Infinite closed Jackson networks

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Abstract

In this paper we prove the results concerning the limiting behaviour (as time parameter $t \to \infty$) of closed Jackson networks with an infinite number of nodes and possibly infinite number of customers. As this system appears to be not ergodic we describe the class of stationary distributions and explore the character of non–stability in particular cases.
Keywords: *particle systems, zero–range interaction process, Bose–Einstein speeds, Jackson networks, Markov chains, excessive measures, potential, supermartingales, majorizing*

1 Introduction

For the moment not so much is known about the long–time behaviour of closed Jackson networks with an infinite number of nodes (ICJN) and customers inside. Although this consideration would be a logical continuation of the finite case (ordinary closed Jackson networks (later on CJN) with N nodes and m customers), which is always ergodic (see [1]), it turns out that the notion of infinity is here essential, and ICJN is never ergodic. The first attempt to investigate the behaviour of finite CJN as $N \to \infty$, $m \to \infty$, $\frac{m}{N} \to c > 0$ was made in [2]. Papers [3, 4] consider the mean–field approximation for the stochastical dynamics of CJN.

ICJN could be also viewed as a transport network system: Poisson streams of customers with intensities γ_i enter the nodes $i \in J$ (J is a countable set of nodes) where the vehicles carrying them to certain destinations (other nodes) are stationed, $\sum\limits_{i \in J} \gamma_i = \gamma \leq \infty$. If there is a car in a node i, it carries a passenger to node j with probability p_{ij}. Then the customer leaves the system. Case $\gamma < \infty$ was considered in [5].

Our approach here is to consider ICJN as an interaction particle process (see, e.g., [6]), namely, as a zero–range process introduced by Spitzer [7]. The case of homogeneous zero–range interaction at Bose – Einstein speeds was investigated by E. Waymire [8], E. D. Andjel [9], A. Galves and H. Guiol [10]. Here we also allow the transition speeds to be dependent on the location of a cell (non–homogeneity).

The paper is organized as follows: in section 2 the model of zero–range interaction is introduced, necessary definitions are made and the existence theorem is proved. The results of section 3 yield invariant measures and the character of transience. Section 4 is devoted to the investigation of special case $\gamma < \infty$.

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2 Model: Existence and Monotonicity

Consider the following non–homogeneous zero–range interaction model at Bose – Einstein speeds: a number of indistinguishable particles is located in a countable set of sites J. Transitions of particles at a site $i \in J$ are made after random periods of time τ — i. i. exponentially d. r. v. with parameter γ_i: if site i is not empty, then one (and only one) particle chosen randomly at this site instantly moves to any site j with probability p_{ij}. Transitions are made independently for any $i \in J$. Probabilities of jumps from i to j form together single particle law matrix $\mathbf{P} = (p_{ij})_{i,j\in J}\,,\ \forall\, i \in J \sum\limits_{j\in J} p_{ij} = 1$. One can describe the state of the system by process $\eta(t) = \{\eta_i(t)\}_{i\in J}\,,\ t \geq 0$, where $\eta_i(t)$ is a number of particles in cell i at time t.

Let $\mathbf{Z}_+ = \mathbf{N} \cup \{\mathbf{0}\},\ \ \overline{\mathbf{Z}}_+ = \mathbf{Z}_+ \cup \{\infty\}$. Then the state space of our system $\mathbf{W} = \overline{\mathbf{Z}}_+^J$ is a compact metrizable space. Let $\mathcal{B}$ be a σ – algebra generated by open sets in the product topology. Let $C(\mathbf{W})$ be the Banach space of all real–valued continuous functions on $\mathbf{W}$ with the uniform norm.

Let $\forall\ f \in C(\mathbf{W})\ \forall\ i \in J$ the "measure of dependence on coordinate i" be

$$\Delta_f(i) = \sup\left\{\mid f(\eta) - f(\zeta) \mid : \eta, \zeta \in \mathbf{W},\ \eta_j = \zeta_j\ \forall\ i \neq j\right\}.$$

Introduce $D(\mathbf{W}) = \left\{ f \in C(\mathbf{W}) : |||\ f\ ||| \stackrel{def}{=} \sum\limits_{i\in J} \Delta_f(i) < \infty \right\}$. $D(\mathbf{W})$ is dense in $C(\mathbf{W})$ (*see* [6]). Define the following operator on $D(\mathbf{W})$:

$$A\,f(\eta) = \sum_{i\in J}\sum_{j\in J}\left[I\{\eta_i > 0\}\gamma_i p_{ij}\left(f(\ldots\eta_i - 1\ldots\eta_j + 1\ldots) - f(\eta)\right)\right].$$

Theorem 2.1 (Existence) *If*

$$\sup_{i\in J}\gamma_i < \infty,\ \ \sup_{i\in J}\sum_{j\in J}\gamma_j p_{ji} < \infty, \tag{2.1}$$

then

1. *There exists unique Feller process $\eta(t) : (\Omega, \Im, \mathbf{P}) \to (\mathbf{W}, \mathcal{B})$ that describes our system; $(\Omega, \Im, \mathbf{P})$ — some probability space.*
2. *Operator $\overline{A}$ (closure of A) is an infinitesimal operator for process η. $D(\mathbf{W})$ is a core of $\overline{A}$.*

Proof:
The infinitesimal properties of any particle system are described by the collection of transition intensity measures $c_T(\eta, \xi)$ on $\overline{\mathbf{Z}}_+^T$: here $\eta = (\eta_i)_{i\in J}\,,\ T = \{i_1, \ldots i_n\}$ is any finite subset of J, and $c_T(\eta, \xi)$ is the intensity of transition from state η of the system to $\eta^\xi = (\zeta_j,\ j \in J : \zeta_j = \eta_j\ \forall j \notin T,\ \ \zeta_{i_k} = \xi_k\ \forall k = 1\ldots n)$ involving only $T = \{i_1, \ldots i_n\}$ coordinates. We have for

$$|T| > 2,\ |T| = 1 : \qquad c_T(\eta, \xi) = 0$$

$$|T| = 2,\ T = \{i, j\} : \ \ c_T(\eta, \xi) = \begin{cases} \gamma_i p_{ij} I\left\{\eta_i > 0\right\},\ \text{if } \xi_1 = \eta_i - 1,\ \xi_2 = \eta_j + 1 \\ \gamma_j p_{ji} I\left\{\eta_j > 0\right\},\ \text{if } \xi_1 = \eta_i + 1,\ \xi_2 = \eta_j - 1 \\ 0,\ \text{otherwise} \end{cases}. \tag{2.2}$$

Construct $A\,f(\eta) = \sum\limits_{T} \int\limits_{\overline{\mathbf{Z}}_+^T} c_T(\eta, d\zeta)\left(f(\eta^\zeta) - f(\eta)\right)$. Let $c_T(i) = \sup\{\|c_T(\eta_1, d\zeta) - c_T(\eta_2, d\zeta)\|_T \,|\, \eta_1, \eta_2 :\ \eta_1(j) = \eta_2(j)\ \forall j \neq i\}$ where $\|\cdot\|_T$ is a total variation norm of a measure on $\overline{\mathbf{Z}}_+^T$. Let us define $\nu(i,j) = \sum\limits_{T \ni i} c_T(j)$, $i \neq j$, and $\nu(i,i) = 0$. In order to apply existence theorem 3.9 [6] we need to verify the following conditions:

$$\sup_{i\in J} \sum_{T\ni i} \sup_{\eta\in\overline{\mathbf{Z}}_+^T} c_T\left(\eta, \overline{\mathbf{Z}}_+^T\right) < \infty, \tag{2.3}$$

$$\sup_{i\in J} \sum_{j\in J} \nu(i,j) < \infty$$

Taking into account (2.2) one can rewrite these conditions as follows:

$$\sup_{i\in J} \sum_{j\in J} (\gamma_i p_{ij} + \gamma_j p_{ji}) < \infty. \tag{2.4}$$

Since by our assumptions $\sum\limits_{j\in J} p_{ij} = 1\ \forall i$, condition (2.4) is obviously equivalent to (2.1). Then A is a pregenerator of the semigroup of transition operators that uniquely define the desired Markov process. The application of the general existence theory in [6] fulfils the proof.

Corollary 2.1 *If one of the following holds:*

- $\sup\limits_{i\in J} \gamma_i < \infty$, $\sup\limits_{i\in J} \sum\limits_{j\in J} p_{ji} < \infty$
- $\sum\limits_{i\in J} \gamma_i < \infty$,

then conditions (2.1) are satisfied.

Later on assume that (2.1) holds.

Introduce a partial ordering on $\mathbf{W}$: we say that $\xi \prec \xi'$ for ξ, $\xi' \in \mathbf{W}$ if $\xi_i \leq \xi'_i$ for all $i \in J$ (if $\xi'_i = \infty$ the last inequality always holds).

Lemma 2.1 (Monotonicity) *Consider two ICJN processes $\eta(t)$ and $\eta'(t)$ with the same $\gamma = \{\gamma_i\}_{i\in J}$ and single particle law P such that $\eta(0) = \xi \in \mathbf{W}$, $\eta'(0) = \xi' \in \mathbf{W}$, and $\xi \prec \xi'$. Then there exists a process $S(t) = (\xi(t), \xi'(t))$ on the phase space $\mathbf{W} \times \mathbf{W}$ such that $S(0) = (\xi, \xi')$ and $\xi(t) \prec \xi'(t)$ a.s., and processes $\xi(t)$ and $\xi'(t)$ are the stochastic copies of $\eta(t)$ and $\eta'(t)$, respectively (in the sense of finite-dimensional distributions).*

The proof of the lemma is standard and involves the construction of processes $\eta(t)$ and $\eta'(t)$ on the same probability space (see [8], theorem 4.2 and [13], lemma 2).

We shall say (under the conditions of lemma 2.1) that $\eta'(t)$ *stochastically majorizes* $\eta(t)$: $\eta(t) \prec \eta'(t)$.

Suppose the Markov chain with state space J and transition matrix P to be homogeneous, aperiodic, and irreducible. Introduce

Conjecture A: There exists a non–trivial invariant measure $\pi = (\pi_i)_{i\in J}$ for P (not necessarily finite).
Conjecture B: There exists unique non–trivial invariant probability measure $\pi = (\pi_i)_{i\in J}$ for P (later on called *stationary*).
Conjecture C: A countable Markov chain with transition matrix P is transient.

3 General case $\gamma \leq \infty$

Denote

$$a \stackrel{def}{=} \sup_{i\in J} \frac{\pi_i}{\gamma_i} < \infty. \tag{3.1}$$

Let $\rho_{\max} = 1/a$. For any $\rho \in [0; \rho_{\max}]$ and $\rho = \infty$ introduce product measures $L_\rho(\cdot)$ on $(\mathbf{W}, \mathcal{B})$ with marginal factors $l^i_\rho(\cdot),\ i \in J$ defined in the following two cases:
if $\rho \in [0; \rho_{\max}]$ and $\rho(\pi_i/\gamma_i) < 1$, then

$$l^i_\rho(k) = \begin{cases} (1 - \rho(\pi_i/\gamma_i))\,(\rho(\pi_i/\gamma_i))^k,\ k \in \mathbf{Z}_+, \\ 0,\ k = \infty \end{cases}$$

else we have $\rho = \rho_{\max}$, $\rho(\pi_i/\gamma_i) = 1$ or $\rho = \infty$, and

$$l^i_\rho(k) = \begin{cases} 0,\ k \in \mathbf{Z}_+, \\ 1,\ k = \infty. \end{cases}$$

Introduce $\mathbf{L} = \{L_\rho(\cdot) :\ \rho \in [0; \rho_{\max}],\ \rho = \infty\}$. Denote by $\tilde{\mathbf{L}}$ the closed convex hull of $\mathbf{L}$:

$$\tilde{\mathbf{L}} = \bigcap_{K\text{–closed convex set},\ K\supseteq\mathbf{L}} K.$$

Let $\mathcal{M}$ be the class of all invariant measures for Markov process $\{\eta(t)\}_{t\geq 0}$.

The following four assertions could be proved by slight modification of the proofs of propositions 2.14, 3.1, 2.15, 2.16 given in [8]; to see that one should take measure $\left\{\frac{\pi_i}{\gamma_i}\right\}_{i\in J}$ instead of $\bar{a} = (\bar{a}(x) : x \in J)$ (using the notations of [8]) there.

Theorem 3.1 (Invariant measures) *Suppose that conjecture A and (3.1) hold. Then the closed convex hull* $\tilde{\mathbf{L}}$ *of the set of measures* $\{L_\rho(\cdot) :\ \rho \in [0; \rho_{\max}],\ \rho = \infty\}$ *belongs to the class* $\mathcal{M}$ *of all invariant measures for Markov process* $\{\eta(t)\}_{t\geq 0}$.

Proof:
The class of measures $\mathbf{L}$ belongs to $\mathcal{M}$ according to theorem 2.14 [8]. But $\mathcal{M}$ itself is a compact closed subset of the set $\mathcal{P}$ of all probability measures on $(\mathbf{W}, \mathcal{B})$ (see [6], proposition 1.8). Then $\tilde{\mathbf{L}} \subseteq \mathcal{M}$ by definition of $\tilde{\mathbf{L}}$.

Let

$$\sum_{i\in J} \frac{\pi_i}{\gamma_i} < \infty. \tag{3.2}$$

Proposition 3.1 *Suppose that conjecture A and (3.2) hold. Then* $\forall\ \rho \in [0; \rho_{\max}]$

$$L_\rho\left(\eta : \sum_{i\in J} \eta_i < \infty\right) = 1.$$

Theorem 3.2 (Clustering) *Let conjecture A and 3.2 hold. Suppose* $j_0 \in J$ *satisfies* $\pi_{j_0}/\gamma_{j_0} = \max_{i\in J} \pi_i/\gamma_i$. *Then* $\forall\ k \in \mathbf{Z}_+\ \ \forall \eta^0 : \sum_{i\in J} \eta_i^0 = \infty$

$$\lim_{t\to\infty} \mathbf{P}\left\{\eta_{j_0}(t) > k \mid \eta(0) = \eta^0\right\} = 1.$$

Corollary 3.1 (All invariant measures on $\left(\mathbf{Z}_+^J, \mathcal{B}\cap\mathbf{Z}_+^J\right)$**)**

If conjecture A and (3.2) hold, then

$$\left\{L_\rho^N(\cdot) = L_\rho\left(\cdot \mid \sum_{i\in J} \eta_i = N\right) :\ N \in \mathbf{N},\ \rho \in (0; \rho_{\max})\right\}$$

are all invariant measures on $\left(\mathbf{Z}_+^J, \mathcal{B}\cap\mathbf{Z}_+^J\right)$.

Say $G_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)}\ \ \forall i, j \in J$ where $p_{ij}^{(n)}$ is the probability to reach state i from state j in n steps.

Denote by $\eta_{\bar{J}}(t)$ the restriction of $\eta(t)$ to some subset $\bar{J}$ of the nodes of J. Let $\{J_n\}_{n=1}^{\infty}$ be the sequence of non–decreasing finite subsets of J: $J_1 \subset J_2 \subset J_3 \ldots, \bigcup_n J_n = J$, chosen so that matrices $P^{(n)} = (p_{ij})_{i,j\in J_n}$ are irreducible. For given $\{\gamma_i\}_{i\in J}$, P and $\xi \in \mathbf{Z}_+^J$ let $\eta(t)$ be the ICJN–process starting from $\eta(0) = \xi$, and $\eta^{(n)}(t)$ be the one that starts from

$$\eta^{(n)}(0) = \begin{cases} \xi_j,\ j \in J_n, \\ \infty,\ j \notin J_n. \end{cases}$$

It follows from lemma 2.1 that $\eta(t) \prec \eta^{(n)}(t)$. One can see that the restriction $\eta_{J_n}^{(n)}(t)$ is an opened Jackson network on the set J_n with input intensities $\Delta^{(n)} = (\Delta_j^{(n)})_{j\in J_n}$, $\Delta_j^{(n)} = \sum_{i\in J\setminus J_n} \gamma_i p_{ij}$. Let $\rho^{(n)} = (\rho_j^{(n)})_{j\in J_n}$ be the unique solution of the stream conservation equation (see [1]):

$$\rho^{(n)} = \rho^{(n)} P^{(n)} + \Delta^{(n)}.$$

Matrix $I - P^{(n)}$ is invertible as $P^{(n)}$ is an irreducible submatrix of $P^{(n+1)}$ with the spectral radius not greater then 1. Due to that one can write

$$\rho^{(n)} = \Delta^{(n)}(I - P^{(n)})^{-1} = \Delta^{(n)}(I + P^{(n)} + (P^{(n)})^2 + \ldots).$$

Now we are ready to formulate the following proposition that would give us the tool to obtain the conditions under which all particles in the system vanish as $t \to \infty$.

Proposition 3.2 *Suppose* $\eta(0) = \xi$. *If for any* n_0 *and* $k \in J_{n_0}$ $\rho_k^{(n)} \to 0$ *as* $n \to \infty$, *then for all* $\xi \in \mathbf{Z}_+^J$ $\eta(t) \longrightarrow 0$ *weakly as* $t \to \infty$.

Proof:
As the convergence of any finite marginal distributions of $\eta(t)$ is obviously equivalent to the weak convergence of the distribution measures of $\eta(t)$ we shall actually prove that for any finite $\bar{J} \subset J$

$$\lim_{t\to\infty} P(\eta_{\bar{J}}(t) = 0 \mid \eta(0) = \xi) = 1.$$

In what follows we shall use the next result given in [11]: let $u_{n,j}$ be the unique solution of the finite system of equations

$$u_{n,j} = \sum_{k\in J_n} \min[1, u_{n,k}]p_{ij} + \Delta_j^{(n)}.$$

Say $J_n^0 = \{j \in J_n : u_{n,j} < 1\}$, $J_n^1 = \{j \in J_n : u_{n,j} \geq 1\}$. Then the distributions of $\eta_{J_n}^{(n)}(t) \in \mathbf{Z}_+^{J_n}$ weakly converge to the invariant measure

$$\pi_{u_n}(z) = \prod_{j\in J_n^0} (1 - u_{n,j})u_{n,j}^{z_j} \prod_{j\in J_n^1} \delta_\infty(z_j),$$

where $\delta_\infty(\cdot)$ is a Dirac measure concentrated in ∞. The inequality $u_n \leq \rho^{(n)}$ yields $u_{n,k} \to 0$ for any $k \in J_{n_0}$.

It follows from $\eta(t) \prec \eta^{(n)}(t)$ that

$$P(\eta_{J_{n_0}}(t) = 0 \mid \eta(0) = \xi) \geq P(\eta_{J_{n_0}}^{(n)}(t) = 0 \mid \eta(0) = \xi).$$

Consequently,

$$\liminf_{t\to\infty} P(\eta_{J_{n_0}}(t) = 0 \mid \eta(0) = \xi) \geq \lim_{t\to\infty} P(\eta_{J_{n_0}}^{(n)}(t) = 0 \mid \eta(0) = \xi).$$

The r.h.s. term tends to 1 as $n \to \infty$. Therefore, $P(\eta_{J_{n_0}}(t) = 0 \mid \eta(0) = \xi) \to 1$ as $t \to \infty$. Then we note that for arbitrary finite subset $\bar{J}$ one can choose n_0 such that $J_{n_0} \supset \bar{J}$; thus the proposition is proven.

Theorem 3.3 (Devastation) *Suppose that one of the following conditions holds:*

1. $\gamma < \infty$ *and conjecture C*
2. $\gamma < \infty$ *and* $\forall j \in J$ $\sum_{i\in J} p_{ij} \leq 1$
3. $\forall j \in J$ $\sum_{i\in J} p_{ij} \leq 1$ *and there exists a sequence of non–decreasing finite sets* $\{J_n\}_{n=1}^\infty$, $J_1 \subset J_2 \subset \ldots$, $\bigcup_n J_n = J$ *such that* $\sup_{j\in J\setminus J_n} \gamma_j \longrightarrow 0$ *as* $n \to \infty$ *and matrices* $P^{(n)} = (p_{ij})_{i,j\in J_n}$ *are irreducible*
4. *Let* $\forall j \in J$ $\sum_{i\in J} p_{ij} \leq 1$. *Introduce the terminating Markov chain* $Y = \{Y_m\}_{m=1}^\infty$ *with state space* J *and transition matrix* $P^T = (p_{ji})_{i,j\in J}$. *Denote by* $\bar{P}_j(J)$ *the probability of terminating for* Y *provided that* $Y_0 = j$. *Let* $\bar{P}_j(J) = 1$ *for all* $j \in J$ *and* $\sup_{j\in J} \gamma_j < \infty$.

Then for any $\eta(0) \in \mathbf{Z}_+^J$ *the process* $\eta(t) \to 0$ *weakly as* $t \to \infty$.

Proof:
Let us verify that the conditions of proposition 3.2 hold in any of the cases 1) – 4). Then immediately applying the above proposition one can come to the desired conclusion.

1) Because of the transience of P the sum $I+P+P^2+\ldots = G = (G_{ij})_{i,j\in J} < \infty$ is finite (i. e. $G_{ij} < \infty\ \forall i,j \in J$). Obviously,

$$G^{(n)} = (I - P^{(n)})^{-1} = I + P^{(n)} + (P^{(n)})^2 + \ldots \le G_{J_n} = (G_{ij})_{i,j\in J_n}.$$

It is a well–known fact (see [12], theorem 13 of Chapter 2) that $G_{jk} \le G_{kk}\ \forall j,k \in J$.
The explicit form of $\rho_k^{(n)}$ is

$$\rho_k^{(n)} = (\Delta^{(n)} G^{(n)})_k = \sum_{i\in J\setminus J_n} \sum_{j\in J_n} \gamma_i p_{ij} G_{j,k}^{(n)}. \tag{3.3}$$

It follows from the inequalities $G_{j,k}^{(n)} \le G_{jk} \le G_{kk}$ that

$$\rho_k^{(n)} \le G_{kk} \sum_{i\in J\setminus J_n} \sum_{j\in J_n} \gamma_i p_{ij},$$

and as $\sum\limits_{j\in J_n} p_{ij} \le 1$, we get

$$\rho_k^{(n)} \le G_{kk} \sum_{i\in J\setminus J_n} \gamma_i. \tag{3.4}$$

Due to the convergence of the series $\sum\limits_{i\in J} \gamma_i$ the right–hand side of (3.4) tends to zero as $n \to \infty$. Hence $\rho_k^{(n)} \to 0$ as $n \to \infty$.

2) Let for any $i \in J_n$ $\mathcal{L}_{n,j}$ be the set of all sequences $(j, j_1, \ldots, j_l)$ of arbitrary length l such that $j_k \in J_n$, $k < l$, while $j_l \in J \setminus J_n$. Then one can rewrite (3.3) in the following way:

$$\rho_j^{(n)} = \sum_{\mathcal{L}_{n,j}} (\prod_{k=0}^{l-2} p^T_{j_k j_{k+1}}) p^T_{j_{l-1} j_l} \gamma_{j_l},$$

where p^T_{ij} are the elements of the transpose of P: $p^T_{ij} = p_{ji}$. Evidently,

$$\rho_j^{(n)} \le \bar\gamma_{J_n} \sum_{\mathcal{L}_{n,j}} (\prod_{k=0}^{l-2} p^T_{j_k j_{k+1}}) p^T_{j_{l-1} j_l} = \bar\gamma_{J_n} x_j^{(n)},$$

where $\bar\gamma_{J_n} = \sup\limits_{j\in J\setminus J_n} \gamma_j$. One can interpret $x_j^{(n)}$ as the probability of the event that the terminating Markov chain Y with state space J and transition matrix P^T starting from j ever enters the set $J \setminus J_n$. In view of that $x_j^{(n)} \le 1$. Hence, $\rho_j^{(n)} \le \bar\gamma_{J_n}$. Then $\bar\gamma_{J_n} \to 0$ as $n \to \infty$ because of the convergence of $\sum\limits_{j\in J} \gamma_j$, that yields $\rho_k^{(n)} \to 0$ as $n \to \infty$.

3) The above reasoning holds also for this case, as the requirement $\bar\gamma_{J_n} \to 0$ is stated now in the assumptions of 3).

4) One can adopt the proof in 2) for the case $\gamma = \infty$. Namely, it is clear that $x_j^{(n)} \to 1 - \bar P_j(J)$ as $n \to \infty$. Then it is sufficient to require $\bar P_j(J) = 1$, $\sup\limits_{j\in J} \gamma_j < \infty$ to get $\rho_j^{(n)} \to 0$.

Definition 3.1 *The i-th coordinate $\eta_i(\cdot)$ of the process η (the i-th cell of the system) is* stochastically bounded *if* $\lim_{m\to\infty} \sup_{t\in\mathbf{R}_+} P\{\eta_i(t) > m\} = 0$.

Theorem 3.4 (Stochastical boundedness) *If $\exists i_0 \in J$: $\gamma_{i_0} > \sum_{j\in J} \gamma_j p_{ji_0}$ and $\eta_{i_0}(0) \in \mathbf{Z}_+$, then $\eta_{i_0}(t)$ is stochastically bounded, and*

$$\limsup_{t\to\infty} E\eta_{i_0}(t) \le \frac{\sum\limits_{j\in J,\, j\ne i_0} \gamma_j p_{ji_0}}{\gamma_{i_0} - \sum\limits_{j\in J} \gamma_j p_{ji_0}}.$$

Proof:
Construct the process $\eta'(t)$ such that

$$\eta'(0) = \begin{cases} \eta_j(0), & j = i_0, \\ \infty, & j \ne i_0. \end{cases}$$

Again, using lemma 2.1, we obtain $\eta(t) \prec \eta'(t)$. Note that $\eta'(0)$ is an opened finite Jackson network with only one node i_0. The intensity of the input stream $\Delta^{(1)}_{i_0}$ of this network is equal to $\sum_{j\in J, j\ne i_0} \gamma_j p_{ji_0}$, the intensity of service equals γ_{i_0}. The probability to quit the system is $1-p_{i_0i_0}$. Then the stream conservation equation would be $\rho^{(1)}_{i_0} = \Delta^{(1)}_{i_0} + \rho^{(1)}_{i_0} p_{i_0i_0}$, and $\eta'_{i_0}(\cdot)$ is ergodic iff $\rho^{(1)}_{i_0} < \gamma_{i_0}$, i. e., $\Delta^{(1)}_{i_0}/(1-p_{i_0i_0}) < \gamma_{i_0}$, or

$$\sum_{j\in J,\, j\ne i_0} \gamma_j p_{ji_0} < \gamma_{i_0}(1 - p_{i_0i_0}).$$

The last inequality holds due to the assumptions of the theorem. Consequently, $\eta'_{i_0}(\cdot)$ is ergodic and stochastically bounded. Therefore, $\eta_{i_0}(\cdot)$ is stochastically bounded. Let $\eta'_{i_0}(\infty) = \lim_{t\to\infty} \eta'_{i_0}(t)$. Then $\eta'_{i_0}(\infty)$ has a geometrical distribution $P\left\{\tilde{\eta_{i_0}}(\infty) = n\right\} = (1-\alpha)\alpha^n,\ n \in \mathbf{Z}_+$ with parameter (see [1])

$$\alpha = \frac{\sum\limits_{j\in J,\, j\ne i_0} \gamma_j p_{ji_0}}{(1 - p_{i_0i_0})\gamma_{i_0}}.$$

It follows from $\eta(t) \prec \eta^{(n)}(t)$ that $E\eta_{i_0}(t) \le E\eta'_{i_0}(t)$ for all $t \ge 0$ and

$$\limsup_{t\to\infty} E\eta_{i_0}(t) \le E\eta'_{i_0}(\infty) = \frac{\alpha}{1-\alpha}.$$

Substituting α, we obtain the last statement of the theorem.

4 Case $\gamma < \infty$

In this case one can prove that the transitions of particles occur a. s. only in discrete random moments of time t_n, $t_n - t_{n-1} = \tau_n \sim$ i. i. exponentially d. r. v. with parameter $\gamma = \sum_{i\in J} \gamma_i$, n — number of transition since $t = 0$. Then

$$P\{n\text{–th transition is made from cell } i \mid \eta_i(t_n - 0) > 0\} = \frac{\gamma_i}{\gamma} \stackrel{def}{=} \beta_i, \quad \sum_{i\in J} \beta_i = 1.$$

Introduce embedded Markov chain $\eta(n) = \{\eta_i(t_n)\}_{i\in J}$ $\forall n\in \mathbf{N}$. Later on we shall write $\eta_i(n)$ instead of $\eta_i(t_n)$. Suppose $\eta_i(0) = x_i$ $\forall$ i, $\sum\limits_{i\in J} x_i = \infty$. Let us introduce the non–decreasing family of $\sigma-$algebras $\{\mathcal{F}_n\}_{n\in\mathbf{N}}$,

$$\mathcal{F}_n = \sigma\left(\eta(m),\ m \le n\right) = \sigma\left(\eta_i(m),\ i\in J,\ m\le n\right) \quad \forall n\,.$$

Definition 4.1 *Measure $\mu(\cdot)$ on J (possibly σ-finite) is called (strictly) excessive for transition matrix $P = (p_{ij})_{i,j\in J}$ iff $\mu \ge \mu P$: $\forall i$ $\mu_i \ge \sum\limits_{j\in J}\mu_j p_{ji}$ ($\mu > \mu P$, respectively).*

Definition 4.2 *Function f on J is called harmonious (resp. excessive) for transition matrix $P = (p_{ij})_{i,j\in J}$ iff $Pf = f$: $\forall i$ $f_i = \sum\limits_{j\in J} p_{ij}f_j$ ($Pf \le f$).*

Lemma 4.1 *Let $\mu(\cdot)$ and $\nu(\cdot)$ be finite measures on the same measurable space $(\Omega, \mathcal{L})$, $\mu(\Omega) = \nu(\Omega) = a < \infty$. If $\forall A \in \mathcal{L}$ $\quad \nu(A) \ge \mu(A)$, then $\mu(\cdot) = \nu(\cdot)$.*

Proof:
Suppose the contrary holds: $\exists\, A \in \mathcal{L} : \nu(A) > \mu(A)$. But by condition of the lemma $\nu(\bar{A}) \ge \mu(\bar{A})$. Summing these inequalities up one can obtain $a > a$. We arrived at a contradiction. Thus the statement of lemma is proved.

Lemma 4.2 *If conjecture B holds and $\{\beta_i\}_{i\in J} \ne \{\pi_i\}_{i\in J}$ or if conjecture B does not hold, then*

$$\exists\, i_0,\, j_0 : \quad \gamma_{i_0} > \sum_{j\in J}\gamma_j p_{ji_0}, \quad \gamma_{j_0} < \sum_{j\in J}\gamma_j p_{jj_0}$$

Proof:
If $\gamma < \infty$, then measures $\beta(\cdot) : \beta(i) = \beta_i, \quad \mu(\cdot) = (\beta P)(\cdot) : \mu_i = \sum\limits_{j\in J}\beta_j p_{ji}$ are probability measures on J. Suppose one of the following inequalities holds:

$$\beta \ge \mu \tag{4.1}$$

$$\beta \le \mu \tag{4.2}$$

Then lemma 4.1 yields $\beta = \mu, \quad \beta = \beta P$. If conjecture B is true, it means $\beta = \pi$, otherwise it implies that β is a stationary distribution for P . Both cases are prohibited by the conditions of lemma 4.2. Then none of the inequalities (4.1), (4.2) holds which fulfils the proof.

Remark 4.1 *In case $\gamma < \infty$ under general assumptions on $\{\gamma_i\}_{i\in J}$ (see lemma 4.2) there exists i_0 such that the conditions of theorem 3.4 are satisfied. So at least one node of the system is always stochastically bounded. On the other hand, lemma 4.2 shows that this situation might not hold for all $i \in J$.*

Remark 4.2 *Lemma 4.2 states that if $\gamma < \infty$ then excessive probability measures β do not exist for P. But if $\gamma = \infty$, conjecture C holds, then there exist infinitely many $\sigma-$finite strictly excessive measures μ : $\sum\limits_{i\in J}\mu_i = \infty, \quad \mu > \mu P$. Then in case $\gamma = \infty$ for $\{\gamma_i\}_{i\in J}$: $\gamma_i > \sum\limits_{j\in J}\gamma_j p_{ji} \quad \forall i \in J$ all coordinates $\eta_i(t)$ are stochastically bounded.*

Proof:
If conjecture C holds, then for some i $G_{ii} < \infty$. The assumptions concerning the properties of a Markov chain with the transition matrix P guarantee the last inequality to hold $\forall i \in J$. Then for any $\{\nu_i\}_{i\in J} : \nu_i \geq 0 \quad \forall i \in J$ construct the measure $\mu = \{\mu_i\}_{i\in J}$ in the following way: $\mu_i = \sum_{j\in J} \nu_j G_{ji}$. One can choose $\{\nu_i\}_{i\in J}$ such that $\forall i \quad \mu_i < \infty$ (for example, $\nu_j = \delta_{jj_0} \Rightarrow \mu_i = G_{j_0 i} < \infty \quad \forall i$), $\{\mu_i\}_{i\in J}$ is strictly excessive. The application of theorem 3.4 fulfils the proof.

For some $\{a_i\}_{i\in J}, \quad a_i \geq 0, \quad \sum_{i\in J} a_i < \infty$ and for all $x \in \mathbf{W}$ define $f(x) = \sum_{i\in J} a_i x_i \leq \infty$.

Theorem 4.1 (Existence of supermartingales) *If conjecture C holds and there exist numbers* $\{\varphi_i\}_{i\in J} : \varphi_i \geq 0, \sum_{i\in J}\sum_{j\in J} G_{ij}\varphi_j < \infty$, *then for* $a_i = (G\varphi)_i = \sum_{j\in J} G_{ij}\varphi_j$ *and initial distributions of Markov chain* η *such that* $\sum_{i\in J} a_i\eta_i(0) < \infty$ *a. s. and* $E\sum_{i\in J} a_i\eta_i(0) < \infty$ *sequence* $(f(\eta(n)), \mathcal{F}_n)_{n=1}^{\infty}$ *is a supermartingale.*

Proof:
First let us prove that $f(\eta(n))$ defined above is finite $\forall n \in \mathbf{N}$ provided that the conditions of theorem 4.1 hold: $f(\eta(0))$ is clearly finite; then on each step n at most two coordinates are changed. It means that if $\eta(0) = x$, then $|\eta_i(n) - x_i| \leq n$ a. s. for all i. Therefore $f(\eta(n))$ is also finite $\forall n$. $(f(\eta(n)), \mathcal{F}_n)_{n=1}^{\infty}$ is a supermartingale iff by definition

$$E\left(f\left(\eta(n+1)\right) \mid \mathcal{F}_n\right) - f\left(\eta(n)\right) \leq 0 \quad \text{a. s.}$$

This inequality could be rewrited as

$$\sum_{i:\,\eta_i(n)>0} \gamma_i \left(\sum_{j\in J} p_{ij}a_j - a_i\right) \leq 0 \tag{4.3}$$

So if

$$\forall i \quad a_i \geq 0, \quad \sum_{j\in J} p_{ij}a_j \leq a_i, \tag{4.4}$$

$$\sum_{i\in J} a_i < \infty \tag{4.5}$$

(function a_i is summable and excessive for P), then inequality (4.3) holds and our theorem is proved. By the well-known criterion of ergodicity of countable Markov chains if P is transient, then there exist finite excessive functions for P that are not constant. Then by Riesz theorem (see [14], p.22) and condition (4.5) this function a_i must have the following representation:

$$a_i = (G\varphi)_i = \sum_{j\in J} G_{ij}\varphi_j < \infty \tag{4.6}$$

for some $\{\varphi_i\}_{i\in J} : \varphi_i \geq 0 \quad \forall i$. And vice versa, all $\{a_i\}_{i\in J}$ introduced in (4.6) are excessive. This representation is correct as $\forall i,j$ $G_{ij} < \infty$. For $\{a_i\}_{i\in J}$ to be summable we require the following condition:

$$\sum_{i\in J}\sum_{j\in J} G_{ij}\varphi_j < \infty.$$

Then the function a_i introduced in the statement of theorem 4.1 satisfies (4.4) and (4.5), thus $f(\eta(n))$ appears to be a supermartingale.

Proposition 4.1 *If* $\exists\, j_0 : \sum_{i\in J} G_{ij_0} < \infty$, *then there exist numbers* $\{\varphi_i\}_{i\in J} : \ \varphi_i \geq 0$ *such that the conditions of theorem 4.1 hold.*

Proof:
It is sufficient to choose $\varphi_j = \delta_{jj_0}$. If the statement of proposition 4.1 holds, then $\sum_{i\in J}\sum_{j\in J} G_{ij}\varphi_j < \infty$, and thus the conditions of theorem 4.1 are satisfied.

Proposition 4.2 *There exist matrices P for which the conditions of proposition 4.1 are satisfied.*

Proof:
Consider $J = \mathbf{N}$,

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ q & 0 & p & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & q & 0 & p & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & q & 0 & p & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \begin{matrix} p+q=1 \\ p,q>0 \\ p>q \end{matrix} .$$

This matrix P as a transition operator of a Markov chain describes the behaviour of the integer–valued random walk with impenetrable barrier at origin:

$$W_k = \max(0, W_{k-1}+B_k),\ k \geq 1,\ W_0 = i, \quad B_k = \begin{cases} 1, \text{ with probability } p, \\ -1, \text{ with probability } q. \end{cases}$$

This random walk is an irreducible aperiodic homogeneous Markov chain. According to the well–known results for random walks it is also transient (as $p > q$, see [15], p.28). Therefore for $i,\ j \in J\ G_{ij} < \infty$. Suppose $j_0 = 1$ and prove that $\sum_{i=1}^{\infty} G_{i,j_0} < \infty$. $G_{i,1} = E\left(A \mid W(0) = i\right)$ where $A = \sharp\left\{k : W(k) = 1\right\}$ denotes the number of k such that $W(k) = 1$. Introduce the following sequence of moments $\left\{\tau_k^{(i)}\right\}_{k=1}^{\infty}$:

$\tau_1^{(i)} = \inf\left\{n \geq 1 : W(n) = 1\right\},\ \tau_k^{(i)} = \inf\left\{n > \tau_{k-1}^{(i)} : W(n) = 1\right\},\ k > 1$, in which our chain W returns to state 1 provided that $W(0) = i$. These moments $\tau_k^{(i)}$ form the sequence of regeneration times for W (which could be also viewed as a regeneration process). Then $\theta_1 = \tau_1^{(i)},\ \theta_k = \tau_k^{(i)} - \tau_{k-1}^{(i)},\ k > 1$ are independent and $\{\theta_k\}_{k\geq 2}$ are also identically distributed as the duration of regeneration cycles (see [16], p.90). If $p > q$, then $\theta_1,\ \theta_2,\ \theta_3, \dots$ consist together the terminating renewal process. Let

$$b_1 = P\left\{\theta_1 = \infty\right\} > 0,\ b = P\left\{\theta_k = \infty\right\} > 0,\ k > 1.$$

Then the number of the last renewal has a distribution $P\left\{A = k\right\} = (1-b_1)b(1-b)^{k-1},\ k \geq 1$, $P\left\{A = 0\right\} = b_1$. b_1 depends on $W(0) = i$, but b does not. Calculate $E\left(A \mid W(0) = i\right)$:

$E(A \mid W(0)=i) = b(1-b_1)\sum\limits_{k\geq 1} k(1-b)^{k-1} = (1-b_1)\left(\sum\limits_{k\geq 0} kb(1-b)^{k-1} + b\sum\limits_{k\geq 0}(1-b)^k\right) = (1-b_1)\left(\frac{1-b}{b} + \frac{b}{1-(1-b)}\right) = \frac{1-b_1}{b}$.

Then

$$G_{i,1} = \frac{P\left\{\tau_1^{(i)} < \infty\right\}}{b}.$$

Let us prove that $P\left\{\tau_1^{(i)} < \infty\right\} = \left(\frac{q}{p}\right)^{i-1}$ $\forall i \geq 1$. In order to do that we shall use the method of generating functions developed in [17], p.84, lemma 1. Denote

$$\tau_1^{(i)}(n) = \begin{cases} \inf\{k \geq 1 : W(k) = 1\}, \text{ if } W_k < n \ \forall k < \tau_1^{(i)} \\ \infty, \text{ otherwise} \end{cases}.$$

(as before $W(0) = i$). Let $C_n = \left\{\tau_1^{(i)}(n) < \infty\right\}$, $C_n \subseteq C_{n+1}$ $\forall n$, $\bigcup\limits_{n=1}^{\infty} C_n = \left\{\tau_1^{(i)} < \infty\right\}$, hence $P\{C_n\} \underset{n\to\infty}{\longrightarrow} P\left\{\tau_1^{(i)} < \infty\right\}$. Let us calculate $P\{C_n\}$ $\forall n$. Introduce the generating function of $\tau_1^{(i)}(n)$:

$F_i^n(z) = E\left(z^{\tau_1^{(i)}(n)} \mid \tau_1^{(i)}(n) < \infty\right) = \sum\limits_{k=1}^{\infty} z^k P\left\{\tau_1^{(i)}(n) = k\right\}$. Then the following recurrent formulae could be deduced by means of the total probability formula:

$$F_i^n(z) = \begin{cases} pzF_{i+1}^n(z) + qzF_{i-1}^n(z), \ 1 < i < n \\ 1, \ i = 1 \\ 0, \ i = n \end{cases}. \tag{4.7}$$

Solving this equation one can see that $F_i^n(z) = A(z)\left(\frac{1-\sqrt{1-4pqz^2}}{2pz}\right)^i + B(z)\left(\frac{1+\sqrt{1-4pqz^2}}{2pz}\right)^i$, where $A(z)$ and $B(z)$ could be found from boundary conditions (4.7). And as $P\{C_n\} = F_i^n(1)$ one can calculate that $P\{C_n\} = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^i}{\left(\frac{q}{p}\right)^n - \frac{q}{p}} \underset{n\to\infty}{\longrightarrow} \left(\frac{q}{p}\right)^{i-1}$. Then $\sum\limits_{i=1}^{\infty} G_{i,1} = \frac{1}{b}\sum\limits_{i=1}^{\infty}\left(\frac{q}{p}\right)^{i-1} = \frac{p}{b(p-q)} < \infty$, so the conditions of proposition (4.1) are satisfied.

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