Joint Estimators for the Specific Intrinsic Volumes of Stationary Random Sets

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Abstract. Stationary random closed sets Ξ in $\mathbb{R}^d$ are considered whose realizations belong to the extended convex ring. A new approach is proposed to joint estimation of the specific intrinsic volumes $\overline{V}_0(\Xi), \dots, \overline{V}_d(\Xi)$ of Ξ, including the specific Euler–Poincaré characteristic $\overline{V}_0(\Xi)$, the specific surface area $2\overline{V}_{d-1}(\Xi)$, and the volume fraction $\overline{V}_d(\Xi)$ of Ξ. Nonparametric estimators are constructed, which can be represented by integrals of some stationary random fields. This implies in particular that these estimators are unbiased. Moreover, conditions are derived which ensure that they are mean–square consistent. A positive definite and consistent estimator for their asymptotic covariance matrix is derived.

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1 Introduction

We consider stationary random closed sets (RACS) Ξ in $\mathbb{R}^d$, $d \geqslant 2$, assuming that each realization of Ξ belongs to the extended convex ring and that the convex components of these realizations satisfy certain (local) integrability conditions. For this class of stationary RACS Ξ, we propose a new approach to joint estimation of the specific intrinsic volumes $\overline{V}_0(\Xi), \dots, \overline{V}_d(\Xi)$ of Ξ including the specific Euler–Poincaré characteristic $\overline{V}_0(\Xi)$, the specific surface area $2\overline{V}_{d-1}(\Xi)$, and the volume fraction $\overline{V}_d(\Xi)$, where

$$\overline{V}_j(\Xi) = \lim_{n\to\infty} \frac{E\, V_j(\Xi \cap W_n)}{V_d(W_n)}, \quad j = 0, \dots, d,$$

$V_j(\Xi \cap W_n)$ denotes the j–th intrinsic volume of $\Xi \cap W_n$, and $\{W_n\}$ is some sequence of unboundedly increasing (convex and compact) sets $W_n \subset \mathbb{R}^d$.

Based on an explicit extension of the classical Steiner formula to the convex ring $\mathcal{R}$, which has been proven by Schneider [21], this new approach yields a natural estimator $\widehat{v} = (\widehat{v}_0, \dots, \widehat{v}_d)^\top$ for all specific intrinsic

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volumes $\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)$ simultaneously as a solution of a simple system of $d+1$ linear equations. We show that $\widehat{v}$ is asymptotically unbiased. Furthermore, we consider a slightly modified estimator $\widetilde{v} = (\widetilde{v}_0, \ldots, \widetilde{v}_d)^\top$ for the vector $\big(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)\big)^\top$ of specific intrinsic volumes of Ξ, which can be represented by integrals of some stationary random fields $Z_r = \big\{Z_r(x),\, x \in \mathbb{R}^d\big\}$; $r > 0$. This implies in particular that $\widetilde{v}$ is unbiased. Moreover, using a proving technique which has been considered in [1] and [2] for similarly structured functionals of stationary RACS, we derive conditions which ensure that the estimators $\widehat{v}$ and $\widetilde{v}$ are mean–square consistent, and that a positive definite and consistent estimator for the asymptotic covariance matrix of $\widetilde{v}$ can be provided.

Notice however that there is yet another explicit extension of Steiner's formula, introduced by Matheron [12], which leads to nonnegative functionals V'_j on $\mathcal{R}$, whereas Schneider's approach mentioned above is based on an inclusion–exclusion formula and therefore leads to additive functionals which can take negative values as well. Thus, for $j = 0, \ldots, d-2$, the functionals V'_j on $\mathcal{R}$ are different from the intrinsic volumes $V_j : \mathcal{R} \to \mathbb{R}$. Nevertheless, for $j = 0, \ldots, d$, the limits

$$\overline{V}'_j(\Xi) = \lim_{n\to\infty} \frac{E\, V'_j(\Xi \cap W_n)}{V_d(W_n)}$$

can be jointly estimated by considering some stationary random fields $Z'_r = \big\{Z'_r(x),\, x \in \mathbb{R}^d\big\}$, which are similar to the fields Z_r used in the present paper in order to construct the estimator $\widetilde{v} = (\widetilde{v}_0, \ldots, \widetilde{v}_d)^\top$ for $\big(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)\big)^\top$. Besides that, a more general concept of additive as well as nonnegative extensions of this type has been developed in Hug and Last [8] for so–called support measures.

In Rataj [19], intrinsic volumes of unions of sets of positive reach are approximated by the volumes of corresponding parallel sets. Estimators for the Euler–Poincaré characteristic, based on integral–geometric formulae, can be found e.g. in Nagel et al. [14], Ohser and Nagel [16]. Recently, Tscheschel and Stoyan [25] studied the variance of an estimator for the Euler–Poincaré characteristic of random networks, which can be used to approximate stationary RACS with realizations from the extended convex ring.

In the case of the underlying stationary RACS Ξ being induced by an independently marked germ–grain process, we discuss examples for which our conditions on Ξ are fulfilled and for which $\widetilde{v}$ is asymptotically normal. Thus, asymptotic tests can be constructed in order to check hypotheses about the vector $\big(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)\big)^\top$ of specific intrinsic volumes of Ξ. In particular, we discuss the Boolean model $\Xi = \bigcup_{i=1}^\infty \Xi_i$ with compact and convex grains $\Xi_1, \Xi_2, \ldots$. In this case, an asymptotic Gaussian test can be derived for simultaneous verification of hypotheses about $\big(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)\big)^\top$. The asymptotic normality of estimators for certain specific intrinsic volumes such as the specific Euler–Poincaré characteristic, specific surface area or the volume fraction has been discussed e.g. in Heinrich [5, 6], Heinrich and Molchanov [7], Mase [11], Böhm et al. [1]; see also [13], pp. 30–43 for further references.

Algorithms for practical computation of the estimators $\widehat{v}$ and $\widetilde{v}$ for discretized sets on a grid will be discussed in a forthcoming paper; see [10]. Other estimation algorithms for specific intrinsic volumes and, in particular, the Euler–Poincaré characteristic have been described e.g. in Ohser and Mücklich [15], Ohser, Nagel and Schladitz [17, 18], Vogel [26]. In Robins [20], the determination of the Euler–Poincaré characteristic of discretized sets by means of Betty numbers is discussed using the approach of homology groups.

2 Preliminaries

2.1 Intrinsic volumes

We begin with recalling some basic notions from convex geometry which will be used in the following; see e.g. [21], [22].

Denote the family of all convex bodies in $\mathbb{R}^d$ by $\mathcal{K}$ and let $V_d(K)$ be the usual volume of $K \in \mathcal{K}$. Furthermore, let $B_r(x)$ be the closed ball in $\mathbb{R}^d$ with radius $r > 0$ and center at $x \in \mathbb{R}^d$. By $o \in \mathbb{R}^d$ we denote the origin in $\mathbb{R}^d$, and by k_j the volume of the unit ball in $\mathbb{R}^j$; $j = 0, \ldots, d$. For any convex bodies K_1, $K_2 \in \mathcal{K}$, let $K_1 \oplus K_2 = \{x_1 + x_2 : x_1 \in K_1,\, x_2 \in K_2\}$ and $K_1 \ominus K_2 = \{x \in \mathbb{R}^d : K_2 + x \subset K_1\}$ be the Minkowski sum and the Minkowski difference, respectively. One can prove that for each $j = 0, \ldots, d$ there exist nonnegative functionals $V_j : \mathcal{K} \to [0, \infty)$ such that for each $r > 0$ the volume $V_d(K \oplus B_r(o))$ of the so–called *parallel body* $K \oplus B_r(o)$ of any $K \in \mathcal{K}$ is given by

$$V_d(K \oplus B_r(o)) = \sum_{j=0}^{d} r^{d-j} k_{d-j} V_j(K)\,. \tag{2.1}$$

This polynomial expansion in r is often referred to as *Steiner's formula*. The functionals V_j, $j = 0, \ldots, d$ are called *intrinsic volumes*. Let $\mathcal{R}$ be the *convex ring*, i.e., the family of all finite unions of convex bodies. It can be proven that for each $j = 0, \ldots, d$ there exists a unique additive extension of the functional $V_j : \mathcal{K} \to [0, \infty)$ to the convex ring $\mathcal{R}$ given by the *inclusion–exclusion formula*

$$V_j(K_1 \cup \ldots \cup K_n) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{j_1 < \ldots < j_i} V_j(K_{j_1} \cap \ldots \cap K_{j_i}) \tag{2.2}$$

for any $K_1, \ldots, K_n$, where $K_1 \cup \ldots \cup K_n \in \mathcal{R}$. Notice that the value $V_j(K_1 \cup \ldots \cup K_n)$ in (2.2) does not depend on the representation of the set $K_1 \cup \ldots \cup K_n \in \mathcal{R}$ by its convex components $K_1, \ldots, K_n \in \mathcal{K}$. Moreover, intrinsic volumes have a nice geometric interpretation based on Crofton's formula (cf. e.g. [22], p. 78–79).

2.2 Stationary random closed sets and specific intrinsic volumes

Let Ξ be a stationary RACS in $\mathbb{R}^d$ (see e.g. [12], [23], [24]) with realizations ξ from the extended convex ring $\mathcal{S}$ almost surely. This means that $\xi \cap K$ belongs to the usual convex ring $\mathcal{R}$ for any (compact and convex) $K \in \mathcal{K}$ and almost every realization ξ of Ξ.

For $K \in \mathcal{R} \setminus \{\emptyset\}$, let $N(K) = \min\{m \in \mathbb{N} : K = \bigcup_{i=1}^{m} K_i\,, \quad K_i \in \mathcal{K}\}$ denote the minimal number of convex components of the set K, where we put $N(K) = 0$ if $K = \emptyset$. One can show that the mapping $N(\Xi \cap [0,1]^d) : \Omega \to \mathbb{R}$ is measurable, i.e. $N(\Xi \cap [0,1]^d)$ is a random variable. Assume that $EN(\Xi \cap [0,1]^d) < \infty$. Then, Ξ can be represented by a point process $\{\Xi_i\}$ of compact and convex sets, i.e. a sequence of compact and convex RACS $\Xi_1, \Xi_2, \ldots$ such that $\#\{i : \Xi_i \cap B_r(o) \neq \emptyset\} < \infty$ for each $r > 0$, and $\Xi = \bigcup_{i=1}^{\infty} \Xi_i$ (see [23], Satz 4.4.2). Furthermore, if

$$E\, 2^{N(\Xi \cap [0,1]^d)} < \infty \tag{2.3}$$

holds, then it follows that for any (monotonously increasing) sequence $\{W_n\}$ of compact and convex observation windows W_n with

$$W_n = nK_0 \qquad \text{for some } K_0 \in \mathcal{K} \text{ such that } V_d(K_0) > 0 \text{ and } o \in \mathrm{int}(K_0), \tag{2.4}$$

the expectations $E\,V_j(\Xi \cap W_n)$ are well defined and the limit

$$\overline{V}_j(\Xi) = \lim_{n\to\infty} \frac{E\,V_j(\Xi \cap W_n)}{V_d(W_n)} \tag{2.5}$$

exists for each $j = 0, \ldots, d$ (see [23], Satz 5.1.3). The functional $\overline{V}_j(\Xi)$ is called the *intensity* of the intrinsic volume V_j or the *specific intrinsic j–volume* of Ξ.

Notice that (2.4) implies in particular that for each $r > 0$

$$\lim_{n\to\infty} \frac{V_d(W_n \oplus B_r(o))}{V_d(W_n)} = \lim_{n\to\infty} \frac{V_d(W_n \ominus B_r(o))}{V_d(W_n)} = 1\,, \qquad \lim_{n\to\infty} \frac{V_d(\partial W_n \oplus B_r(o))}{V_d(W_n)} = 0\,. \tag{2.6}$$

2.3 Explicit extension of intrinsic volumes

In order to construct a joint estimator for the vector $(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))^\top$ of specific intrinsic volumes introduced in (2.5), we use an explicit extension of Steiner's formula (2.1) to the convex ring $\mathcal{R}$, which has been proven by Schneider [21]. The idea of this extension is based on the index of sets from the convex ring $\mathcal{R}$, where for any $K \in \mathcal{R}$ and $q, x \in \mathbb{R}^d$, $q \neq x$, the *index* of K is defined by

$$J(K,q,x) = \begin{cases} 1 - \lim\limits_{\delta\to+0}\lim\limits_{\varepsilon\to+0} V_0\left(K \cap B_{|x-q|-\varepsilon}(x) \cap B_\delta(q)\right) & \text{if } q \in K, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

In particular, we have $J(\emptyset, q, x) = 0$ for arbitrary $q, x \in \mathbb{R}^d$, $q \neq x$ and $J(K,q,x) = 0$ for $q \notin \partial K$. For any $r > 0$, define the functional $\rho_r : \mathcal{R} \to \mathbb{R}$ by

$$\rho_r(K) = \int\limits_{\mathbb{R}^d} I_r(K,x)\,dx\,, \qquad K \in \mathcal{R}\,, \tag{2.8}$$

where

$$I_r(K,x) = \sum_{q\neq x} J\left(K \cap B_r(x), q, x\right)\,. \tag{2.9}$$

The last sum consists of finitely many summands (being different from zero). It can be proven that the quantities introduced in (2.7) to (2.9) are well defined. By this, the following extension

$$\rho_r(K) = \sum_{j=0}^{d-1} r^{d-j} k_{d-j} V_j(K)\,, \qquad K \in \mathcal{R} \tag{2.10}$$

of Steiner's formula (2.1) to the convex ring $\mathcal{R}$ holds. Notice that the functionals V_j that arise from the latter construction coincide with those introduced in (2.2).

Since (2.10) holds for any $r > 0$, by choosing d pairwise different $r_0, \dots, r_{d-1} > 0$, we are able to get the following system of linear equations

$$\rho_{r_i}(K) = \sum_{j=0}^{d-1} r_i^{d-j} k_{d-j} V_j(K)\,, \qquad i = 0, \dots, d-1 \tag{2.11}$$

with respect to $V_0(K), \dots, V_{d-1}(K)$. Thus, instead of computing $(V_0(K), \dots, V_{d-1}(K))^\top$ directly, we can first compute the vector $(\rho_{r_0}(K), \dots, \rho_{r_{d-1}}(K))^\top$ given by (2.7)–(2.9) and solve (2.11) subsequently. This approach has the advantage that the computation of $\rho_{r_0}(K), \dots, \rho_{r_{d-1}}(K)$ only requires the computation of the local Euler–Poincaré characteristics of K, which can be done much easier than the direct computation of (global) intrinsic volumes $V_0(K), \dots, V_{d-1}(K)$. The reason for this is that the local Euler–Poincaré characteristic $V_0\big(K \cap B_{|x-q|-\varepsilon}(x) \cap B_\delta(q)\big)$ in (2.7) can be easily computed by the inclusion–exclusion formula (2.2) for small $\varepsilon, \delta > 0$, since then the right–hand side of (2.2) consists of a small number of summands being either 1 or -1; see also [10]. The idea to represent the intrinsic volumes of convex sets by a system of linear algebraic equations similar to (2.11) is due to [22], p. 45. Furthermore, this indirect approach leads to joint estimators for all $\overline{V}_0(\Xi), \dots, \overline{V}_d(\Xi)$ simultaneously, possessing an integral representation by stationary random fields and, therefore, having useful asymptotic properties; see Section 3 below.

For $x \notin K$, the sum $I_r(K, x)$ in (2.9) exhibits a nice geometric interpretation (see [21], p. 224), namely, $I_r(K, x) = V_0(K \cap B_r(x))$ for all $x \notin K$. In particular, for each $K \in \mathcal{K}$

$$I_r(K, x) = \mathbf{1}\big(x \in (K \oplus B_r(o)) \setminus K\big) \tag{2.12}$$

holds and consequently $\rho_r(K) = V_d\big((K \oplus B_r(o)) \setminus K\big)$. Besides this, we have

$$I_r(K, x) = \sum_{k=1}^{N(K \cap B_r(x))} (-1)^{k-1} \sum_{i_1 < \ldots < i_k} I_r(K_{i_1} \cap \ldots \cap K_{i_k}, x) \tag{2.13}$$

for each $K \in \mathcal{R}$, where the compact and convex sets $K_1, K_2, \ldots$ are the convex components of a minimal decomposition of $K \cap B_r(x)$. Notice that (2.13) follows from the fact that the functional $I_r : \mathcal{R} \times \mathbb{R}^d \to \mathbb{R}$ defined in (2.7) and (2.9) is additive in the first argument. Furthermore, for each $K \in \mathcal{R}$, the representation formulae (2.12) and (2.13) imply that

$$|I_r(K, x)| \leqslant \sum_{k=1}^{N(K \cap B_r(x))} \binom{N(K \cap B_r(x))}{k} = 2^{N(K \cap B_r(x))} - 1 \leqslant 2^{N(K \cap B_r(x))}\,. \tag{2.14}$$

With the notation $[z, z+e] = [0,1]^d + z$ for any $z \in \mathbb{Z}^d$ and $A(x, r) = \{z \in \mathbb{Z}^d : [z, z+e] \cap B_r(x) \neq \emptyset\}$ for any $x \in \mathbb{R}^d$, $r > 0$, the following lemma yields a useful upper bound on $|I_r(K, x)|$.

Lemma 2.1. *For any $K \in \mathcal{R}$, $x \in \mathbb{R}^d$ and $r > 0$,*

$$|I_r(K, x)| \leqslant \sum_{k=1}^{2^d} \sum_{z \in A(x,r)} \sum\nolimits^{\neq}_{z_1, \ldots, z_k \in A(x,r):\, z \in \bigcap_{i=1}^k [z_i, z_i+e]} 2^{N\left(\bigcap_{i=1}^k [z_i, z_i+e] \cap K\right)}\,, \tag{2.15}$$

where the inner sum extends over all pairwise disjoint $z_1, \ldots, z_k \in A(x, r)$ such that $z \in \bigcap_{i=1}^k [z_i, z_i + e]$.

Proof. Since $K = \bigcup_{z\in\mathbb{Z}^d}(K \cap [z, z + e])$ and I_r is additive on $\mathcal{R}$, we have

$$I_r(K,x) = \sum_{k=1}^{2^d}(-1)^{k-1} \quad {\sum}^{\neq}_{z_1,\ldots,z_k\in A(x,r)} I_r(\bigcap_{i=1}^{k}[z_i, z_i + e] \cap K, x)\,. \tag{2.16}$$

The first sum on the right–hand side of (2.16) runs from 1 to 2^d since $\bigcap_{i=1}^{k}[z_i, z_i + e] \cap K = \emptyset$ for $k > 2^d$ if all z_i are different. Besides this, for any $A, B \in \mathcal{K}$ with $A \subset B$, we have $N(K \cap A) \leqslant N(K \cap B)$ for each $K \in \mathcal{R}$. Thus, (2.14) and (2.16) imply that

$$\begin{aligned} |I_r(K,x)| &\leqslant \sum_{k=1}^{2^d} \ {\sum}^{\neq}_{z_1,\ldots,z_k\in A(x,r)} 2^{N(\bigcap_{i=1}^{k}[z_i,z_i+e]\cap K)} \\ &= \sum_{k=1}^{2^d} \sum_{z\in A(x,r)} \ {\sum}^{\neq}_{z_1,\ldots,z_k\in A(x,r):\ z\in\bigcap_{i=1}^{k}[z_i,z_i+e]} 2^{N\left(\bigcap_{i=1}^{k}[z_i,z_i+e]\cap K\right)}\,. \end{aligned}$$

□

2.4 Representation of specific intrinsic volumes

Assume that condition (2.3) is fulfilled. Furthermore, let $\{W_n\}$ be a sequence of compact and convex observation windows W_n such that (2.4) holds. Then, putting $K = \Xi \cap W_n$ in (2.10) for each $n = 1, 2, \ldots$, taking expectation and dividing by $V_d(W_n)$ on both sides of Steiner's formula (2.10), we get

$$\overline{\rho}_r(\Xi) = \sum_{j=0}^{d-1} r^{d-j} k_{d-j} \overline{V}_j(\Xi), \qquad r > 0\,, \tag{2.17}$$

where

$$\overline{\rho}_r(\Xi) = \lim_{n\to\infty} \frac{E\,\rho_r(\Xi \cap W_n)}{V_d(W_n)}\,.$$

The limit on the left–hand side of (2.17) exists due to the existence of the specific intrinsic volumes $\overline{V}_0(\Xi), \ldots, \overline{V}_{d-1}(\Xi)$ on the right–hand side of (2.17) as defined in (2.5).

Considering (2.17) for any positive radii r_i, $i = 0, \ldots, d-1$, one gets the following system of linear equations

$$\overline{\rho}_{r_i}(\Xi) = \sum_{j=0}^{d-1} {r_i}^{d-j} k_{d-j} \overline{V}_j(\Xi)\,, \qquad i = 0, \ldots, d-1 \tag{2.18}$$

with respect to $\overline{V}_j(\Xi)$, $j = 0, \ldots, d-1$. If we add one more equation, namely

$$\lim_{n\to\infty} \frac{E\,V_d(\Xi \cap W_n)}{V_d(W_n)} = \overline{V}_d(\Xi)\,, \tag{2.19}$$

which holds since $E\,V_d(\Xi \cap W_n) = \overline{V}_d(\Xi)V_d(W_n)$ for each $n = 1, 2, \ldots$, we get a system of $d + 1$ linear equations on the variables $\overline{V}_j(\Xi)$, $j = 0, \ldots, d$.

Suppose that the radii $r_0, \ldots, r_{d-1} > 0$ are pairwise different. Then, there exists a unique solution $v = \big(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)\big)^\top$ of the system of linear equations (2.18)–(2.19) which can be written in matrix form as

$$A_{r_0 \ldots r_{d-1}}\, v = c, \tag{2.20}$$

where

$$c = \Big(\overline{\rho}_{r_0}(\Xi), \ldots, \overline{\rho}_{r_{d-1}}(\Xi), \overline{V}_d(\Xi)\Big)^\top \tag{2.21}$$

and the deterministic $(d+1) \times (d+1)$–dimensional matrix

$$A_{r_0 \ldots r_{d-1}} = \begin{pmatrix} r_0^d k_d & r_0^{d-1} k_{d-1} & \ldots & r_0^2 k_2 & r_0 k_1 & 0 \\ r_1^d k_d & r_1^{d-1} k_{d-1} & \ldots & r_1^2 k_2 & r_1 k_1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{d-1}^d k_d & r_{d-1}^{d-1} k_{d-1} & \ldots & r_{d-1}^2 k_2 & r_{d-1} k_1 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix} \tag{2.22}$$

is regular by the properties of Vandermonde's determinant.

3 Estimators for specific intrinsic volumes

3.1 An indirect estimator

In order to estimate the vector of specific intrinsic volumes $v = \big(\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi)\big)^\top$ from a single realization of Ξ observed in a certain window W_n of the type given in (2.4), we proceed in the following way. First, we choose an appropriate estimator $\widehat{c}_n$ for the vector of limits c as given in (2.21). Then, in view of (2.20), we use $\widehat{c}_n$ for (indirect) estimation of v. More precisely, we define the estimator $\widehat{v}_n$ of v by

$$\widehat{v}_n = A^{-1}_{r_0 \ldots r_{d-1}} \widehat{c}_n\,, \tag{3.1}$$

where $A^{-1}_{r_0 \ldots r_{d-1}}$ is the inverse of the matrix $A_{r_0 \ldots r_{d-1}}$ given in (2.22).

An estimator $\widehat{c}_n$ for c can be constructed by considering the following (natural) estimators $\widehat{c}_{ni}$ for the components

$$c_i = \overline{\rho}_{r_i}(\Xi)\,, \qquad i = 0, \ldots, d-1\,. \tag{3.2}$$

For any $n \geqslant 1$ and $i = 0, \ldots, d-1$, let

$$\widehat{c}_{ni} = \frac{\rho_{r_i}(\Xi \cap W_n)}{V_d(W_n)}\,, \tag{3.3}$$

where

$$\rho_{r_i}(\Xi \cap W_n) = \int\limits_{W_n \oplus B_{r_i}(o)} \Bigg(\sum_{q \in \partial(\Xi \cap W_n) \cap B_{r_i}(x),\ q \neq x} J\big(\Xi \cap W_n \cap B_{r_i}(x), q, x\big) \Bigg)\, dx\,. \tag{3.4}$$

The integration in (3.4) is performed over $W_n \oplus B_{r_i}(o)$ or, to be more precise, over $\partial(\Xi \cap W_n) \oplus B_{r_i}(o)$ for the following reasons. Firstly, the index function $J\big(\Xi \cap W_n \cap B_{r_i}(x), q, x\big)$ is equal to zero for any point $q \notin \partial(\Xi \cap W_n)$. Hence, the sum in (3.4) runs over (finitely many) $q \in \partial(\Xi \cap W_n)$, $q \neq x$. Secondly, points x with $\partial(\Xi \cap W_n) \cap B_{r_i}(x) \neq \emptyset$ necessarily belong to $\partial(\Xi \cap W_n) \oplus B_{r_i}(o)$.

In addition, for $i = d$ we put

$$\widehat{c}_{nd} = \frac{V_d(\Xi \cap W_n)}{V_d(W_n)}\,. \tag{3.5}$$

Theorem 3.1. *The estimators $\widehat{c}_n = \left(\widehat{c}_{n0}, \ldots, \widehat{c}_{nd}\right)^\top$ for c and $\widehat{v}_n = \left(\widehat{v}_{n0}, \ldots, \widehat{v}_{nd}\right)^\top$ for v given in* (3.3) *to* (3.5) *and* (3.1)*, respectively, are asymptotically unbiased as $n \to \infty$.*

Proof. The last component $\widehat{c}_{nd}$ of $\widehat{c}_n$ given in (3.5) is an unbiased estimator for c_d by definition. Besides this, by (3.2) to (3.4) we obtain that for each $i = 0, \ldots, d-1$

$$\lim_{n\to\infty} E\widehat{c}_{ni} = \lim_{n\to\infty} \frac{E\rho_{r_i}(\Xi \cap W_n)}{V_d(W_n)} \;=\; c_i\,.$$

Thus, the estimator $\widehat{c}_n = \left(\widehat{c}_{n0}, \ldots, \widehat{c}_{nd}\right)^\top$ for c is asymptotically unbiased as $n \to \infty$. Together with (2.20) and (3.1), this implies that $\lim\limits_{n\to\infty} E\,\widehat{v}_n = \lim\limits_{n\to\infty} E\,A^{-1}_{r_0\ldots r_{d-1}}\widehat{c}_n = A^{-1}_{r_0\ldots r_{d-1}} \lim\limits_{n\to\infty} E\,\widehat{c}_n = A^{-1}_{r_0\ldots r_{d-1}} c = v\,.$ $\square$

3.2 Modified estimators induced by stationary random fields

Besides the estimators $\widehat{c}_n$ and $\widehat{v}_n$ introduced in Section 3.1, it is useful to consider some (slightly modified) versions $\widetilde{c}_n$ and $\widetilde{v}_n$ of these estimators. To be precise, for each $i = 0, \ldots, d-1$ and for any $n \geqslant 1$ such that $V_d(W_n \ominus B_{r_i}(o)) > 0$, let

$$\widetilde{c}_{ni} = \frac{\widetilde{\rho}_{n,r_i}(\Xi)}{V_d(W_n \ominus B_{r_i}(o))} \tag{3.6}$$

where

$$\widetilde{\rho}_{n,r_i}(\Xi) = \int\limits_{W_n \ominus B_{r_i}(o)} \left(\sum_{q \in \partial\Xi \cap B_{r_i}(x),\; q \neq x} J\big(\Xi \cap B_{r_i}(x), q, x\big) \right) dx\,. \tag{3.7}$$

For $i = d$, we put

$$\widetilde{c}_{nd} = \widehat{c}_{nd} = \frac{V_d(\Xi \cap W_n)}{V_d(W_n)}\,. \tag{3.8}$$

Finally, let $\widetilde{c}_n = (\widetilde{c}_{n0}, \ldots, \widetilde{c}_{nd})$ and

$$\widetilde{v}_n = A^{-1}_{r_0\ldots r_{d-1}} \widetilde{c}_n\,. \tag{3.9}$$

An advantage of the random variable $\widetilde{\rho}_{n,r}(\Xi)$ defined in (3.7) is that it can be expressed as an integral of a certain (measurable and stationary) random field. For this, define

$$Z_r(x) = \sum_{q \in \partial\Xi \cap B_r(x),\; q \neq x} J\big(\Xi \cap B_r(x), q, x\big)\,, \qquad x \in \mathbb{R}^d\,. \tag{3.10}$$

Notice that $Z_r(x) = V_0(\Xi \cap B_r(x))$ for each $x \notin \Xi$. Moreover, the random field $Z_r = \{Z_r(x),\, x \in \mathbb{R}^d\}$ is stationary due to the stationarity of Ξ. Hence, the defining equation (3.7) of $\widetilde{\rho}_{n,r}(\Xi)$ can be rewritten as

$$\widetilde{\rho}_{n,r}(\Xi) = \int\limits_{W_n \ominus B_r(o)} Z_r(x)\, dx\,. \tag{3.11}$$

From Lemma 2.1 we immediately get the following auxiliary result.

Lemma 3.1. *For arbitrary $x \in \mathbb{R}^d$ and $r > 0$, it holds almost surely that*

$$|Z_r(x)| \leqslant \sum_{k=1}^{2^d} \sum_{z \in A(x,r)} \sum\nolimits^{\neq}_{z_1,\ldots,z_k \in A(x,r):\, z \in \bigcap_{i=1}^k [z_i, z_i+e]} 2^{N\left(\bigcap_{i=1}^k [z_i,z_i+e] \cap \Xi\right)}\,. \tag{3.12}$$

Due to the stationarity of Ξ, Lemma 3.1 yields an upper bound on $E\,|Z_r(x)|$ given by

$$E\,|Z_r(x)| \leqslant \sum_{k=1}^{2^d} \sum_{z \in A(x,r)} \binom{2^d}{k} E\, 2^{N(\Xi \cap [0,1]^d)} = \left(2^{2^d} - 1\right) |A(x,r)| E\, 2^{N(\Xi \cap [0,1]^d)}\,. \tag{3.13}$$

Thus, using Minkowski's inequality, it becomes clear that $E\,|Z_r(x)|^s < \infty$ holds for an arbitrary fixed $s \geqslant 1$ if $E\, 2^{sN(\Xi \cap (0,1]^d)} < \infty$. In particular, by (2.3) and (3.13), the expectation

$$z_r = E\, Z_r(x) \tag{3.14}$$

is well–defined and finite. However, it does not depend on $x \in \mathbb{R}^d$ due to the stationarity of Z_r.

Lemma 3.2. *For $i = 0, \ldots, d-1$ and for each n sufficiently large, it holds*

$$E\widetilde{c}_{ni} = z_{r_i}\,. \tag{3.15}$$

Proof. Using (3.11) and (3.14), we have

$$E\widetilde{c}_{ni} = \frac{E\,\widetilde{\rho}_{r_i}(\Xi \cap W_n)}{V_d(W_n \ominus B_{r_i}(o))} = \frac{1}{V_d(W_n \ominus B_{r_i}(o)))} \int\limits_{W_n \ominus B_{r_i}(o)} E\, Z_{r_i}(x)\, dx = z_{r_i}$$

for each n such that $V_d(W_n \ominus B_r(o)) > 0$ for $r = \max\{r_0, \ldots, r_{d-1}\}$. □

Theorem 3.2. *For each $n \geqslant 1$ such that $V_d(W_n \ominus B_r(o)) > 0$ for $r = \max\{r_0, \ldots, r_{d-1}\}$, the estimators $\widetilde{c}_n = \left(\widetilde{c}_{n0}, \ldots, \widetilde{c}_{nd}\right)^\top$ and $\widetilde{v}_n = \left(\widetilde{v}_{n0}, \ldots, \widetilde{v}_{nd}\right)^\top$ defined in (3.6) to (3.9) are unbiased for c and v, respectively, i.e.,*

$$E\,\widetilde{c}_n = c \qquad \textit{and} \qquad E\,\widetilde{v}_n = v\,. \tag{3.16}$$

Proof. For any $n \geqslant 1$, the estimator $\widetilde{c}_{nd} = \widehat{c}_{nd}$ is obviously unbiased for the volume fraction $c_d = v_d = \overline{V}_d(\Xi)$ of Ξ. For $i = 0, \ldots, d-1$, we showed in Lemma 3.2 that the expectation $E\,\widetilde{c}_{ni}$ does not depend on n. Furthermore, for any fixed n, we have $\left|E\,\widetilde{c}_{ni} - c_i\right| \leqslant E\left|\widetilde{c}_{ni} - \widehat{c}_{ni}\right| + E\left|\widehat{c}_{ni} - c_i\right|$, where the second summand of the latter bound converges to zero by Theorem 3.1 as $n \to \infty$. Thus, in order to prove the first part of the assertion, it remains to show that the first summand in this bound converges to zero as well. By (3.4), (3.11), and (3.13), we have

$$E\left|\widetilde{\rho}_{n,r_i}(\Xi)\right| \leqslant \int\limits_{W_n \ominus B_{r_i}(o)} \left(2^{2^d}-1\right)|A(x,r)|\,E\,2^{N(\Xi\cap[0,1]^d)}\,dx \leqslant aV_d(W_n \ominus B_{r_i}(o)) \tag{3.17}$$

and

$$E\left|\widetilde{\rho}_{n,r_i}(\Xi)-\rho_{r_i}(\Xi\cap W_n)\right| \leqslant \int\limits_{\partial W_n \oplus B_{r_i}(o)} \left(2^{2^d}-1\right)|A(x,r)|\,E\,2^{N(\Xi\cap[0,1]^d)}\,dx \leqslant aV_d(\partial W_n \oplus B_{r_i}(o)) \tag{3.18}$$

where $a = k_d r^d (2^{2^d}-1)\,E\,2^{N(\Xi\cap[0,1]^d)}$ is a constant. Notice that the integration in (3.18) is performed over $\partial W_n \oplus B_{r_i}(o)$ since the sums in (3.4) and (3.7) are equal for all x with $B_{r_i}(x) \subset \mathrm{int}(W_n)$. This implies that

$$\begin{aligned} E\left|\widetilde{c}_{ni}-\widehat{c}_{ni}\right| &\leqslant \left(\frac{1}{V_d(W_n\ominus B_{r_i}(o))}-\frac{1}{V_d(W_n)}\right)E\left|\widetilde{\rho}_{n,r_i}(\Xi)\right| + \frac{1}{V_d(W_n)}\,E\left|\widetilde{\rho}_{n,r_i}(\Xi)-\rho_{r_i}(\Xi\cap W_n)\right| \\ &\leqslant a\left(1-\frac{V_d(W_n\ominus B_{r_i}(o))}{V_d(W_n)}+\frac{V_d(\partial W_n\oplus B_{r_i}(o))}{V_d(W_n)}\right). \end{aligned}$$

Thus, by (2.6), the first part of the assertion follows. Consequently, the second part of the assertion holds by (2.20) and (3.9). □

3.3 Mean–square consistency

By the integral representation given in (3.11), it turns out that the random vector $\widetilde{c}_n = \left(\widetilde{c}_{n0},\ldots,\widetilde{c}_{nd}\right)$ defined in (3.6) to (3.8) is similarly structured as the functionals of the stationary RACS Ξ considered in [1] and [2]. Moreover, estimators of the form (3.11) appear in the theory of stationary random fields as least–square estimators for the mean; see e.g. Section 3.1 in [9]. It is well known that such estimators are mean–square consistent; see [9], p. 131. By (3.1) and (3.9), this leads to conditions for mean–square consistency of the empirical intrinsic volumes $\widehat{v}_n$ and $\widetilde{v}_n$.

Assume that the stationary random field $Z_r = \left\{Z_r(x),\, x \in \mathbb{R}^d\right\}$ defined in (3.10) is of *second order* for each $r > 0$, i.e. $E\,Z_r^2(x) < \infty$ for any $x \in \mathbb{R}^d$. Due to (3.12), a sufficient condition for this is $E\,4^{N(\Xi\cap[0,1]^d)} < \infty$. Denote the covariance function of the random field $Z_r = \{Z_r(x)\}$ by $Cov_r(x)$, where $Cov_r(x) = E\left(Z_r(o)Z_r(x)\right) - z_r^2$ for any $x \in \mathbb{R}^d$. Besides this, we consider the covariance $Cov_\Xi(x) = P(o \in \Xi,\, x \in \Xi) - p_\Xi^2$ of the stationary RACS Ξ, where $p_\Xi = P(o \in \Xi) = \overline{V}_d(\Xi)$ is the volume fraction of Ξ. Assume that

$$E\,4^{N(\Xi\cap[0,1]^d)} < \infty \qquad \text{and} \qquad \int\limits_{\mathbb{R}^d} |Cov_r(x)|\,dx < \infty\,, \qquad r > 0\,, \tag{3.19}$$

and let

$$\int\limits_{\mathbb{R}^d} |Cov_\Xi(x)|\, dx < \infty\,. \tag{3.20}$$

Theorem 3.3. *Under the above assumptions, the estimators $\widehat{c}_n$ and $\widetilde{c}_n$ given in* (3.3) *to* (3.5) *and in* (3.6) *to* (3.8), *respectively, are mean–square consistent for c as $n \to \infty$. Moreover, the estimators $\widehat{v}_n$ and $\widetilde{v}_n$ given in* (3.1) *and* (3.9) *are mean–square consistent for v as $n \to \infty$.*

Proof. The mean–square consistency of $\widetilde{c}_n$ easily follows from the integral representation (3.11); see also [9], p. 131. Thus, for each $i = 0, \ldots, d$, we have

$$\lim_{n\to\infty} E\big(\widetilde{c}_{ni} - c_i\big)^2 = 0\,. \tag{3.21}$$

Using the bounds (3.17) and (3.18), we get by similar arguments as in the proof of Theorem 3.2 that

$$\begin{aligned} E\left(\widetilde{c}_{ni} - \widehat{c}_{ni}\right)^2 \;&\leqslant\; \Big(\frac{1}{V_d(W_n \ominus B_{r_i}(o))} - \frac{1}{V_d(W_n)}\Big)^2 E\,\widetilde{\rho}^{\,2}_{n,r_i}(\Xi) + \frac{1}{V_d^2(W_n)}\, E\left(\widetilde{\rho}_{n,r_i}(\Xi) - \rho_{r_i}(\Xi \cap W_n)\right)^2 \\ &\quad + \Big(\frac{1}{V_d(W_n \ominus B_{r_i}(o))} - \frac{1}{V_d(W_n)}\Big)\, \frac{2}{V_d(W_n)}\, E\left|\widetilde{\rho}_{n,r_i}(\Xi)\Big(\widetilde{\rho}_{n,r_i}(\Xi) - \rho_{r_i}(\Xi \cap W_n)\Big)\right| \\ &\leqslant\; b\left(\Big(1 - \frac{V_d(W_n \ominus B_{r_i}(o))}{V_d(W_n)}\Big) + \frac{V_d(\partial W_n \oplus B_{r_i}(o))}{V_d(W_n)}\right)^2 \end{aligned}$$

for each $i = 0, \ldots, d-1$, where $b < \infty$ is some constant. Together with (3.21), this implies that $\lim_{n\to\infty} E\left(\widehat{c}_{ni} - c_i\right)^2 = 0$ for each $i = 0, \ldots, d$. Furthermore, in view of (2.20), (3.1), and (3.9), it follows that $\lim_{n\to\infty} E\left(\widetilde{v}_{ni} - v_i\right)^2 = 0$ and $\lim_{n\to\infty} E\left(\widehat{v}_{ni} - v_i\right)^2 = 0$ for each $i = 0, \ldots, d$. $\square$

3.4 Second order characteristics

If the stationary RACS Ξ is induced by an independently marked germ–grain process with compact and convex grains, conditions can be derived such that the vector $\widetilde{v}_n = \big(\widetilde{v}_{n0}, \ldots, \widetilde{v}_{nd}\big)^\top$ of empirical intrinsic volumes is asymptotically normal as $n \to \infty$ (see Section 4.4 below). Regarding this, it is useful to determine the asymptotic covariance matrix of the (scaled) random vectors $\sqrt{V_d(W_n)}\,\widetilde{c}_n$ as $n \to \infty$. However, this asymptotic covariance matrix can still be investigated under the general assumptions on the RACS Ξ, which have been made in this paper up to now.

In addition to the covariance functions $Cov_r(x)$ and $Cov_\Xi(x)$ introduced in Section 3.3, we consider the (centered) cross–covariance functions $Cov_{rr'}(x)$ and $Cov_{r,\Xi}(x)$ of the pairs of random fields $(Z_r, Z_{r'})$ and $(Z_r, \mathbf{1}_\Xi)$ for any $r, r' > 0$ where

$$Cov_{rr'}(x) = E\left(Z_r(o) Z_{r'}(x)\right) - z_r z_{r'}\,, \qquad Cov_{r,\Xi}(x) = E\left(Z_r(o)\mathbf{1}(x \in \Xi)\right) - z_r\, p_\Xi$$

for each $x \in \mathbb{R}^d$. Notice that $Cov_{rr'}(x) = Cov_{r'r}(-x)$ and $Cov_{r,\Xi}(x) = Cov_{\Xi,r}(-x)$ for arbitrary $r, r' > 0$ and $x \in \mathbb{R}^d$. Furthermore, for any $K \in \mathcal{K}$, we consider the *geometric covariogram* $\gamma_K(x)$ of K, where $\gamma_K(x) = V_d\big(K \cap (K - x)\big)$, $x \in \mathbb{R}^d$.

Theorem 3.4. *Let* $E\,4^{N(\Xi\cap[0,1]^d)}<\infty$*. Then, for* $0\leqslant i,j<d$,

$$Cov(\widetilde{c}_{ni},\widetilde{c}_{nj})=\frac{\displaystyle\int_{\mathbb{R}^d}Cov_{r_ir_j}(x)\,V_d\big((W_n\ominus B_{r_i}(o))\cap(W_n\ominus B_{r_j}(o)-x)\big)\,dx}{V_d(W_n\ominus B_{r_i}(o))V_d(W_n\ominus B_{r_j}(o))}\tag{3.22}$$

and

$$Cov(\widetilde{c}_{ni},\widetilde{c}_{nd})=\frac{\displaystyle\int_{\mathbb{R}^d}Cov_{r_i,\Xi}(x)\,V_d\left((W_n\ominus B_{r_i}(o))\cap(W_n-x)\right)\,dx}{V_d(W_n\ominus B_{r_i}(o))V_d(W_n)}\,.\tag{3.23}$$

Moreover,

$$Var(\widetilde{c}_{ni})=\frac{\displaystyle\int_{\mathbb{R}^d}Cov_{r_i}(x)\,\gamma_{W_n\ominus B_{r_i}(o)}(x)\,dx}{V_d^2(W_n\ominus B_{r_i}(o))}\,,\qquad Var(\widetilde{c}_{nd})=\frac{1}{V_d^2(W_n)}\int\limits_{\mathbb{R}^d}Cov_\Xi(x)\,\gamma_{W_n}(x)\,dx\,.\tag{3.24}$$

Proof. By (3.6), (3.7), and (3.10), we have

$$\begin{aligned}Cov(\widetilde{c}_{ni},\widetilde{c}_{nj}) &= \frac{1}{V_d(W_n\ominus B_{r_i}(o))V_d(W_n\ominus B_{r_j}(o))}\Bigg\{E\int\limits_{W_n\ominus B_{r_i}(o)}\int\limits_{W_n\ominus B_{r_j}(o)}Z_{r_i}(x_1)Z_{r_j}(x_2)\,dx_2\,dx_1\\ &\qquad -E\Bigg(\int\limits_{W_n\ominus B_{r_i}(o)}Z_{r_i}(x_1)\,dx_1\Bigg)\cdot E\Bigg(\int\limits_{W_n\ominus B_{r_j}(o)}Z_{r_j}(x_2)\,dx_2\Bigg)\Bigg\}.\end{aligned}$$

By Fubini's theorem, this implies

$$\begin{aligned}Cov(\widetilde{c}_{ni},\widetilde{c}_{nj}) &= \frac{\displaystyle\int_{W_n\ominus B_{r_i}(o)}\int_{W_n\ominus B_{r_j}(o)}Cov_{r_ir_j}(x_2-x_1)\,dx_2\,dx_1}{V_d(W_n\ominus B_{r_i}(o))V_d(W_n\ominus B_{r_j}(o))}\\ &= \frac{\displaystyle\int_{\mathbb{R}^d}Cov_{r_ir_j}(x)\,V_d\big((W_n\ominus B_{r_i}(o))\cap(W_n\ominus B_{r_j}(o)-x)\big)\,dx}{V_d(W_n\ominus B_{r_i}(o))V_d(W_n\ominus B_{r_j}(o))}\,,\end{aligned}$$

where the latter equality follows from an appropriate substitution of the variables x_1,x_2. Similar arguments can be applied to prove formulae (3.23) and (3.24). □

Corollary 3.1. *Assume that conditions* (3.19) *and* (3.20) *are fulfilled and the cross–covariances* $Cov_{rr'}(x)$ *and* $Cov_{r,\Xi}(x)$ *are absolutely integrable for any* $r,r'>0$*, i.e.*

$$\int\limits_{\mathbb{R}^d}|Cov_{rr'}(x)|\,dx<\infty\,,\qquad\int\limits_{\mathbb{R}^d}|Cov_{r,\Xi}(x)|\,dx<\infty\,.\tag{3.25}$$

Then, for $0\leqslant i<d$ *and* $0\leqslant j\leqslant d$,

$$\lim_{n\to\infty}Cov(\sqrt{V_d(W_n)}\widetilde{c}_{ni},\sqrt{V_d(W_n)}\widetilde{c}_{nj})=\begin{cases}\int\limits_{\mathbb{R}^d}Cov_{r_ir_j}(x)\,dx & \textit{if } j<d,\\ \int\limits_{\mathbb{R}^d}Cov_{r_i,\Xi}(x)\,dx & \textit{if } j=d.\end{cases}$$

Moreover,

$$\lim_{n\to\infty} Var(\sqrt{V_d(W_n)}\widetilde{c}_{nd}) = \int\limits_{\mathbb{R}^d} Cov_{\Xi}(x)\, dx\,.$$

Proof. Using (2.6), the statements immediately follow from Theorem 3.4 and from the dominated convergence theorem. $\square$

Notice that results similar to those of Corollary 3.1 can be found e.g. in [9]. However, instead of our integrability conditions, rather restrictive mixing conditions are assumed in that case; see Theorem 1.7.6 of [9].

3.5 Consistent estimation of the asymptotic covariance matrix

In this section, we construct mean–square consistent estimators for the integrals of covariances considered in Corollary 3.1. Our construction is similar to the technique used in [1] for mean–square consistent estimation of the asymptotic covariance matrix of properly normalized volume fractions of stationarily connected RACS. However, instead of indicator–valued random fields as investigated in [1], in the present paper we consider a rather general class of stationary random fields, where we merely assume that they satisfy some (mild) integrability conditions.

Let $(\Omega, \mathfrak{F}, P)$ be an arbitrary probability space and $\mathfrak{B}(\mathbb{R}^d)$ the Borel σ–algebra on $\mathbb{R}^d$. Assume that the random fields $Y_0 = \{Y_0(x),\, x \in \mathbb{R}^d\}, \ldots, Y_d = \{Y_d(x),\, x \in \mathbb{R}^d\}$ defined on this probability space are *stationarily connected*, i.e., the vector field $Y = \{Y(x),\, x \in \mathbb{R}^d\}$ with $Y(x) = (Y_0(x), \ldots, Y_d(x))^\top$ is stationary. For each $i = 0, \ldots, d$, suppose that $Y_i : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a measurable mapping with respect to the product σ–algebra $\mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{F}$.

Assume that for each $i = 0, \ldots, d$,

$$E\, Y_i^4(0) < \infty\,. \tag{3.26}$$

Then, in particular, the expectation $m_i = E\, Y_i(x)$ and the covariance $Cov_{ij}(x) = E\,(Y_i(o)Y_j(x)) - m_i m_j$ are well defined for any $i, j = 0, \ldots, d$ and $x \in \mathbb{R}^d$. Notice that $Cov_{ij}(x) = Cov_{ji}(-x)$ holds for each $x \in \mathbb{R}^d$. Rewriting condition (3.25), we assume that

$$\int\limits_{\mathbb{R}^d} |Cov_{ij}(x)|\, dx < \infty \tag{3.27}$$

for any $i, j = 0, \ldots, d$. Thus, the matrix $\Sigma = (\sigma_{ij})$ of integrated cross–covariances is also well defined, where $\sigma_{ij} = \int\limits_{\mathbb{R}^d} Cov_{ij}(x)\, dx$ for $0 \leqslant i, j \leqslant d$. On the space of real–valued $(d+1) \times (d+1)$–matrices $A = (a_{ij})$, we consider the norm

$$\|A\| = \sqrt{\sum_{i,j=0}^{d} a_{ij}^2}\,. \tag{3.28}$$

The aim of this section is to construct a sequence of estimators $\widehat{\Sigma}_n = \left(\widehat{\sigma}_{nij}\right)$ for $\Sigma = (\sigma_{ij})$ such that $\lim_{n\to\infty} E\, \|\widehat{\Sigma}_n - \Sigma\|^2 = 0$ holds. Suppose that for each $n \in \mathbb{N}$, the estimator $\widehat{\Sigma}_n$ is based on the observation of the vector field $Y = \{Y(x),\, x \in \mathbb{R}^d\}$ within a certain sampling window $W_n \subset \mathbb{R}^d$, where $\{W_n\}$ is a sequence of (monotonously increasing, compact and convex) sets of the kind given in (2.4). For each $n \in \mathbb{N}$, we additionally consider some compact and convex sets $W_{n0}, \ldots, W_{nd} \subset W_n$, where we assume that the i–th component Y_i of the vector field Y is observable only on W_{ni}, for each $i = 0, \ldots, d$. Suppose that for each $n \in \mathbb{N}$, these subsets of W_n are ordered with respect to inclusion, i.e. $W_{n0} \subset \ldots \subset W_{nd} = W_n$.

For any $i, j = 0, \ldots, d$, let $W_{nij} = W_{ni} \cap W_{nj}$ and let $\{U_{nij}\}$ be a monotonously increasing sequence of compact and convex (averaging) sets such that $U_{nij} \subset W_{nij}$ and $V_d(U_{nij}) > 0$ for each $n \in \mathbb{N}$. Furthermore, assume that $\lim\limits_{n\to\infty} U_{nij} = \mathbb{R}^d$ and that

$$\lim_{n\to\infty} \frac{V_d^2(U_{nij})}{V_d(W_{nij})} = 0\,, \tag{3.29}$$

$$\lim_{n\to\infty} \frac{\min\limits_{x\in U_{nij}} V_d\big(W_{nij} \cap (W_{nij} + x)\big)}{V_d(W_{nij})} = 1 \tag{3.30}$$

for any $i, j = 0, \ldots, d$. Notice that conditions (3.29) and (3.30) keep the averaging sets U_{nij} small enough in comparison with W_{nij} ensuring, for instance, that the volume of U_{nij} grows slower than the square root of the volume of W_{nij}.

For any $n \in \mathbb{N}$ and $i, j = 0, \ldots, d$, we consider the following estimator

$$\widehat{\sigma}_{nij} = \frac{1}{V_d(W_{nij})} \int\limits_{U_{nij}} \widehat{Cov}_{nij}(x)\, V_d\big(W_{nij} \cap (W_{nij} - x)\big)\, dx \tag{3.31}$$

for σ_{ij}, where

$$\widehat{Cov}_{nij}(x) = \frac{\int\limits_{W_{nij}\cap(W_{nij}+x)} Y_j(y) Y_i(y-x)\, dy}{V_d(W_{nij} \cap (W_{nij} + x))} - \frac{\int\limits_{W_{nij}} Y_i(y)\, dy \int\limits_{W_{nij}} Y_j(y)\, dy}{V_d^2(W_{nij})} \tag{3.32}$$

is the standard estimator for the cross–covariance $Cov_{ij}(x)$ of Y_i and Y_j; see e.g. [9], Chapter 4. Notice that the values $\widehat{Cov}_{nij}(x)$ of the integrand in (3.31) are weighted by $V_d\big(W_{nij} \cap (W_{nij} - x)\big)/V_d(W_{nij})$. Thus, besides restricting the area of averaging in (3.31) to the set $U_{nij} \subset W_{nij}$, this *edge correction* leads to smaller weights for those $x \in U_{nij}$ being further away from the origin, that is *not* in the central part of $U_{nij} \subset W_{nij} \subset W_n$, but closer to the boundary of these sets.

Theorem 3.5. *Assume that*

$$\sup_{x_1,x_2\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \big| E\big(Y_i(o)Y_j(x_1)Y_i(y)Y_j(y+x_2)\big) - E\big(Y_i(o)Y_j(x_1)\big)\, E\big(Y_i(o)Y_j(x_2)\big)\big|\, dy < \infty \tag{3.33}$$

and

$$\int\limits_{\mathbb{R}^{3d}} \Big| E\Big((Y_i(o) - m_i)(Y_j(x_1) - m_j)(Y_i(x_2) - m_i)(Y_j(x_3) - m_j)\Big)\Big|\, d(x_1, x_2, x_3) < \infty \tag{3.34}$$

for any $i,j = 0,\ldots,d$. Then, the estimator $\widehat{\Sigma}_n = (\widehat{\sigma}_{nij})$ introduced in (3.31)–(3.32) is mean–square consistent for the matrix $\Sigma = (\sigma_{ij})$, i.e.

$$\lim_{n\to\infty} E\,\|\widehat{\Sigma}_n - \Sigma\|^2 = 0\,. \tag{3.35}$$

Proof. By (3.28), it suffices to show that for any $i,j = 0,\ldots,d$

$$\lim_{n\to\infty} E\,\left(\widehat{\sigma}_{nij} - \sigma_{ij}\right)^2 = 0\,. \tag{3.36}$$

By Minkowski's inequality, we have

$$\begin{aligned}\sqrt{E\,\left(\widehat{\sigma}_{nij} - \sigma_{ij}\right)^2} \;&\leqslant\; \left(E\Big(\int\limits_{U_{nij}} (\widehat{Cov}_{ij}(x) - Cov_{ij}(x))\frac{\gamma_{W_{nij}}(x)}{V_d(W_{nij})}\,dx\Big)^2\right)^{1/2} \\ &\quad + \int\limits_{U_{nij}} |Cov_{ij}(x)|\left(1 - \frac{\gamma_{W_{nij}}(x)}{V_d(W_{nij})}\right)dx + \int\limits_{\mathbb{R}^d\setminus U_{nij}} |Cov_{ij}(x)|\,dx\,, \end{aligned}\tag{3.37}$$

where $\gamma_{W_{nij}}(x) = V_d(W_{nij}\cap(W_{nij}+x))$. Due to the integrability of the cross–covariances (3.27), the last term in (3.37) tends to zero as $n\to\infty$. By (3.30), we get

$$\int\limits_{U_{nij}} |Cov_{ij}(x)|\left(1 - \frac{\gamma_{W_{nij}}(x)}{V_d(W_{nij})}\right)dx \leqslant \left(1 - \frac{\min\limits_{x\in U_{nij}} \gamma_{W_{nij}}(x)}{V_d(W_{nij})}\right)\int\limits_{\mathbb{R}^d} |Cov_{ij}(x)|\,dx \longrightarrow 0$$

as $n\to\infty$. It remains to show that the first right–hand term in (3.37) tends to zero, i.e.

$$\lim_{n\to\infty} E\Big(\frac{1}{V_d(W_{nij})}\int\limits_{U_{nij}} \left(\widehat{Cov}_{nij}(x) - Cov_{ij}(x)\right)\gamma_{W_{nij}}(x)\,dx\Big)^2 = 0\,.$$

With the abbreviating notation

$$\widehat{C}_{nij}(x) = \frac{\int\limits_{W_{nij}\cap(W_{nij}+x)} Y_j(y)Y_i(y-x)\,dy}{V_d(W_{nij}\cap(W_{nij}+x))} \qquad \text{and} \qquad \widehat{m}_{ni} = \frac{\int\limits_{W_{nij}} Y_i(y)\,dy}{V_d(W_{nij})}\,, \tag{3.38}$$

the estimator $\widehat{Cov}_{nij}(x)$ defined in (3.32) can be rewritten as $\widehat{Cov}_{nij}(x) = \widehat{C}_{nij}(x) - \widehat{m}_{ni}\widehat{m}_{nj}$. By Minkowski's inequality, we have

$$\begin{aligned}&\Big(E\Big(\int\limits_{U_{nij}} (\widehat{Cov}_{nij}(x) - Cov_{ij}(x))\frac{\gamma_{W_{nij}}(x)}{V_d(W_{nij})}\,dx\Big)^2\Big)^{1/2} \\ &\leqslant \Big(E\Big(\int\limits_{U_{nij}} \left(\widehat{C}_{nij}(x) - E\left(Y_i(o)Y_j(x)\right)\right)\frac{\gamma_{W_{nij}}(x)}{V_d(W_{nij})}\,dx\Big)^2\Big)^{1/2} \\ &\quad + \Big(E\Big(\int\limits_{U_{nij}} (m_i m_j - \widehat{m}_{ni}\widehat{m}_{nj})\frac{\gamma_{W_{nij}}(x)}{V_d(W_{nij})}\,dx\Big)^2\Big)^{1/2}.\end{aligned}\tag{3.39}$$

For the square of the first term on the right–hand side in (3.39), we get by stationarity of Y and an appropriate change of variables that

$$E\Big(\int\limits_{U_{nij}}\Big(\widehat{C}_{nij}(x)-E\big(Y_i(o)Y_j(x)\big)\Big)\frac{\gamma_{W_{nij}}(x)}{V_d(W_{nij})}\,dx\Big)^2$$
$$\leqslant\ \frac{V_d^2(U_{nij})}{V_d(W_{nij})}\sup_{x_1,x_2\in\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\big|\,Cov\big(Y_i(o)Y_j(x_1),Y_i(y)Y_j(y+x_2)\big)\big|\,dy\,.$$

By (3.29) and (3.33), the last expression converges to zero as $n\to\infty$. Using the inequality of Cauchy–Schwarz, the square of the second term on the right–hand side in (3.39) can be bounded by

$$E\Big(\int\limits_{U_{nij}}\big(m_im_j-\widehat{m}_{ni}\widehat{m}_{nj}\big)\frac{\gamma_{W_{nij}}(x)}{V_d(W_{nij})}\,dx\Big)^2$$
$$\leqslant\ \frac{V_d^2(U_{nij})}{V_d^{3/2}(W_{nij})}\Big(\sqrt{a\,EY_j^4(o)}+2|m_i|\sqrt{a\,EY_j^2(o)}\Big)+\frac{V_d^2(U_{nij})}{V_d(W_{nij})}\,m_i^2\int\limits_{\mathbb{R}^d}|Cov_{jj}(x)|\,dx\,,$$

where $a<\infty$ is the maximum value of the integrals considered in (3.34). By (3.29), the latter bound converges to zero as $n\to\infty$. □

The mean–square consistency (3.35) proved above obviously implies the asymptotical unbiasedness of the estimator $\widehat{\Sigma}_n$, i.e. $\lim_{n\to\infty}\|E\,\widehat{\Sigma}_n-\Sigma\|=0$. However, it should be noted that this holds under much weaker conditions than those of Theorem 3.5. Namely, it is enough to require (3.27) and (3.30).

Letting $Y_0(x)=Z_{r_0}(x),\ldots,Y_{d-1}(x)=Z_{r_{d-1}}(x)$, and $Y_d(x)=\mathbf{1}(x\in\Xi)$, the general model discussed in the present section turns out to be the one considered in Sections 3.2 to 3.4. Recall that we assumed in (2.4) that the sampling windows W_n are given by $W_n=nK_0$ for some $K_0\in\mathcal{K}$ such that $V_d(K_0)>0$ and $o\in K_0\setminus\partial K_0$. Furthermore, in Sections 3.2 to 3.4, we considered the reduced sampling windows $W_{ni}=W_n\ominus B_{r_i}(o)$. If, for example, $K_0=B_1(o)$ and the averaging sets U_{nij} are given by $U_{nij}=B_{\varepsilon_n\sqrt{n}-\max\{r_i,r_j\}}(o)$ for some sequence $\{\varepsilon_n\}$ with $0<\varepsilon_n\leqslant 1$, $\varepsilon_n\downarrow 0$ and $\sqrt{n}\varepsilon_n\uparrow\infty$ then $U_{nij}\subset W_{nij}$ holds for any $i,j=0,\ldots,d$ and the conditions (3.29) and (3.30) are fulfilled. Examples of stationary RACS Ξ, for which the integrability conditions (3.26)–(3.27) and (3.33)–(3.34) are fulfilled, will be discussed in Section 4.

4 Germ–grain processes

Let Ξ be a stationary RACS induced by a *germ–grain process* $(X,M)=\{(X_i,M_i)\}$ in $\mathbb{R}^d$, where $X=\{X_i\}$ is a stationary point process in $\mathbb{R}^d$ with positive finite intensity λ and $M=\{M_i\}$ is a sequence independent of X consisting of independent and identically distributed copies of a non–empty compact and convex RACS M_0 in $\mathbb{R}^d$ (called the *typical grain*) such that $o\in M_0$ and

$$E\,V_d(M_0\oplus K)<\infty\qquad\text{for each }K\in\mathcal{K}.\tag{4.1}$$

In other words, we assume that with probability 1

$$\Xi = \bigcup_{i=1}^{\infty} \Xi_i \qquad \text{where } \Xi_i = M_i + X_i. \tag{4.2}$$

Notice that condition (4.1) implies that for almost every realization of the germ–grain process $(X, M) = \{(X_i, M_i)\}$, only finitely many sets $M_i + X_i$ hit a fixed set $K \in \mathcal{K}$. This means in particular that the infinite union Ξ in (4.2) is a closed set with probability 1 and, therefore, Ξ is a RACS; see e.g. Lemma 3 of Heinrich [4]. Besides this, we consider the following set $A_k \subset \partial\Xi$ for any $k \in \mathbb{N}$. Let

$$A_k = \{q \in \partial\Xi : \exists\, i_1, \ldots, i_k \text{ such that } q \in \bigcap_{j=1}^{k} \partial\Xi_{i_j}; \quad \forall\, i_1, \ldots, i_{k+1} \quad q \notin \bigcap_{j=1}^{k+1} \partial\Xi_{i_j}\} \tag{4.3}$$

and assume that

$$P(A_k = \emptyset) = 1, \qquad k > d\,. \tag{4.4}$$

Conditions similar to (4.4) can be found e.g. in Heinrich and Molchanov [7]. In particular, condition (4.4) is fulfilled if $X = \{X_i\}$ is Poisson.

4.1 Alternative bound on $|Z_r(x)|$

Under the above (additional) assumptions on the structure of Ξ, we can get deeper insight into the properties of the random field $Z_r = \{Z_r(x),\, x \in \mathbb{R}^d\}$ defined in (3.10). In particular, the results of Section 3 can be obtained under weaker integrability conditions than those of (2.3) and (3.19), respectively. Indeed, instead of (2.3), a sufficient condition for the existence of specific intrinsic volumes is (4.1); see [23], Satz 5.1.4. In order to weaken the first integrability assumption in (3.19), i.e. $E\, 4^{N(\Xi\cap[0,1]^d)} < \infty$, we consider an upper bound for $|Z_r(x)|$ which is different from that in (3.12). Denote by $N_r(x)$ the number of grains of Ξ that intersect the ball $B_r(x)$, i.e., $N_r(x) = \#\{i : \Xi_i \cap B_r(x) \neq \emptyset\}$.

Lemma 4.1. *For any dimension $d \geqslant 2$, there exists a constant $a_d < \infty$ such that*

$$|Z_r(x)| \leqslant a_d N_r^d(x) \tag{4.5}$$

almost surely for any $x \in \mathbb{R}^d$ and $r > 0$.

Proof. By definition of $N_r(x)$, it holds $\Xi \cap B_r(x) = \bigcup_{i=1}^{N_r(x)} \Xi_i \cap B_r(x)$. Due to condition (4.4), we have $A_k = \emptyset$ almost surely for all $k > d$, where A_k denotes the set introduced in (4.3). Thus, the definition of $Z_r(x)$ given in (3.10) can be rewritten as

$$Z_r(x) = \sum_{k=1}^{d} \sum_{q \in A_k} J(\Xi \cap B_r(x), q, x)\,. \tag{4.6}$$

If $q \in A_k$ then the absolute value of the index $J(\Xi \cap B_r(x), q, x)$ can obviously be bounded from above by $|J(\Xi \cap B_r(x), q, x)| \leqslant k$. Together with relation (4.6), this inequality yields

$$|Z_r(x)| \leqslant \sum_{k=1}^{d} k \binom{N_r(x)}{k} \leqslant a_d N_r^d(x)\,,$$

where $a_d = \sum_{k=1}^{d} 1/(k-1)!$. $\square$

Using the result of Lemma 4.1, the bound (3.12) on $|Z_r(x)|$ can be replaced by (4.5). Thus, the first integrability assumption in (3.19) can be replaced by

$$E\,N_r^{2d}(o) < \infty \tag{4.7}$$

in order to ensure that the stationary random field $Z_r = \{Z_r(x),\, x \in \mathbb{R}^d\}$ is of second order; $r > 0$. Notice that a similar condition for the existence of specific intrinsic volumes has been derived in [27], p. 339. Furthermore, using Minkowski's inequality, it is not difficult to see that (4.7) holds for any $r > 0$ if and only if $E\,N_1^{2d}(o) < \infty$.

4.2 Finite total variation of factorial moment measures

Sufficient conditions for (4.7) can be derived as follows. Consider the j–th *factorial moment measure* α_j of $X = \{X_i\}$ given by $\alpha_j\big(A_1 \times \ldots \times A_j\big) = E\sum_{i_1,\ldots,i_j}^{\neq} \mathbf{1}\big(X_{i_1} \in A_1, \ldots, X_{i_j} \in A_j\big)$ for arbitrary bounded Borel sets $A_1, \ldots, A_j \in \mathcal{B}(\mathbb{R}^d)$, where the summation $\sum^{\neq}$ extends over all j–tuples of distinct indices. Since X is stationary with intensity λ, the factorial moment measure α_j of X admits the decomposition

$$\alpha_j(A_1 \times \ldots \times A_j) = \lambda \int_{A_1} \alpha_{j-1}^{(0)}\big((A_2 - x) \times \ldots \times (A_j - x)\big)\, dx\,, \tag{4.8}$$

where $\alpha_{j-1}^{(0)}$ denotes the $(j-1)$–th factorial moment measure with respect to the reduced Palm distribution of X (see [3], Chapters 10 and 12).

Lemma 4.2. *Let $m \in \mathbb{N}$ be arbitrary, but fixed. Assume that* (4.1) *holds and that*

$$\int_{\mathbb{R}^{dj}} |\alpha_j - \alpha_j^{(0)}|\, d(x_1, \ldots, x_j) < \infty \tag{4.9}$$

for each $j \in \{1, \ldots, m-1\}$. Then, $EN_r^m(0) < \infty$ for each $r > 0$.

Proof. Notice that

$$N_r(o) = \#\{i : \Xi_i \cap B_r(o) \neq \emptyset\} = \#\{i : -X_i \in (M_i \oplus B_r(o))\} = \sum_{(X_i,M_i)\in(X,M)} \mathbf{1}\big(-X_i \in (M_i \oplus B_r(o))\big)$$

where $\mathbf{1}\big(-X_i \in (M_i \oplus B_r(o))\big)$ denotes the indicator of the set $\{\omega : -X_i(\omega) \in (M_i(\omega) \oplus B_r(o))\}$. Thus, the expectation $E\,N_r^m(o)$ can be rewritten as

$$E\,N_r^m(o) = \sum_{j=1}^{m} j!\, E \sum_{i_1<\ldots<i_j} \mathbf{1}\big(-X_{i_1} \in (M_{i_1} \oplus B_r(o))\big) \cdot \ldots \cdot \mathbf{1}\big(-X_{i_j} \in (M_{i_j} \oplus B_r(o))\big)\,.$$

Furthermore, by (4.8) we get for each $j \in \{1,\dots,m\}$ that

$$j!\,E \sum_{i_1<\ldots<i_j} \prod_{k=1}^{j} \mathbf{1}\big(-X_{i_k} \in (M_{i_k} \oplus B_r(o))\big) = \int\limits_{\mathbb{R}^{dj}} \prod_{i=1}^{j} P(-x_i \in M_0 \oplus B_r(o))\, \alpha_j\big(d(x_1,\dots,x_j)\big)$$

$$= \lambda \int\limits_{\mathbb{R}^d} P(-x_j \in M_0 \oplus B_r(o)) \int\limits_{\mathbb{R}^{(j-1)d}} \prod_{i=1}^{j-1} P(-x_i \in M_0 \oplus B_r(o))\, \alpha_{j-1}^{(0)}\big(d(x_1-x_j,\dots,x_{j-1}-x_j)\big)\, dx_j$$

$$= \lambda \int\limits_{\mathbb{R}^d} P(-x_j \in M_0 \oplus B_r(o))$$

$$\times \int\limits_{\mathbb{R}^{(j-1)d}} \prod_{i=1}^{j-1} P(-x_i \in M_0 \oplus B_r(o))\, (\alpha_{j-1}^{(0)} - \alpha_{j-1})\big(d(x_1-x_j,\dots,x_{j-1}-x_j)\big)\, dx_j$$

$$+\lambda \int\limits_{\mathbb{R}^d} P(-x_j \in M_0 \oplus B_r(o)) \int\limits_{\mathbb{R}^{(j-1)d}} \prod_{i=1}^{j-1} P(-x_i \in M_0 \oplus B_r(o))\, \alpha_{j-1}\big(d(x_1-x_j,\dots,x_{j-1}-x_j)\big)\, dx_j\,.$$

Thus,

$$j!\,E \sum_{i_1<\ldots<i_j} \prod_{k=1}^{j} \mathbf{1}\big(-X_{i_k} \in (M_{i_1} \oplus B_r(o))\big)$$

$$\leqslant \lambda\, E\, V_d(M_0 \oplus B_r(o)) \int\limits_{\mathbb{R}^{(j-1)d}} |\alpha_{j-1}^{(0)} - \alpha_{j-1}|\big(d(x_1,\dots,x_{j-1})\big) + \lambda \int\limits_{\mathbb{R}^d} P(-x_j \in M_0 \oplus B_r(o))$$

$$\times \int\limits_{\mathbb{R}^{(j-1)d}} \prod_{i=1}^{j-1} P(-x_i \in M_0 \oplus B_r(o))\, \alpha_{j-1}\big(d(x_1-x_j,\dots,x_{j-1}-x_j)\big)\, dx_j\,.$$

By induction with respect to j, this leads to the inequality

$$j!\,E \sum_{i_1<\ldots<i_j} \prod_{k=1}^{j} \mathbf{1}\big(-X_{i_k} \in (M_{i_1} \oplus B_r(o))\big)$$

$$\leqslant \Big(\lambda\, E\, V_d(M_0 \oplus B_r(o))\Big)^j + \sum_{k=1}^{j-1} \Big(\lambda\, E\, V_d(M_0 \oplus B_r(o))\Big)^{j-k} \int\limits_{\mathbb{R}^{kd}} |\alpha_k^{(0)} - \alpha_k|\big(d(x_1,\dots,x_k)\big)\,.$$

Thus, conditions (4.1) and (4.9) are sufficient for $E\, N_r^m(o) < \infty$. $\square$

4.3 The Boolean model

If X is Poisson, i.e. $\Xi = \bigcup_{i=1}^{\infty}(M_i + X_i)$ is the Boolean model, then condition (4.9) is obviously fulfilled, since $\alpha_j = \alpha_j^{(0)}$ for any $j \geqslant 1$ in this case. Similarly, one can show that (3.20) is fulfilled for the Boolean model provided that

$$E\,V_d^2(M_0) < \infty\,. \tag{4.10}$$

Indeed, for the covariance $Cov_\Xi(x)$ of the Boolean model Ξ we have (see e.g. [23, 24])

$$Cov_\Xi(x) = (1-p_\Xi)^2\,\Big(e^{\lambda\,E\,V_d(M_0\cap(M_0-x))} - 1\Big) \geqslant 0\,, \qquad x \in \mathbb{R}^d\,.$$

Together with condition (4.10), this implies that $\int_{\mathbb{R}^d} |Cov_\Xi(x)|\,dx \leqslant \lambda(1-p_\Xi)^2\,E\,V_d^2(M_0) < \infty$. Furthermore, one can provide similar sufficient conditions for (absolute) integrability of the covariance function $Cov_r(x)$ of the random field Z_r. In particular, it is easy to see that $\int_{\mathbb{R}^d} |Cov_r(x)|\,dx < \infty$ holds for the Boolean model with uniformly bounded generic grain M_0, because in this case we have $Cov_r(x) = 0$ if $|x| > a$ for some constant $a < \infty$. Thereby,

$$\int\limits_{\mathbb{R}^d} |Cov_r(x)|\,dx = \int\limits_{\{x:\,|x|\leqslant a\}} |Cov_r(x)|\,dx \leqslant V_d(B_a(o))\;E\,Z_r^2(o) < \infty\,.$$

In the same way, it can be shown that (3.25) holds, i.e., the cross–covariances $Cov_{rr'}(x)$ and $Cov_{r,\Xi}(x)$ are absolutely integrable for the Boolean model with uniformly bounded generic grain M_0. Furthermore, the integrability conditions (3.33) and (3.34) are fulfilled in this case as well.

4.4 Asymptotic normality of empirical intrinsic volumes

Using the integral representation of $\widetilde{c}_n = (\widetilde{c}_{n0},\ldots,\widetilde{c}_{nd})$ derived in (3.11) and requiring some mixing conditions on the stationary point process $X = \{X_i\}$ of germs (see e.g. Heinrich and Molchanov [7], Ivanov and Leonenko [9], Mase [11]), the following central limit theorem for the random vector $\widetilde{c}_n$ can be shown. For $n \to \infty$,

$$\begin{pmatrix} \sqrt{V_d(W_n)}\,(\widetilde{c}_{no} - c_0) \\ \vdots \\ \sqrt{V_d(W_n)}\,(\widetilde{c}_{nd} - c_d) \end{pmatrix} \Longrightarrow N(0,\Sigma)\,, \tag{4.11}$$

where $\Longrightarrow$ denotes convergence in distribution, $c = (c_0,\ldots,c_d)^\top$, and the asymptotic covariance matrix Σ is given by

$$\Sigma = \begin{pmatrix} \int\limits_{\mathbb{R}^d} Cov_{r_0}(x)\,dx & \int\limits_{\mathbb{R}^d} Cov_{r_0r_1}(x)\,dx & \ldots & \int\limits_{\mathbb{R}^d} Cov_{r_0,\Xi}(x)\,dx \\ \int\limits_{\mathbb{R}^d} Cov_{r_0r_1}(x)\,dx & \int\limits_{\mathbb{R}^d} Cov_{r_1}(x)\,dx & \ldots & \int\limits_{\mathbb{R}^d} Cov_{r_1,\Xi}(x)\,dx \\ \vdots & \vdots & & \vdots \\ \int\limits_{\mathbb{R}^d} Cov_{r_0,\Xi}(x)\,dx & \int\limits_{\mathbb{R}^d} Cov_{r_1,\Xi}(x)\,dx & \ldots & \int\limits_{\mathbb{R}^d} Cov_\Xi(x)\,dx \end{pmatrix}. \tag{4.12}$$

Furthermore, by (3.1), we have

$$\begin{pmatrix} \sqrt{V_d(W_n)}\,(\widetilde{v}_{no} - v_0) \\ \vdots \\ \sqrt{V_d(W_n)}\,(\widetilde{v}_{nd} - v_d) \end{pmatrix} \Longrightarrow N\big(0, A^{-1}_{r_0\ldots r_{d-1}}\,\Sigma\,(A^{-1}_{r_0\ldots r_{d-1}})^\top\big)\,. \tag{4.13}$$

The asymptotic normality (4.13) of the vector $\widetilde{v}_n = (\widetilde{v}_{n0}, \ldots, \widetilde{v}_{nd})^\top$ of empirical intrinsic volumes can be used in order to construct asymptotic Gauss tests for simultaneous verification of hypotheses about the vector $v = (v_0, \ldots, v_d)^\top$ of specific intrinsic volumes of Ξ, provided that the transformed covariance matrix $A^{-1}_{r_0 \ldots r_{d-1}} \Sigma \, (A^{-1}_{r_0 \ldots r_{d-1}})^\top$ is positive definite and can be estimated consistently.

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