ON THE LOCAL CONNECTIVITY NUMBER OF STATIONARY RANDOM CLOSED SETS

Evgueni Spodarev,[1] and Volker Schmidt[1]

[1] *Universität Ulm*
Abteilung Stochastik
D-89069 Ulm, Germany
spodarev@mathematik.uni-ulm.de, schmidt@mathematik.uni-ulm.de

Abstract Random closed sets (RACS) in the d–dimensional Euclidean space are considered, whose realizations belong to the extended convex ring. A family of nonparametric estimators is investigated for the simultaneous estimation of the vector of all specific Minkowski functionals (or, equivalently, the specific intrinsic volumes) of stationary RACS. The construction of these estimators is based on a representation formula for the expected local connectivity number of stationary RACS intersected with spheres, whose radii are small in comparison with the size of the whole sampling window. Asymptotic properties of the estimators are given for unboundedly increasing sampling windows. Numerical results are provided as well.

Keywords: Mathematical morphology; random closed sets; stationarity; Minkowski functionals; intrinsic volumes; nonparametric estimation; local Euler–Poincaré characteristic; principal kinematic formula; Boolean model

Introduction

The theory of random closed sets (RACS) and its morphological aspects with emphasis on applications to image analysis have been developed in the second half of the 20th century. This scientific process has been significantly influenced by the pioneering monographs of G. Matheron [6] and J. Serra [15, 16]. It turned out that *Minkowski functionals* or, equivalently, *intrinsic volumes* are important characteristics in order to describe binary images, since they provide useful information about the morphological structure of the underlying RACS. In particular, the so–called *specific intrinsic volumes* of stationary RACS have been intensively studied for various models from stochastic geometry.

There exist several approaches to the construction of statistical estimators for particular specific intrinsic volumes of stationary RACS in two and three dimensions. However, in many cases, only little is known about goodness properties of these estimators, like unbiasedness, consistency, or distributional properties. Furthermore, an extra algorithm has to be designed for the estimation of each specific intrinsic volume separately.

In contrast to this situation, the method of moments proposed in the present paper provides a unified theoretical and algorithmic framework for simultaneous nonparametric estimation of all specific intrinsic volumes, in an arbitrary dimension $d \geqslant 2$. The construction principle of these estimators, which is similar to the approach considered in [11], is based on a representation formula for the (expected) local connectivity number of stationary RACS intersected with spheres, whose radii are small in comparison with the size of the whole sampling window. It can be considered as a statistical counterpart to a method for the simultaneous computation of all intrinsic volumes of a deterministic polyconvex set based on the principal kinematic formula.

Our estimators are unbiased by definition. Moreover, under suitable integrability and mixing conditions, they are mean–square consistent and asymptotically normal distributed. This can be used in order to establish asymptotic tests for the vector of specific intrinsic volumes.

Notice that the method of moments (which is also called the method of intensities by some authors) has been used in the analysis of various further statistical aspects of models from stochastic geometry, for example, in order to estimate the intensity of germs and other characteristics of the Boolean model; see e.g. [7], and Sections 5.3–5.4 in [13].

The present paper is organized as follows. Some necessary preliminaries on Minkowski functionals and intrinsic volumes, respectively, are given in Section 1. In Section 2, the computation of intrinsic volumes of deterministic polyconvex sets is briefly discussed. The above–mentioned representation formula for the (expected) local connectivity number of stationary RACS is stated in Section 3; see Proposition 3.1. We give an alternative proof of this representation formula which makes use of an explicit extension of Steiner's formula for convex bodies to the convex ring. The result of Proposition 3.1 is then used in Section 4 in order to construct a family of nonparametric estimators for all $d + 1$ specific intrinsic volumes simultaneously. The construction principle of these estimators is described and their asymptotic properties are discussed. A related family of least–squares estimators is also provided in Section 4. In Section 5, some aspects of variance reduction using *kriging of the mean* are touched upon. Finally, in Section 6 numerical results are given for the planar Boolean model with spherical primary grains. They are compared with those obtained by another method described in [10] for the computation of specific intrinsic volumes.

1. Minkowski functionals and intrinsic volumes

Let $d \geqslant 2$ be an arbitrary fixed integer and let $\mathcal{K}$ be the family of all *convex bodies*, i.e., compact convex sets in $\mathbb{R}^d$. The *convex ring* $\mathcal{R}$ in $\mathbb{R}^d$ is the family of all finite unions $\bigcup_{i=1}^m K_i$ of convex bodies $K_1, \ldots, K_m \in \mathcal{K}$. The elements of $\mathcal{R}$ are called *polyconvex sets*. Furthermore, the *extended convex ring* $\mathcal{S}$ is the family of all subsets $A \subset \mathbb{R}^d$ such that $A \cap K \in \mathcal{R}$ for any $K \in \mathcal{K}$. For $A, B \subset \mathbb{R}^d$, the *Minkowski sum* $A \oplus B$ and the *Minkowski difference* $A \ominus B$ are defined by $A \oplus B = \{x + y : x \in A, y \in B\}$ and $A \ominus B = \{x \in \mathbb{R}^d : B + x \subset A\}$, respectively. For any Borel set $B \subset \mathbb{R}^d$, denote by $V_d(B)$ its Lebesgue measure. It is well known that there exist nonnegative functionals $V_j : \mathcal{K} \to [0, \infty)$, $j = 0, \ldots, d$ such that for each $r > 0$ the volume $V_d(K \oplus B_r(o))$ of the so–called *parallel body* $K \oplus B_r(o)$ of any $K \in \mathcal{K}$ is given by *Steiner's formula*

$$V_d(K \oplus B_r(o)) = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K)\,, \tag{1}$$

where $o \in \mathbb{R}^d$ denotes the origin, $B_r(x) = \{y \in \mathbb{R}^d : |y-x| \leqslant r\}$ is the closed ball with midpoint $x \in \mathbb{R}^d$ and radius r, and κ_j is the volume of the unit ball in $\mathbb{R}^j$. The functionals V_j are called *intrinsic volumes*. They are closely related to the widely known *quermassintegrals* or *Minkowski functionals* W_j given by $W_j(K) = V_{d-j}(K)\, \kappa_j \big/ \binom{d}{j}$, $K \in \mathcal{K}$. There exists a unique additive extension of the functionals V_j to the convex ring $\mathcal{R}$ given by the *inclusion–exclusion formula*

$$V_j(K_1 \cup \ldots \cup K_m) = \sum_{k=1}^{m} (-1)^{k-1} \sum_{i_1 < \ldots < i_k} V_j(K_{i_1} \cap \ldots \cap K_{i_k}) \tag{2}$$

for any $K_1, \ldots, K_m \in \mathcal{K}$. The intrinsic volumes $V_j(K)$, $j = 0, \ldots, d$ provide information about the morphological structure of the polyconvex set $K \in \mathcal{R}$. For example, $V_d(K)$ is the usual volume, $2V_{d-1}(K)$ is the surface area, and the *Euler–Poincaré characteristic* $V_0(K)$ reflects the connectivity properties of K. Notice that in the planar case, that is $d = 2$, $V_0(K)$ is equal to the number of "clumps" minus the number of "holes" of $K \in \mathcal{R}$, i.e., the number of connected outer boundary components of K minus the number of its inner boundary components. In particular, $V_0(K) = 1$ for any convex body $K \neq \emptyset$. Furthermore, for any $K \in \mathcal{R}$ and $q, x \in \mathbb{R}^d$, $q \neq x$, the so–called *index* $J(K, q, x)$ of K is given by

$$J(K, q, x) = 1 - \lim_{\delta \to +0} \lim_{\varepsilon \to +0} V_0\left(K \cap B_{|x-q|-\varepsilon}(x) \cap B_\delta(q)\right) \tag{3}$$

for $q \in K$. For all $q \notin K$, we put $J(K, q, x) = 0$. In particular, $J(\emptyset, q, x) = 0$ for arbitrary $q, x \in \mathbb{R}^d$.

2. Computation of intrinsic volumes of a polyconvex set

Given a polyconvex set $K \subset \mathbb{R}^d$, apply the principal kinematic formula of integral geometry (cf. formula (4.5.3) in [12]) to the Euler–Poincaré characteristic of the intersection of K with an arbitrary translation of the ball $B_r(o)$. This yields

$$\int_{K \oplus B_r(o)} V_0(K \cap B_r(x))\, dx = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K)\,, \tag{4}$$

where the integration domain is $K \oplus B_r(o)$ since $V_0(K \cap B_r(x)) = 0$ for $x \notin K \oplus B_r(o)$. Introduce the notation $R_r = \int_{K \oplus B_r(o)} V_0(K \cap B_r(x))\, dx$.

Writing equation (4) for $d+1$ pairwise different radii $r_0, \dots, r_d$ yields the following system of $d+1$ linear equations:

$$A_{r_0 \dots r_d} V = R\,, \tag{5}$$

where $V = \big(V_0(K), \dots, V_d(K)\big)^\top$, $R = (R_{r_0}, \dots, R_{r_d})^\top$ and

$$A_{r_0 \dots r_d} = \begin{pmatrix} r_0^d \kappa_d & r_0^{d-1} \kappa_{d-1} & \dots & r_0^2 \kappa_2 & r_0 \kappa_1 & 1 \\ r_1^d \kappa_d & r_1^{d-1} \kappa_{d-1} & \dots & r_1^2 \kappa_2 & r_1 \kappa_1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_d^d \kappa_d & r_d^{d-1} \kappa_{d-1} & \dots & r_d^2 \kappa_2 & r_d \kappa_1 & 1 \end{pmatrix} \tag{6}$$

is a regular matrix. Then, $V = A^{-1}_{r_0 \dots r_d} R$ is the unique solution of (5). The integrals R_{r_i} can be approximated by

$$\widehat{R}_{r_i} = \Delta^d \sum_{k=1}^{m} V_0(K \cap B_r(x_k))\,, \tag{7}$$

where the points $x_1, \dots, x_m$ belong to a d–dimensional cubic lattice with mesh size Δ. Thus, the vector V can be computed numerically as

$$V \approx A^{-1}_{r_0 \dots r_d} \widehat{R}\,, \tag{8}$$

where $\widehat{R}$ is the vector $(\widehat{R}_{r_0}, \dots, \widehat{R}_{r_d})^\top$ of approximations given in (7). This numerical solution heavily depends on the choice of radii $r_0, \dots, r_d$. To reduce this dependence, a least–squares method can be used; see also [5]. Instead of (5), consider the (overdetermined) system of linear equations $\widehat{R} = A_{r_0 \dots r_{k-1}}\, x$ for $k > d+1$ pairwise different radii $r_0, \dots, r_{k-1}$ where $x = (x_0, \dots, x_d)^\top \in \mathbb{R}^{d+1}$. It is well known that the vector

$$v^* = \big(A^\top_{r_0 \dots r_{k-1}} A_{r_0 \dots r_{k-1}}\big)^{-1} A^\top_{r_0 \dots r_{k-1}} \widehat{R} \tag{9}$$

is the unique solution of the *least–squares minimization* problem

$$|\,\widehat{R} - A_{r_0 \ldots r_{k-1}}\, v^*\,| = \min_{x \in \mathbb{R}^{d+1}} |\,\widehat{R} - A_{r_0 \ldots r_{k-1}}\, x\,|$$

and, therefore, can be regarded as an approximation to the vector V of intrinsic volumes of K. For a discussion of the practical choice of radii $r_0, \ldots, r_{k-1}$, see [4, 5].

In general, the numerical solutions (8) and (9) of (5) do not necessarily preserve the positivity property of the volume $V_d(K)$ and the surface area $2V_{d-1}(K)$. Practically one can cope with this problem by changing the values and the number of radii r_i as well as distances between them. For a detailed discussion, see [4].

3. Stationary random closed sets

Let Ξ be a stationary random closed set (RACS) in $\mathbb{R}^d$ whose realizations belong to the extended convex ring $\mathcal{S}$ with probability 1. Recall that stationarity of Ξ means the invariance of its distribution with respect to arbitrary translations in $\mathbb{R}^d$. More details on stationary RACS can be found in many books; see e.g. [6, 7, 13, 15, 16, 18].

Specific intrinsic volumes

For any $K \in \mathcal{R}$, let $N(K) = \min\{m \in \mathbb{N} : K = \bigcup_{i=1}^m K_i,\ K_i \in \mathcal{K}\}$ denote the minimal number of convex components of the set K, where we put $N(K) = 0$ if $K = \emptyset$. Assume that

$$E\, 2^{N(\Xi \cap [0,1]^d)} < \infty\,. \tag{10}$$

Then, for any sequence $\{W_n\}$ of compact and convex observation windows $W_n = nW$ with $W \in \mathcal{K}$ such that $V_d(W) > 0$ and $o \in \mathrm{int}(W)$, the limit $\overline{V}_j(\Xi) = \lim_{n\to\infty} E\, V_j(\Xi \cap W_n) \big/ V_d(W_n)$ exists for each $j = 0, \ldots, d$ (see [13], Theorem 5.1.3) and is called the jth *specific intrinsic volume* of Ξ.

Local Euler–Poincaré characteristic

The expectation $E\, V_0\,(\Xi \cap B_r(x))$ is called *local Euler–Poincaré characteristic* or, equivalently, *local connectivity number* of Ξ, where $r > 0$ is an arbitrary fixed number. For $r = 1$, the following representation formula for $E\, V_0\,(\Xi \cap B_1(x))$ can be found e.g. in [13], Corollary 5.3.2, where its proof is based on the principal kinematic formula. In the present paper, we give an alternative proof for any $r > 0$, which makes use of an explicit extension of Steiner's formula (1) to the convex ring; see [12].

Proposition 3.1. *For any $r \geqslant 0$ and $x \in \mathbb{R}^d$, it holds*

$$E\,V_0\left(\Xi \cap B_r(x)\right) = \sum_{j=0}^{d} r^{d-j}\kappa_{d-j}\overline{V}_j(\Xi)\,. \tag{11}$$

Proof. Consider the stationary random field $\{Z_r(x),\ x \in \mathbb{R}^d\}$, where

$$Z_r(x) = \sum_{q \in \partial\Xi \cap B_r(x),\ q \neq x} J(\Xi \cap B_r(x), q, x)\,.$$

and $J\big(\Xi \cap B_r(x), q, x\big)$ is given by (3). In [11], we showed that $E\,Z_r(x) = \sum_{j=0}^{d-1} r^{d-j}\kappa_{d-j}\overline{V}_j(\Xi)$ holds for any $x \in \mathbb{R}^d$. Thus, it suffices to prove that $E\,V_0\left(\Xi \cap B_r(x)\right) = E\,Z_r(x) + \overline{V}_d(\Xi)$. Notice that the function $f(r) = E\,Z_r(x)$ is continuously differentiable as a polynomial in r, where $f(r) = \int_0^r f^{(1)}(s)\,ds$ since $f(0) = 0$. Furthermore, for any $s > 0$, we have

$$f^{(1)}(s) = \frac{d}{ds} E\,V_0(\Xi \cap B_s(x))\,, \tag{12}$$

where the derivative on the right–hand side does not depend on x by the stationarity of Ξ. In order to show (12), let A_o be a sufficiently small open cube with diagonals crossing at the origin o such that $A_o \subset \mathrm{int}(B_s(o))$. Then, for any $\Delta s > 0$, we have

$$\begin{aligned}
Z_{s+\Delta s}(o) - Z_s(o) &= \sum_{q \in \partial\Xi \cap (B_{s+\Delta s}(o) \setminus B_s(o))} J\big(\Xi \cap B_{s+\Delta s}(o), q, o\big) \\
&= \sum_{q \in \partial\Xi \cap (B_{s+\Delta s}(o) \setminus B_s(o))} J\big((\Xi \setminus A_o) \cap B_{s+\Delta s}(o), q, o\big) \\
&= V_0((\Xi \setminus A_o) \cap B_{s+\Delta s}(o)) - V_0((\Xi \setminus A_o) \cap B_s(o)) \\
&= V_0(\Xi \cap B_{s+\Delta s}(o)) - V_0(\Xi \cap B_s(o))\,,
\end{aligned}$$

where the third equality follows from the fact that

$$\sum_{q \in \partial A \cap B_r(o)} J((A \setminus A_o) \cap B_r(o), q, o) = V_0((A \setminus A_o) \cap B_r(o))$$

for each $A \in \mathcal{S}$ and for any $r > 0$ such that $A_o \subset \mathrm{int}(B_r(o))$. This gives

$$\begin{aligned}
f^{(1)}(s) &= \lim_{\Delta s \downarrow 0} E\,\frac{Z_{s+\Delta s}(o) - Z_s(o)}{\Delta s} \\
&= \lim_{\Delta s \downarrow 0} E\,\frac{V_0(\Xi \cap B_{s+\Delta s}(o)) - V_0(\Xi \cap B_s(o))}{\Delta s} = \frac{d}{ds} E\,V_0(\Xi \cap B_s(o))\,.
\end{aligned}$$

Now, using (12), $f(r)$ can be rewritten as

$$\begin{aligned} f(r) &= \int_0^r \frac{d}{ds} E\, V_0(\Xi \cap B_s(o))\, ds = E\, V_0(\Xi \cap B_r(o)) - E\, V_0(\Xi \cap \{o\}) \\ &= E\, V_0(\Xi \cap B_r(o)) - E\, \mathbf{1}_\Xi(o) = E\, V_0(\Xi \cap B_r(o)) - \overline{V}_d(\Xi)\,, \end{aligned}$$

where $\mathbf{1}_\Xi$ denotes the indicator of Ξ. □

It is well known that the Minkowski functionals of polyconvex sets can be defined through the Euler–Poincaré characteristics of their lower dimensional sections by means of Crofton's formula; see e.g. [13, 18]. Proposition 3.1 immediately implies that

$$\overline{V}_j(\Xi) = \frac{1}{(d-j)!\,\kappa_{d-j}} \cdot \frac{d^{(d-j)} E\, V_0(\Xi \cap B_r(x))}{dr^{(d-j)}}\Big|_{r=0}$$

for any $j = 0, \ldots, d-1$ and $x \in \mathbb{R}^d$. Thus, similarly to Crofton's formula, the specific intrinsic volumes of stationary RACS can be expressed by their local Euler–Poincaré characteristics.

4. Estimation of specific intrinsic volumes

In this section, similar to the approach considered in [11], the method of moments is used to construct joint nonparametric estimators for the specific intrinsic volumes $\overline{V}_j(\Xi)$, $j = 0, \ldots, d$.

Indirect estimation via local Euler–Poincaré characteristics

For any $d+1$ positive pairwise different radii r_j, Proposition 3.1 yields the following system of $d+1$ linear equations with respect to the variables $\overline{V}_j(\Xi)$, $j = 0, \ldots, d$:

$$A_{r_0 \ldots r_d}\, v = c, \qquad (13)$$

where $A_{r_0 \ldots r_d}$ is the matrix introduced in (6), $v = (\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))^\top$ and $c = (E\, V_0(\Xi \cap B_{r_0}(o)), \ldots, E\, V_0(\Xi \cap B_{r_d}(o)))^\top$. Similar to the deterministic case of Section 2, choose an appropriate estimator $\widehat{c}$ for c and define the estimator $\widehat{v}$ for v by

$$\widehat{v} = A^{-1}_{r_0 \ldots r_d} \widehat{c} \qquad (14)$$

in order to estimate the vector v of specific intrinsic volumes from a single realization of Ξ observed in a certain window $W \in \mathcal{K}$. For any $r > 0$ such that $V_d(W \ominus B_r(o)) > 0$, consider the stationary random field $\{Y_r(x),\, x \in \mathbb{R}^d\}$ with $Y_r(x) = V_0\,(\Xi \cap B_r(x))$. An unbiased estimator for $y_r = E\, Y_r(o)$ is given by $\widehat{y}_r = \int_{W \ominus B_r(o)} Y_r(x)\, \mu(dx)$, where μ is an arbitrary probability measure concentrated on $W \ominus B_r(o) \subset \mathbb{R}^d$. For instance, μ can be the normalized Lebesgue measure $\mu(\cdot) = V_d(\cdot \cap W \ominus B_r(o))/V_d(W \ominus B_r(o))$ on $W \ominus B_r(o)$, or a discrete measure $\mu(\cdot) = \sum_{i=1}^{m} w_i \delta_{x_i}(\cdot)$ with measurements at locations $x_1, \ldots, x_m \in W \ominus B_r(o)$ and weights $w_i > 0$ such that $w_1 + \ldots + w_m = 1$. Notice that integration is performed over the reduced window $W \ominus B_r(o)$ to avoid edge effects, since the computation of $V_0\,(\Xi \cap B_r(x))$ for $x \in W$ requires the knowledge of Ξ in the r–neighborhood of x while Ξ is observed only within W. Thus, assuming that $V_d(W \ominus B_{r_j}(o)) > 0$ for each $j = 0, \ldots, d$, an unbiased estimator $\widehat{c}$ for c is given by

$$\widehat{c} = \Big(\int_{W \ominus B_{r_0}(o)} Y_{r_0}(x)\,\mu(dx)\,, \ldots,\, \int_{W \ominus B_{r_d}(o)} Y_{r_d}(x)\,\mu(dx)\Big)^{\top}.$$

Mean–square consistency and asymptotic normality

For $\mu(\cdot) = V_d(\cdot\, \cap W \ominus B_r(o))/V_d(W \ominus B_r(o))$, the integral

$$\widehat{y}_r = \int_{W \ominus B_r(o)} Y_r(x)\,\mu(dx)$$

is the least–squares estimator for y_r, which is mean–square consistent as $W \uparrow \mathbb{R}^d$ provided that some integrability conditions are satisfied; see e.g. [3], p. 131. This means that for a sequence $\{W_n\}$ of unboundedly increasing sampling windows with $W_n = nW$, we have $E\,(\widehat{y}_{r,n} - y_r)^2 \to 0$ as $n \to \infty$, where $\widehat{y}_{r,n} = \int_{W_n \ominus B_r(o)} Y_r(x)\,\mu_n(dx)$ and

$$\mu_n(\cdot) = V_d(\cdot\, \cap W_n \ominus B_r(o))/V_d(W_n \ominus B_r(o))\,;$$

see also [11]. Assuming that $E\,4^{N(\Xi \cap [0,1]^d)} < \infty$, it can be shown that the *covariance functions* $C_{r_i r_j}(x) = E\,(Y_{r_i}(o) Y_{r_j}(x)) - y_{r_i} y_{r_j}$ are well defined; $i, j = 0, \ldots, d$. Furthermore, under suitable mixing conditions on Ξ and assuming that $\int_{\mathbb{R}^d} |C_{r_i r_j}(x)|\,dx < \infty$, the random vector

$$\sqrt{V_d(W_n \ominus B_r(o))}\,\big(\widehat{y}_{r_0,n} - y_{r_0}, \ldots, \widehat{y}_{r_d,n} - y_{r_d}\big)$$

is asymptotically normal distributed, where the asymptotic covariance matrix is given by $(\int_{\mathbb{R}^d} C_{r_i r_j}(x)\,dx)_{i,j=0}^{d}$ and can be consistently estimated; see [3], Section 3.1, and [11]. Notice that the integrability and mixing conditions mentioned above are fulfilled, for example, for rapidly mixing germ–grain models including the well–known *Boolean model*; see e.g. [6, 7, 14]. We also remark that the estimator $\widehat{v}_n = A^{-1}_{r_0 \ldots r_d}(\widehat{y}_{r_0,n}, \ldots, \widehat{y}_{r_d,n})^{\top}$ for v is mean–square consistent and asymptotically normal distributed, provided that the estimator $(\widehat{y}_{r_0,n}, \ldots, \widehat{y}_{r_d,n})^{\top}$ for c possesses these properties.

Least–squares estimator

The least-squares approach of Section 2 also applies (with minor changes) to the case of stationary RACS. For $k > d{+}1$ pairwise different radii $r_0, \ldots, r_{k-1}$

such that $V_d(W \ominus B_{r_j}(o)) > 0$, $j = 0, \ldots, k$, the corresponding solution of the least squares minimization problem is $v^* = (A^\top_{r_0 \ldots r_{k-1}} A_{r_0 \ldots r_{k-1}})^{-1} A^\top_{r_0 \ldots r_{k-1}} \widehat{y}$, where $\widehat{y} = (\widehat{y}_{r_0}, \ldots, \widehat{y}_{r_{k-1}})^\top$ with $\widehat{y}_{r_j} = \int_{W \ominus B_{r_j}(o)} Y_{r_j}(x)\, \mu(dx)$. Notice that the estimator $v^* = (v^*_0, \ldots, v^*_d)^\top$ for the vector $v = (\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))^\top$ of specific intrinsic volumes of Ξ is much more robust with respect to the choice of radii $r_0, \ldots, r_{k-1}$ than the estimator $\widehat{v}$ given in (14).

5. Estimation variance and spatial sampling designs

Besides unbiasedness, another important criterion for goodness of the estimator $\widehat{v}$ given in (14) is related to its variance properties, where the radii $r_0, \ldots, r_d$ and the averaging probability measure μ should be chosen in such a way that the *estimation variance* $\sigma^2 = Var(\widehat{v}) = E\,|\widehat{v} - v|^2$ is possibly small.

Bound on the estimation variance

Unfortunately, it seems to be impossible to determine the estimation variance $\sigma^2 = Var(\widehat{v}) = E\,|A^{-1}_{r_0 \ldots r_d}(\widehat{c} - c)|^2$ explicitly. However, it is easy to get an upper bound for σ^2. Namely, (14) implies that

$$\sigma^2 \leqslant \|A^{-1}_{r_0 \ldots r_d}\|^2 E\,|\widehat{c} - c|^2 = \|A^{-1}_{r_0 \ldots r_d}\|^2 \sum_{j=0}^{d} Var(\widehat{y}_{r_j})\,, \tag{15}$$

where

$$\|A^{-1}_{r_0 \ldots r_d}\|^2 = \max_{i=0,\ldots,d} \lambda_i\big((A_{r_0 \ldots r_d} A^\top_{r_0 \ldots r_d})^{-1}\big) = \frac{1}{\min\limits_{i=0,\ldots,d} \lambda_i(A_{r_0 \ldots r_d} A^\top_{r_0 \ldots r_d})}$$

is the squared matrix norm of $A^{-1}_{r_0 \ldots r_d}$ and $\lambda_i(A)$ is the ith eigenvalue of the matrix A. Notice that $\|A^{-1}_{r_0 \ldots r_d}\|$ is finite. Thus, it is reasonable to choose $r_0, \ldots, r_d$ and μ such that the bound in (15) becomes small. Consider the variance $Var(\widehat{y}_{r_j}) = E(\widehat{y}_{r_j} - y_{r_j})^2$ appearing in (15). For any fixed $r > 0$, let $\mathcal{P}$ denote the family of all probability measures on $W \ominus B_r(o)$ and let the function $L : \mathcal{P} \to (0, \infty)$ be defined by $L(\mu) = E(\widehat{y}_r - y_r)^2$ for each $\mu \in \mathcal{P}$. By Fubini's theorem, we can write

$$E(\widehat{y}_r - y_r)^2 = \int_{W \ominus B_r(o)} \int_{W \ominus B_r(o)} C_{rr}(x - x')\, \mu(dx)\mu(dx')\,. \tag{16}$$

Suppose that $L(\mu_0) = \min_{\mu \in \mathcal{P}} L(\mu)$ holds for some $\mu_0 \in \mathcal{P}$. Then, using the methods of variational analysis developed e.g. in [8] (see also [17]), it can be shown that the function $g(x) = \int_{W \ominus B_r(o)} C_{rr}(x - h)\, \mu_0(dh)$ necessarily has the following properties:

$$g(x) = L(\mu_0) \quad \mu_0\text{–a.e.} \qquad \text{and} \qquad g(x) \geqslant L(\mu_0) \quad \text{for all } x \in \mathbb{R}^d\,.$$

Discrete sampling designs

Suppose now that $L(\mu_0) = \min_{\mu\in\mathcal{P}} L(\mu)$ holds for some discrete probability measure $\mu_0 \in \mathcal{P}$ such that $\mu_0(\cdot) = \sum_{i=1}^m w_i\delta_{x_i}(\cdot)$ for some integer $m \geqslant 1$, where $x_1,\dots,x_m \in W \ominus B_r(o)$ and $w_1,\dots,w_m > 0$ with $w_1+\ldots+w_m = 1$. Then, it can be shown that $L(\mu_0) = (e^\top Q_r^{-1}e)^{-1} = \left(\sum_{i,j=1}^n q_{ij}^{-1}\right)^{-1}$ holds provided that the number of atoms m and the atoms $x_1,\dots,x_m$ themselves satisfy the condition

$$q_r^\top(x)Q_r^{-1}e \geqslant 1 \qquad \text{for all } x\in\mathbb{R}^d \tag{17}$$

and the covariance matrix $Q_r = (C_{rr}(x_i - x_j))_{i,j=1}^m$ is regular, where $e = (1,\dots,1)^\top$, $Q_r^{-1} = \left(q_{ij}^{-1}\right)_{i,j=1}^m$ denotes the inverse matrix of Q_r and $q_r(x) = (C_{rr}(x-x_1),\dots,C_{rr}(x-x_n))^\top$ for any $x\in\mathbb{R}^d$. Moreover, in this case, the vector of weights $w = (w_1,\dots,w_m)^\top$ is given by

$$w = L(\mu_0)Q_r^{-1}e\,. \tag{18}$$

Notice that, for fixed sampling points $x_1,\dots,x_m$, formula (18) coincides with the *kriging of the mean*; see [19]. In this case, the estimator $\widehat{y}_r$ with weights given by (18) is also known as the generalized least–squares estimator of the trend; see [9], p. 11. On the other hand, the locations $x_1,\dots,x_n$ can be chosen iteratively using gradient algorithms described e.g. in [9].

Anyhow, the choice of an appropriate number m of sampling points, locations $x_1,\dots,x_m$ and weights $w_1,\dots,w_m$ depends on the covariance function $C_{rr}:\mathbb{R}^d\to\mathbb{R}$ which is unknown in general. Therefore, $C_{rr}(h)$, $h\in\mathbb{R}^d$ has to be estimated from data. Sometimes it is preferable to consider the *variogram function* $\gamma_r:\mathbb{R}^d\to\mathbb{R}$ with $\gamma_r(h) = \frac{1}{2}\,E\,(Y_r(x) - Y_r(x+h))^2$, $h,x\in\mathbb{R}^d$, instead of C_{rr} since it can be estimated more easily. For corresponding estimation techniques and algorithms, see e.g. [1, 2, 19]. Since $\gamma_r(h) = C_{rr}(o) - C_{rr}(h)$ holds for any $h\in\mathbb{R}^d$, (17) and (18) can be rewritten as

$$p_r^\top(x)\Gamma_r^{-1}e \leqslant 1 \qquad \text{for all } x\in\mathbb{R}^d \tag{19}$$

and

$$w = \gamma_0\Gamma_r^{-1}e\,, \tag{20}$$

respectively, where $\Gamma_r = (\gamma_r(x_i - x_j))_{i,j=1}^m$, $\gamma_0 = (e^\top\Gamma_r^{-1}e)^{-1}$ and $p_r(x) = (\gamma_r(x-x_1),\dots,\gamma_r(x-x_m))^\top$.

6. Numerical results

To test the performance of the above estimation method, 200 realizations of a planar Boolean model Ξ ($d=2$) with circular grains were generated in the observation window $W = [0,1000]^2$. Let λ be the intensity of the stationary

Poisson point process $X = \{X_i\}$ of germs and let the grains Ξ_i be independent circles with radii that are uniformly distributed on $[20, 40]$. Then, Ξ is given by $\Xi = \bigcup_{i=1}^{\infty}(\Xi_i + X_i)$. The intensity λ was chosen to fit the volume fractions $\overline{V}_2(\Xi) = 0.2,\ 0.5,\ 0.8$, respectively. For each realization, the vector v of specific intrinsic volumes of Ξ was estimated using the radii $r_0 = 10$, $r_{i+1} = r_i + 1.3$, $i = 0, \ldots, 49$ in the least–squares method. In the estimation, sampling was performed on the regular square lattice of points $x_1, \ldots, x_m$ with mesh size $\Delta = 5$. Finally, vector $\overline{v}^* = (\overline{v}_0^*, \overline{v}_1^*, \overline{v}_2^*)$ was built being the arithmetic mean over the results of 200 realizations. Its values are compared with the theoretical counterparts $v = (\overline{V}_0(\Xi), \overline{V}_1(\Xi), \overline{V}_2(\Xi))$ in Table 1. Additionally, the specific intrinsic volumes were estimated by the method described in [10] from the same 200 realizations of Ξ. The resulting arithmetic means $\widetilde{v}_0, \widetilde{v}_1, \widetilde{v}_2$ are also presented in Table 1. To compare the precision of both algorithms, the relative error $\delta_{A,B} = \frac{B-A}{A} \cdot 100\%$ of an estimated quantity B with respect to the theoretical value A is given. It is clear from Table 1 that the performance of

Table 1. Theoretical and estimated values of specific intrinsic volumes

$\overline{V}_2(\Xi)$	0.2	0.5	0.8
$\overline{v}_2^*$	0.194299	0.490611	0.793328
$\widetilde{v}_2$	0.199362	0.498217	0.798085
$\delta_{\overline{V}_2,\overline{v}_2^*}$, %	-2.85	-1.88	-0.83
$\delta_{\overline{V}_2,\widetilde{v}_2}$,%	-0.32	-0.36	-0.24
$2\overline{V}_1(\Xi)$	0.011476	0.02228	0.020693
$2\overline{v}_1^*$	0.012123	0.023402	0.021547
$2\widetilde{v}_1$	0.011361	0.021947	0.02022
$\delta_{\overline{V}_1,\overline{v}_1^*}$, %	5.64	5.04	4.13
$\delta_{\overline{V}_1,\widetilde{v}_1}$, %	-1.0	-1.5	-2.29
$\overline{V}_0(\Xi) \times 10^4$	0.4778163	0.3919529	-0.6059316
$\overline{v}_0^* \times 10^4$	0.4348681	0.1555031	-0.10798772
$\widetilde{v}_0 \times 10^4$	0.4312496	0.1595565	-0.10672334
$\delta_{\overline{V}_0,\overline{v}_0^*}$, %	-8.99	-60.33	78.22
$\delta_{\overline{V}_0,\widetilde{v}_0}$, %	-9.75	-59.29	76.13

our algorithm is comparable to that of the method described in [10]. Hovewer, the above results can be improved by taking e.g. $\Delta = 1$. In fact, the precision of our computations can be controlled by changing the sampling design as well as the number and values of dilation radii. The increase of the numbers of radii and sampling points results in a higher precision. This implies longer run times. Hence, the parameters of the algorithm should be tuned in accordance with the needs of specific applications; see [4] for an extensive discussion.

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