{S}^{d-1}$.

Limit theorems for various characteristics of RACS

For the volume fraction, specific boundary surface, number of connected components, etc.:

- Baddeley (1980)
- Mase (1982)
- Heinrich (1993)
- Heinrich, Molchanov (1999)
- Böhm, Heinrich, Schmidt (2004)
- ...

For all specific intrinsic volumes:

- Pantle, Schmidt, S. (2006, 2009)

Intrinsic volumes

Steiner's formula in $\mathbb{R}^d$

- There exist functionals $V_j : \mathcal{K} \to [0, \infty)$, $j = 0, \ldots, d$, (Minkowski functionals, quermassintegrals or intrinsic volumes) such that for any $r > 0$ and $K \in \mathcal{K}$ it holds

$$|K \oplus B_r(o)| = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K) \,.$$

- Functionals $V_0, \ldots, V_d$ are additive, motion invariant, monotone with respect to inclusion, and continuous with respect to d_H.

Intrinsic volumes

Theorem (Hadwiger (1957))

Let $F : \mathcal{K} \to \mathbb{R}$ be any additive, motion invariant and continuous functional. Then, F can be represented in the form

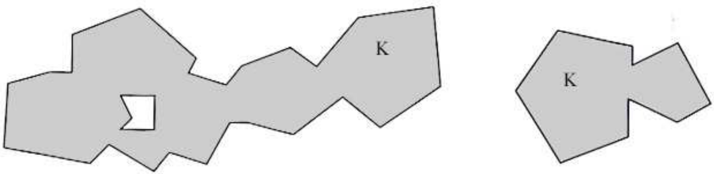
$$F = \sum_{j=0}^{d} a_j V_j$$

for some constants $a_0, \ldots, a_d \in \mathbb{R}$.

Thus, the intrinsic volumes $V_0, \ldots, V_d$ form a basis in the corresponding linear space.

Intrinsic volumes

Additive extension to the convex ring $\mathcal{R}$



For each $j = 0, \ldots, d$, there exists a unique additive extension of $V_j : \mathcal{K} \to [0, \infty)$ to $\mathcal{R}$ given by the inclusion–exclusion formula:

$$V_j(K_1 \cup \ldots \cup K_n) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{j_1 < \ldots < j_i} V_j(K_{j_1} \cap \ldots \cap K_{j_i}), \quad K_1, \ldots, K_n \in \mathcal{K}$$

Intrinsic volumes

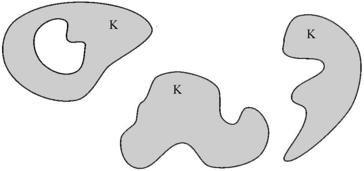
Geometrical interpretation: For any $K \in \mathcal{R}$ with $K \neq \emptyset$,

$V_d(K) = |K|$ (volume)

$2V_{d-1}(K) = \mathcal{H}^{d-1}(\partial K)$ (surface area)

$V_0(K) = \chi(K)$ (Euler number)

In $\mathbb{R}^2$: $\chi(K) = \#\{\text{clumps}\} - \#\{\text{holes}\}$



$\chi(K) = 3 - 1 = 2$

Random closed sets

- Let $(\Omega, \mathcal{F}, \mathcal{P})$ be an arbitrary probability space
- $\mathfrak{C} =$ family of all compact sets in $\mathbb{R}^d$
- $\mathfrak{F} =$ family of all closed sets in $\mathbb{R}^d$
- $\sigma(\mathfrak{F}) = \sigma$–algebra in $\mathfrak{F}$, generated by the sets $F_C = \{F \in \mathfrak{F} : F \cap C \neq \emptyset\}$ for any $C \in \mathfrak{C}$

An $(\mathcal{F}, \sigma(\mathfrak{F}))$–measurable mapping $\Xi : \Omega \to \mathfrak{F}$ is called a random closed set (RACS). Its distribution is uniquely determined by the capacity functional $T_\Xi(C) = P(\Xi \cap C \neq \emptyset)$, $C \in \mathfrak{C}$

Random closed sets

Stationarity and isotropy

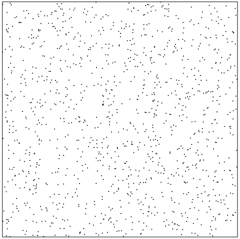
A RACS Ξ is called stationary if $\Xi \stackrel{d}{=} \Xi + x$, $\forall\, x \in \mathbb{R}^d$, and isotropic if $\Xi \stackrel{d}{=} g\Xi$, $\forall\, g \in SO(d)$

Theorem (Matheron (1975))

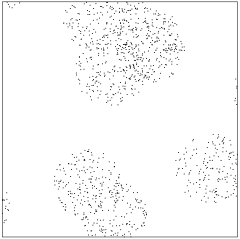
- *The RACS Ξ is stationary (isotropic) $\Longleftrightarrow$ $T_\Xi(C+x) = T_\Xi(C)\ \forall\, x \in \mathbb{R}^d$ and $T_\Xi(gC) = T_\Xi(C)$ $\forall\, g \in SO(d)$, respectively*
- *Each stationary RACS $\Xi \neq \emptyset$ is a.s. unbounded*
- *For any stationary convex RACS Ξ, it holds $\Xi \in \{\emptyset, \mathbb{R}^d\}$ a.s.*

Examples

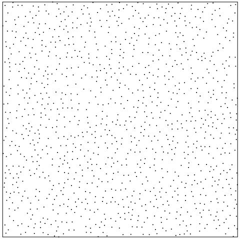
Stationary point processes in $\mathbb{R}^2$



Poisson process



cluster process



hard–core process

Examples

Stationary germ–grain models in $\mathbb{R}^2$

Realizations of germ–grain models: Boolean model with spherical and polygonal grains, respectively; cluster process of segments

Examples

Germ–grain models

The germ–grain model $\Xi = \bigcup_{i=1}^{\infty} (\Xi_i + X_i)$ is called a Boolean model if

- the point process of germs $\{X_1, X_2, \dots\}$ is a stationary Poisson process in $\mathbb{R}^d$ (with intensity λ)
- the grains $\Xi_1, \Xi_2, \dots$ are i.i.d. and independent of $\{X_1, X_2, \dots\}$; $\Xi_i \overset{d}{=} \Xi_0$
- $\mathbb{E}\,|\Xi_0 \oplus K| < \infty, \quad \forall K \in \mathcal{K}$.

Capacity functional: $T_\Xi(C) = 1 - e^{-\lambda \mathbb{E}\,|(-\Xi_0)\oplus C|}, \quad \forall C \in \mathfrak{C}$

Specific intrinsic volumes

- Model assumptions
  - Let Ξ be stationary, $\Xi \in \mathcal{S}$ a.s.
  - $\mathbb{E}\, 2^{N(\Xi \cap [0,1]^d)} < \infty$, where $N(\emptyset) = 0$ and

    $$N(K) = \min\{m \in \mathbb{N} : K = \bigcup_{i=1}^{m} K_i,\ K_i \in \mathcal{K}\} \text{ for } K \in \mathcal{R} \setminus \{\emptyset\}$$

- Specific intrinsic volumes: Let $\overline{V}_j(\Xi) = \lim_{n\to\infty} \frac{\mathbb{E}\, V_j(\Xi \cap W_n)}{|W_n|}$ for $j = 0, \ldots, d$, where $\{W_n\}$ = sequence of monotonously increasing sampling windows $W_n = nW$ with $W \in \mathcal{K}$ and $|W| > 0$

  In particular, $\overline{V}_d(\Xi) = P(o \in \Xi) = \mathbb{E}|\Xi \cap W|/|W|$

Estimation of $(\overline{V}_0(\Xi), \dots \overline{V}_d(\Xi))^\top$

Problem: Estimate $\overline{V}(\Xi) = (\overline{V}_0(\Xi), \dots \overline{V}_d(\Xi))^\top$ on the basis of a single sample from $\Xi \cap W$

Solution: For each $i = 0, \dots, d$, consider a random field $Y_i = \{Y_i(x), x \in \mathbb{R}^d\}$ such that

- Y_i is stationary of second order, i.e. $\mathbb{E}\, Y_i(x) = \mu_i$ and $Cov(Y_i(x), Y_i(x+h)) = Cov_{Y_i}(h) \quad \forall\, x, h \in \mathbb{R}^d$
- $\mu_i = \mathbb{E}\, Y_i(o) = \sum_{j=0}^{d} a_{ij} \overline{V}_j(\Xi)$, where the matrix $A = (a_{ij})_{i,j=0}^d$ is regular

Then, it holds $\overline{V}(\Xi) = A^{-1}\mu$, where $\mu = (\mu_0, \dots, \mu_d)^\top$

Estimation of $(\overline{V}_0(\Xi), \ldots \overline{V}_d(\Xi))^\top$

- For any $i = 0, \ldots, d$ and $x \in U \subset W$, suppose that $Y_i(x)$ can be computed from $\Xi \cap W$
- Let $w(\cdot)$ be a probability measure with support in $U \subset W$

  Examples: $w(\cdot) = |\ \cdot\ \cap U|/|U|$ and $w(\cdot) = \sum_{k=1}^{m} w_k \delta_{x_k}(\cdot)$ with $x_1, \ldots, x_k \in U$, $w_1, \ldots, w_m > 0$ and $w_1 + \ldots + w_m = 1$

Then,

- $\widehat{\mu} = (\widehat{\mu}_0, \ldots, \widehat{\mu}_d)^\top$ with $\widehat{\mu}_i = \int_W Y_i(x)\, w(dx)$ is an unbiased estimator for $\mu = (\mu_0, \ldots, \mu_d)^\top$, and
- $\widehat{V}(\Xi) = A^{-1}\widehat{\mu}$ is unbiased for $\overline{V}(\Xi)$

Estimation of $(\overline{V}_0(\Xi), \ldots \overline{V}_d(\Xi))^\top$

- Method of least squares: Consider $n > d$ random fields Y_i with the properties mentioned above
  - ”Solve” the overdetermined system of linear equations
  - The solution is the LS–estimator for $(\overline{V}_0(\Xi), \ldots \overline{V}_d(\Xi))^\top$
- Minimization of variance
  - Reduction of $Var(\widehat{\mu}_i)$ by an appropriate choice of w
  - For a discrete averaging measure w: optimal weights $w_1, \ldots, w_m$ by kriging of the mean (Wackernagel (1998))
  - sampling points $x_1, \ldots, x_m \in W$ by optimal experimental design for random fields (Näther (1985), Müller (2001))

Consistency

- Let $\mathbb{E}\, 4^{N(\Xi \cap [0,1]^d)} < \infty$ and $\int_{\mathbb{R}^d} |Cov_{Y_i}(h)|\, dh < \infty$ $\forall i = 0, \dots, d$

Then,

- $\widehat{V}(\Xi, W_n) = A^{-1} \widehat{\mu}(W_n)$ with $W_n = nW$ and

$$\widehat{\mu}(W_n) = \left(\int\limits_{W_n} Y_0(x)\, w(dx), \dots, \int\limits_{W_n} Y_d(x)\, w(dx) \right)^{\top}$$

is an L_2–consistent estimator for $\overline{V}(\Xi)$, i.e.,

$$\mathbb{E}|\widehat{V}(\Xi, W_n) - \overline{V}(\Xi)|^2 \to 0, \qquad n \to \infty$$

Asymptotic normality

If Ξ is a stationary germ–grain modell with iid grains $\Xi_i \stackrel{\mathrm{d}}{=} \Xi_0 \in \mathcal{K}$ that are independent of germs $\{X_1, X_2, \ldots\}$ and some additional assumptions on the Y_i are fulfilled, then

$$\sqrt{|W_n|}\big(\widehat{V}(\Xi, W_n) - \overline{V}(\Xi)\big) \xrightarrow{d} \mathrm{N}\big(0, A^{-1}\Sigma\,(A^{-1})^\top\big)$$

as $n \to \infty$ where

- $\Sigma = \left(\int_{\mathbb{R}^d} Cov_{Y_i Y_j}(h)\, dh\right)_{i,j=0}^{d}$ and
- $Cov_{Y_i Y_j}(h) = \mathbb{E}\, Y_i(x) Y_j(x+h) - \mu_i \mu_j$ is the cross covariance function of Y_i and Y_j

Asymptotic normality

Additional assumptions:

- $Y_j(x) = f_j\big((\Xi - x) \cap K_j\big)$, where the f_j are conditionally bounded valuations on $\mathcal{R}$; $K_j \in \mathcal{K}$
- $\mathbb{E}\, 2^{pN(\Xi \cap K_j)} < \infty$, where $N(\Xi \cap K_j) = \#\{i : (M_i + X_i) \cap K_j \neq \emptyset\}$
- $w(\cdot) = |\cdot \cap U_n|/|U_n|$, $U_n \subset W_n$

and either

- $X = \{X_1, X_2, \ldots\}$ is rapidly β–mixing
- Ξ_0 uniformly bounded; $p = 2 + \delta$, $\delta > 0$

or

- $X = \{X_1, X_2, \ldots\}$ has finite range of correlation
- $|Cov_{Y_i Y_j}(h)| \leqslant g_{ij}(\Xi_0, h) \in L_1$ monotonously w.r.t. Ξ_0; $p = 2$

Consistent estimation of covariances

- Let $U_n \subset W_n$ be averaging sets such that $|U_n|^2/|W_n| \to 0$ and $\min_{h\in U_n} |W_n \cap (W_n - h)|/|W_n| \to 1$ for $n \to \infty$
- Then, the L_2–consistency $\lim_{n\to\infty} \mathbb{E}|\widehat{\Sigma}_n - \Sigma|^2 = 0$ holds, where $\widehat{\Sigma}_n = (\widehat{\sigma}_{nij})_{i,j=0}^d$ with

$$\widehat{\sigma}_{nij} = \frac{1}{|W_n|} \int_{U_n} \widehat{Cov}_{nij}(h)\, |W_n \cap (W_n - h)|\, dh\,,$$

$$\widehat{Cov}_{nij}(h) = \frac{\int_{W_n\cap(W_n+h)} Y_j(x) Y_i(x-h)\, dx}{|W_n \cap (W_n + h)|} - \frac{\int_{W_n} Y_i(x)\, dx \int_{W_n} Y_j(x)\, dx}{|W_n|^2}$$

Example of random fields Y_i

Local Euler number

- Let $r_0, \ldots, r_{d-1} > 0$ with $r_i \neq r_j$, $i \neq j$ and $|W \ominus B_{r_i}(o)| > 0$
- $Y_i(x) = V_0(\Xi \cap B_{r_i}(x)) \quad$ for $i = 0, \ldots, d$
- Edge–corrected estimator (minus sampling):
  $U = W \ominus B_r(o)$

$$\widehat{\mu}_i = \int\limits_{W \ominus B_{r_i}(o)} V_0(\Xi \cap B_{r_i}(x))\, w(dx) = \sum_{k=1}^{m} V_0(\Xi \cap B_{r_i}(x_k)) w_k$$

- Discrete averaging measure w, where
  $x_1, \ldots, x_m \in W \ominus B_{r_i}(o)$ and, for example, $w_k = 1/m$,
  $k = 1, \ldots, m$

Scan statistics for point processes

Scan statistic

Let $\Phi = \{X_i\}$ be an independently marked point process in $\mathbb{R}^d$ with iid marks $\{M_i\}$ observed within a cube W. For a (cubic) subwindow $W_o \subset W$, define $S(W_o) = \sum_{i: X_i \in W_o} M_i$.

Scan statistic: $T = \sup_{W_o \in \mathcal{W}} S(W_o)$

- Usual scan statistic of fixed size $r > 0$:
  $\mathcal{W} = \{W_1 = x + r[0,1]^d, x \in \mathbb{R}^d : \ W_1 \subset W\}$.
- Multiscale scan statistic: $\mathcal{W} = \{\text{all cubes } W_1 \subset W\}$

Limit theorems: $T = T_n \xrightarrow{d} ?$ as $W = W_n = n[0,1]^d$, $n \to \infty$

Scan statistics for point processes

- Scan statistics in $\mathbb{R}^1$ and $\mathbb{R}^2$: Glaz, Balakrishnan (1999), Glaz, Naus, Wallenstein (2001)
- LT for the usual scan statistic in $\mathbb{R}^d$: $\Phi =$ stationary compound Poisson process (Chan (2007))
- LT for the multiscale scan statistic in $\mathbb{R}^1$: Cohen (1968), Iglehart (1972), Karlin, Dembo (1992), Doney, Maller (2005)
- LT for the multiscale scan statistic in $\mathbb{R}^d$: independently scattered Lévy measures (Kabluchko, S. (2008))

Scan statistic for Lévy noise

- Lévy noise: Let $\xi = \{\xi(t), t \geq 0\}$ be a Lévy process with $\xi(0) = 0$, $\mathbb{E}\,\xi(1) = \mu$, $\sigma^2 = Var\,\xi(1) > 0$.
  Lévy noise $\mathcal{Z} = \{\mathcal{Z}(B), B \in \mathcal{B}(\mathbb{R}^d)\}$ is an independently scattered stationary random measure on $\mathbb{R}^d$ driven by ξ, i.e. $\mathcal{Z}(B) \stackrel{\mathrm{d}}{=} \xi(|B|)$ for Borel sets $B \in \mathcal{B}(\mathbb{R}^d)$.
- Multiscale scan statistic:

$$T_n = \sup_{W_o \in \mathcal{W}_n} \mathcal{Z}(W_o), \quad n \in \mathbb{N}$$

for $\mathcal{W}_n = \{\text{all cubes within } W_n = [0, n]^d\}$.

LT for the scan statistic of Lévy noise

Theorem (Kabluchko, S. (2008))

- *If* $\mu > 0$ *then* $(T_n - \mu n^d)/(\sigma n^{d/2}) \xrightarrow{d} Y \sim N(0,1)$
- *If* $\mu = 0$ *then* $T_n/(\sigma n^{d/2}) \xrightarrow{d} \sup_{W_o \in \mathcal{W}_1} Z(W_o)$, *where* $Z = \{Z(B),\ B \in \mathcal{B}(\mathbb{R}^d)\}$ *is the standard Gaussian white noise on* $[0,1]^d$.
- *If* $\mu < 0$, *the distribution of* $\xi(1)$ *is non–lattice,* $\varphi(s) = \log \mathbb{E}\, e^{s\xi(1)}$ *exists for* $s \in [0, s_0)$ *with the maximal* $s_0 \in (0, \infty]$ *and* $\exists\, s^* \in (0, s_0)$: $\varphi(s^*) = 0$ *then*

$$s^* T_n - d \log n - (d-1) \log\log n - c \xrightarrow{d} Y,$$

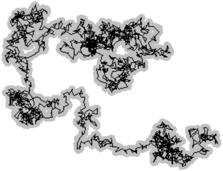
where Y *is standard Gumbel distributed r. v. and* c *is a constant.*

Open problems

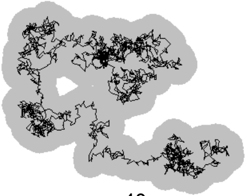
Limit theorems for the Wiener sausage

Let $S(T) = \{X(t) : t \in [0, T]\}$ be the path of the Brownian motion $X \subset \mathbb{R}^d$ with variance σ^2 up to time $T > 0$.

- Wiener sausage S_r of radius $r > 0$: $S_r = S(T) \oplus B_r(o)$



$r = 10$



$r = 40$

A realization of S_r

Intrinsic volumes of the Wiener sausage

- Intrinsic volumes $V_0(S_r), \ldots, V_d(S_r)$ are well–defined a.s. for $d \leqslant 3$, $r > 0$;
  - $V_d(S_r) = |S_r|$
  - $2V_{d-1}(S_r) = \mathcal{H}^{d-1}(\partial S_r)$
  - $V_i(S_r) = (-1)^{d-i-1} V_i\left(\overline{\mathbb{R}^d \setminus S_r}\right)$, $i = 0, \ldots, d-2$, where

    ∂S_r is a Lipschitz manifold with reach$\left(\overline{\mathbb{R}^d \setminus S_r}\right) > 0$ a.s.
- Compute $\mathbb{E}\, V_i(S_r)$, $i = 0, \ldots, d$.
  It is proved that $\mathbb{E}\, V_i(S_r) < \infty$, $i = d, d-1$ for all $d \geqslant 2$ and $\mathbb{E}\, V_0(S_r) < \infty$ for $d = 2$ (Rataj, Schmidt, Meschenmoser, S. (2005, 2009).

Mean volume of the Wiener sausage

- Explicit formulae
  - $d = 2$: Kolmogorov, Leontovich (1933)
  - $d = 3$: Spitzer (1964)
  - $d \geqslant 4$: Berezhkovskii *et al.* (1989)
- Asymptotics of the volume
  - Getoor (1965)
  - Donsker, Varadhan (1975)
  - Le Gall (1988): CLT for shrinking Wiener sausage ($T \to \infty$ or $r \to 0$)
  - van den Berg, Bolthausen (1994)
  - ...

Other mean intrinsic volumes

- Mean surface area: Rataj, Schmidt, S. (2005)
- Support measures and mean curvature functions: Last (2005)
- Mean intrinsic volumes $\mathbb{E}\, V_i(S_r)$ of lower order $i = 0, \ldots, d-2$: an open problem.
  Approximations can be obtained numerically (Rataj, Meschenmoser, S. (2009))

Mean surface area of the Wiener sausage

Theorem (Rataj, Schmidt, S. (2005))

For $d \geqslant 2$, it holds

$$2\mathbb{E}V_{d-1}(S_r) = d\kappa_d r^{d-1} + \frac{4d^2\kappa_d r^{d-1}}{\pi^2}\int_0^\infty \frac{1-e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y^3\left(J_\nu^2(y)+Y_\nu^2(y)\right)}\,dy$$
$$+ d\kappa_d\sigma^2 r^{d-3}T\left(\frac{(d-2)^2}{2} - \frac{4}{\pi^2}\int_0^\infty \frac{e^{-\frac{\sigma^2 y^2 T}{2r^2}}}{y\left(J_\nu^2(y)+Y_\nu^2(y)\right)}\,dy\right)$$

for almost all radii $r > 0$, $\nu = (d-2)/2$. For $d = 2, 3$, this formula holds for all $r > 0$. In the case $d = 3$, it simplifies to

$$\mathbb{E}V_2(S_r) = 4\pi r^2 + 8r\sigma\sqrt{2\pi T} + 2\pi\sigma^2 T\,.$$

Mean surface area of the Wiener sausage

- Asymptotic behaviour (Rataj, Schmidt, S. (2009))

$$2\mathbb{E}V_{d-1}(S_r) \sim \begin{cases} \pi\sigma^2 T r^{-1} \log^{-2} r & \text{if } d = 2, \\ 2\pi\sigma^2 T & \text{if } d = 3, \\ d\kappa_d \sigma^2 T \frac{(d-2)^2}{2} r^{d-3} & \text{if } d \geqslant 4 \end{cases} \quad \text{as } r \to 0.$$

- LT for the volume (Le Gall (1988)): for $d = 2$, it holds

$$(\log r)^2(|S_r| + \pi/\log r)) \xrightarrow{d} c - \pi^2\gamma, \quad r \to 0,$$

  where $\sigma^2 = T = 1$ and γ is the (renormalized) Brownian local time of self–intersections. For $d \geqslant 3$: CLT.
- Open problem: LT for the surface area $2V_{d-1}(S_r)$ and other intrinsic volumes $V_j(S_r)$ of the shrinking Wiener sausage!

Open problems

Limit theorems for excursion sets of stationary random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary C^2–smooth random field, $A(X, u) = \{t \in \mathbb{R}^d : X(t) > u\}$, $u \in \mathbb{R}$ its excursion sets.

- $\mathbb{E}\, V_j(A(X, u))$ for Gaussian and related random fields: Adler, Taylor (2007)
- LT for $V_j(A(X, u))$:
  - Volume $V_d(A(X, u))$:
    - classical results for random processes and Gaussian random fields
    - (BL, θ)-dependent random fields: Bulinski, S., Timmermann; Meschenmoser, Shashkin (2009)
    - Other random fields: Open problem
  - Other intrinsic volumes: Open problem

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