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Isoperimetric problems and roses of neighborhood for stationary flat processes

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The paper yields necessary conditions for the directional distributions of stationary k–flat processes in $\mathbb{R}^d$ that maximize their intersection density of order 2, that is, the mean $(2k-d)$–volume of their self–intersections in an observation window of unit d–volume. The conditions are given in terms of the rose of intersections (i.e., the intensity of intersections of the flat process with test flats). The notion of the *rose of neighborhood* is introduced which is an analogue of the rose of intersections for lower dimensional flat processes. Some properties of the rose of neighborhood are studied and an asymptotically unbiased estimator is given.

1 Introduction

Consider a stationary k–flat process Φ_k^d in $\mathbb{R}^d$, i.e., Φ_k^d is a random point process on the phase space of all k–flats in d–dimensional space, each realization of which is an at most countable "locally finite" collection of k–flats. Stationarity means invariance of its distribution with respect to translations in $\mathbb{R}^d$. The probability distribution θ of the direction of the "typical" flat of the process is called the *rose of directions* or the *directional distribution* of Φ_k^d. The family of all possible pairwise $(2k-d)$–dimensional intersections of the k–flats of Φ_k^d induces a new stationary $(2k-d)$–flat process whose intensity, that is, the mean $(2k-d)$–dimensional volume of its flats in a test window of unit volume, is called the *intersection density of order* 2 of Φ_k^d. Several authors (Davidson [1], Janson and Kallenberg [6], Keutel [7], Mecke and Thomas ([10], [11], [12], [13], [16], [26])) dealt with the following variational problem concerning Φ_k^d: find the directional distributions θ of Φ_k^d that maximize its intersection density under the condition that Φ_k^d has a given intensity.

In the case of hyperplanes ($k = d-1$), the solution is unique and corresponds to the Haar measure on the appropriate Grassmann manifold. In other particular cases, the whole class of extremal directional distributions θ was described. Nevertheless, there are still open problems, e.g., when d is not divisible by $d-k$.

The first part of the paper yields necessary conditions for a maximum of intersection density of order 2 of Φ_k^d. The conditions are given in terms of the rose of intersections of Φ_k^d for arbitrary dimensions d and k (cf. Section 3.4). Section 3 contains the mathematical setting of the problem and an overview of the literature on this subject. The connection to classical isoperimetric problems for centrally symmetric convex bodies is discussed. In Section 3.3, some basic facts of variational calculus needed in the problem discussed here are introduced. The necessary conditions for a maximum proved in Theorem 3.2 are obviously not sufficient. Hence, they do not lead to the solution of the problem of finding the extremal directional distributions θ. Nevertheless, their generality unifies the great variety of solutions given in

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the literature for particular dimensions k. Thus, the common structure in the abundance of extremal measures independent of k becomes clear.

In Section 4, we introduce a counterpart to the classical notion of the *rose of intersections*, the so–called *rose of neighborhood*, for the case that $k + r < d$ for some $r > 0$. The rose of intersections, i.e., the intensity of the process $\Phi_k^d \cap \eta$ (where η is an arbitrary r–flat), $k + r \geq d$, is relevant if the cuts, or sections, of the given experimental pattern are available. This is often the case, for instance, in materials science and biology. Then, the directional distribution of the process Φ_k^d can be computed from its rose of intersections by usual methods, cf. [23] and [24]. But, sometimes, it is impossible to collect information, for example, from the planar sections of the studied pattern for various reasons, because such a cut might destroy the material structure or is just technically impossible (e.g., in geology). Then, some indirect measurements should be performed, such as counting all the objects that intersect a directed laser beam or a drilling path. In our terms, a lower dimensional test flat η $(r < d - k)$ or even a line $(r = 1)$ can be used instead to estimate the directional distribution of Φ_k^d. For this purpose, the rose of neighborhood is introduced and its properties are studied. Namely, we "blow" the test line (or the lower dimensional flat) η up to a test cylinder whose intersection with flats of the process Φ_k^d is not empty anymore. Then, one can count the flats of Φ_k^d that intersect the test cylinder. Collecting such information for the test flats η with various directions, a conclusion about the directional distribution can be made.

Notice that the idea of "blowing" test flats up to their neighborhoods was already mentioned in [6] and [21].

In Section 4.2, an asymptotically unbiased estimator for the rose of neighborhood is proposed. This enables us to estimate the density of the directional distribution of the process Φ_k^d, see Section 4.3.

2 Stationary k–flat processes

In this section, we follow the framework of [16] in introducing the basic notions of k–flat processes (cf. [15], [22], [25] for other constructions).

Let $F(k, d)$ be the set of all k–flats in $\mathbb{R}^d$, $d \geq 2$, $1 \leq k \leq d - 1$. Let $G(k, d)$ be the *Grassmann manifold* of all k–dimensional linear subspaces of $\mathbb{R}^d$. Let $\mathcal{F}$, $\mathcal{G}$ be the σ–algebras of Borel subsets of $F(k, d)$, $G(k, d)$ in their usual topologies. The subset $\varphi \subset F(k, d)$ is called a *flat field* if φ is at most countable and any bounded set $B \subset \mathbb{R}^d$ is intersected by a finite number of k–flats of φ. Let M be the set of all flat fields and $\mathcal{M}$ be the Borel σ–algebra on M.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A measurable mapping $\Phi_k^d : \Omega \to \mathcal{M}$ is called a *k–flat process*. Its distribution is a probability measure on $\mathcal{M}$. Notice that Φ_k^d is an ordinary point process for $k = 0$ and a hyperplane process for $k = d - 1$ in $\mathbb{R}^d$.

A k–flat process Φ_k^d is called *stationary* if its distribution is invariant with respect to all translations in $\mathbb{R}^d$. Denote by ν_k the k–dimensional Lebesgue measure in $\mathbb{R}^d$. The *intensity* of the stationary process Φ_k^d is defined by $\lambda = \frac{E\,\nu_k(\Phi_k^d \cap B)}{\nu_d(B)}$ for any bounded subset B of $\mathbb{R}^d$ with $\nu_d(B) > 0$ where $\Phi_k^d \cap B$ denotes the set $\{\xi \cap B : \xi \in \Phi_k^d\}$. Suppose $0 < \lambda < \infty$. The *rose of directions* (or *directional distribution*) of Φ_k^d is the probability measure θ on $G(k, d)$ given by

$$\theta(\mathcal{C}) \;=\; \frac{E\,\#\{\xi \in \Phi_k^d : \xi \cap \mathbf{S}^{d-1} \neq \emptyset,\; r(\xi) \in \mathcal{C}\}}{\lambda \kappa_{d-k}}\,, \quad \mathcal{C} \in \mathcal{G} \tag{2.1}$$

where $\#A$ denotes the cardinality of the set A and $r(\xi)$ is the direction of the k–flat ξ, i.e., the unique $\bar{\xi} \in G(k, d)$ that is parallel to ξ, $\kappa_d = \frac{2\pi^{\frac{d}{2}}}{d\,\Gamma\left(\frac{d}{2}\right)}$ is the volume of the unit ball, and $\mathbf{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$.

Let $\Phi_k^d(D)$ denote the number of k–flats of Φ_k^d that belong to $D \in \mathcal{F}$. Then the measure $\Lambda : \mathcal{F} \to [0, \infty)$ with

$$\Lambda(D) \;=\; E\,\Phi_k^d(D)\,, \quad D \in \mathcal{F}$$

is called the *intensity measure* of Φ_k^d.

For stationary Φ_k^d, the following factorization of its intensity measure holds (cf. [22]):

$$\Lambda(D) \;=\; \lambda \int\limits_{G(k,d)} \int\limits_{\xi^\perp} I_D(y+\xi)\, \nu_{d-k}^{\xi^\perp}(dy)\, \theta(d\xi)\,, \qquad D \in \mathcal{F}\,, \tag{2.2}$$

where $\nu_{d-k}^{\xi^\perp}(\cdot)$ is the Lebesgue measure on the orthogonal linear subspace $\xi^\perp$ and I_D is the indicator of the set D.

Suppose the intensity λ to be known. For any $\eta \in F\big(d-k+j,d\big)$, the j–flat process $\Phi_k^d \cap \eta$ is again stationary on η. Let $\lambda_{\Phi_k^d \cap \eta}$ be the intensity of $\Phi_k^d \cap \eta$. Due to the stationarity of Φ_k^d, it is sufficient to consider only those affine flats η that contain the origin, i.e., $\eta \in G\big(d-k+j,d\big)$. Then, the intensity $\lambda_{\Phi_k^d \cap \eta}$ as a function of the directional distribution θ and the test flat η rewrites

$$\big(T_{k,d-k+j}\theta\big)(\eta) \;=\; \lambda \int\limits_{G(k,d)} [\xi,\eta]\, \theta(d\xi) \tag{2.3}$$

where $[\xi,\eta]$ is the $\big(d-j\big)$–volume of the parallelepiped spanned by orthonormal bases in $\xi^\perp$ and $\eta^\perp$ (cf. [3], [9]). The function $\big(T_{k,d-k+j}\theta\big)(\eta)$, $\eta \in G\big(d-k+j,d\big)$, is called the *rose of intersections* of Φ_k^d.

3 Isoperimetric problems

3.1 Connection to convex geometry

Suppose the stationary k–flat process Φ_k^d to be Poisson (cf. [15], [25] for the definition). For integers k and d with $2k \geq d$, introduce the $(2k-d)$–flat process

$$X_2\big(\Phi_k^d\big) \;=\; \big\{\xi_1 \cap \xi_2 : \xi_1, \xi_2 \in \Phi_k^d,\ \dim(\xi_1 \cap \xi_2) = 2k-d\big\}\,,$$

which is generated by all $(2k-d)$–dimensional intersections of pairs of k–flats of the original process Φ_k^d. Clearly, $X_2\big(\Phi_k^d\big)$ is stationary. This process is sometimes called the *intersection process of* Φ_k^d *of order* 2. Its intensity $\lambda_{X_2(\Phi_k^d)}$ is known as the *intersection density of* Φ_k^d *of order* 2 (see [25], p. 253 – 255). One can prove by means of the Campbell–Mecke theorem (cf. [22], Corollary 3.1.6) that

$$\lambda_{X_2(\Phi_k^d)} \;=\; \frac{\lambda^2}{2} \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi_1,\xi_2]\, \theta(d\xi_1)\, \theta(d\xi_2) \tag{3.1}$$

where $[\xi_1,\xi_2]$ is the $2(d-k)$–volume of the unit parallelepiped spanned by the bases in $\xi_1^\perp$, $\xi_2^\perp$ (see the proof in [2] and [22], Theorem 4.1.9 for the case of hyperplanes). We shall use the notation $C(\lambda,\theta) = \lambda_{X_2(\Phi_k^d)}$ to emphasize that the intersection density of Φ_k^d is a functional of λ and θ.

One of the so–called *isoperimetric problems* that could be stated for stationary processes is to maximize the intersection density, i.e., for given λ find the set $\mathbf{L_0}$ of directional distributions θ for which $\lambda_{X_2(\Phi_k^d)}$ attains its maximum $c_{\max}$:

$$C(\lambda,\theta) \;=\; \frac{\lambda^2}{2} \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi_1,\xi_2]\, \theta(d\xi_1)\, \theta(d\xi_2) \;\longrightarrow\; \max\,. \tag{3.2}$$

The exact value of $c_{\max}$ is also of interest:

$$c_{\max} \;=\; \max_{\theta \in \mathbf{L}} C(\lambda,\theta) \tag{3.3}$$

where $\mathbf{L}$ is the space of all probability measures on $G(k,d)$.

Problem (3.2) owes its name “isoperimetric” to the connection between the above setting and classical isoperimetric problems for centrally symmetric convex bodies. Let K be a *zonoid*, that is, a centrally symmetric convex body being the limit of finite sums of line segments in the Hausdorff metric. For $d = 2k$, $k \geq 1$, one has the following relationships between the *mixed volume* $V(\cdot, \ldots, \cdot)$, *mixed functional* $\Phi^{(j)}_{m,d-m+j}(\cdot,\cdot)$ (we preserve here the notation of [4] which, as we hope, will not confuse the reader by its similarity to our notation Φ^d_k for the k–flat process) and the integral representation (3.2) for K, cf. [4]:

$$\binom{d}{k}\nu_d(K) \;=\; \binom{d}{k} V(K,\ldots,K) \;=\; \Phi^{(0)}_{k,k}(K,K) \;=\int\limits_{G(k,d)}\int\limits_{G(k,d)} [\xi_1,\xi_2]\,\rho_k(K,d\xi_1)\,\rho_k(K,d\xi_2)\,,$$

where $\rho_k(K,\cdot)$ is the *projection generating measure* of K, cf. [5]. So if we allow measures θ in (3.2) to be chosen from the smaller class of projection generating measures of zonoids in $\mathbb{R}^d$ for $d = 2k$, then problem (3.2) rewrites (up to a constant factor) as follows: find zonoids K of maximum volume $\nu_d(K)$ provided that the total mass of their projection generating measure is λ, i.e., $\rho_k(K, G(k,d)) = \lambda$.

Denote by θ_K the *generating measure* for a zonoid K. By definition, this is the measure on the sphere such that the support function h_K of K is equal to

$$h_K(u) \;=\; \int\limits_{\mathbf{S}^{d-1}} |\langle u,v\rangle|\,\theta_K(dv)\,.$$

If $k = 1$ then $\rho_k(K,\cdot)$ is obviously equal to θ_K. Recalling the fact that in two–dimensional space the class of zonoids coincides with the class of all centrally symmetric convex bodies, one gets that problem (3.2) in $\mathbb{R}^2$ has the following isoperimetric meaning without any further constraints on probability measure θ: find a centrally symmetric convex body $K \subset \mathbb{R}^2$ with maximum volume $\nu_2(K)$ provided that its generating measure θ_K has total mass λ. It can be easily shown that the perimeter of K $p(K)$ is equal to $4\theta_K(\mathbf{S}^1)$. Thus, the perimeter of K is fixed and equal to 4λ. The zonoid K with generating measure proportional to the rose of directions of a stationary line process Φ^2_1 is called the *Steiner compact* of Φ^2_1 (cf. [9]). Hence, the setting (3.2) is a classical isoperimetric problem for Steiner compacts of Φ^2_1, see also [26] for generalizations of these ideas to hyperplane processes Φ^d_{d-1} in arbitrary dimensions d.

3.2 Some bibliographical remarks

The following is an outline of the results known for the problems (3.2) – (3.3). The involved mathematical tools as well as the solutions themselves depend to a large extent on the dimensions d and k:

(a) $d \geq 2$, $k = d-1$: $\mathbf{L_0} = \{\gamma\}$, $c_{\max} = \dfrac{\lambda^2\Gamma^2\left(\frac{d}{2}\right)}{2\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}$, where γ is the unique probability Haar measure on $G(k,d)$ (invariant with respect to all rotations in $\mathbb{R}^d$ around the origin). The case of a line process in the plane ($d = 2$, $k = 1$) was considered in the pioneering paper of Davidson [1]. Janson and Kallenberg [6] investigated the case $d = 3$ using spherical harmonics, while Thomas [26] employed some methods of convex geometry for arbitrary $d \geq 3$.

(b) $k < d-1$: Mecke and Thomas [16] proved that the Haar measure is not extremal. Further developments can be found in [10], [13], and [20].

(i) $d = 2k$, $k > 2$: Mecke [12] showed that $c_{\max} = \frac{\lambda^2}{4}$ and

$$\mathbf{L_0} \;=\; \left\{\theta = \frac{1}{2}\left(\delta_\xi + \delta_{\xi^\perp}\right) \,:\, \xi \in G(k,d)\right\}$$

where δ_ξ is the Dirac measure concentrated in ξ.

(ii) $d = 4$, $k = 2$: the value $c_{\max}$ is the same as above but the class $\mathbf{L_0}$ is essentially larger than in the previous case (cf. Mecke [11]).

(iii) $d-k \mid d$, i.e., $d-k$ divides d, $k < d-2$: Keutel [7] proved that $c_{\max} = \frac{\lambda^2}{2}\frac{k}{d}$ and the class $\mathbf{L_0}$ consists of all measures of the form

$$\theta \;=\; \frac{d-k}{d}\left(\delta_{\xi_1} + \ldots + \delta_{\xi_{\frac{d}{d-r}}}\right)$$

for $\xi_i \in G(k,d)$, $\xi_i^{\perp} \perp \xi_j^{\perp}$, $i \neq j$.

(iv) $d-k \mid d$, $k = d-2$: $c_{\max}$ remains the same as in the previous case, the class $\mathbf{L_0}$ is larger but not yet completely known (cf. [7]).

(v) $d-k$ does not divide d, $k < d-1$: the problem is still open. In [7] some bounds for $c_{\max}$ are given.

In Section 3.4, the functional $C(\lambda,\theta)$ will be extended to a nonlinear functional on the Banach space $\widetilde{\mathcal{L}}$ of all signed measures with finite total variation on $G(k,d)$. Then, we shall use variational methods to describe the extremal class $\mathbf{L_0}$. Appropriate necessary conditions of extremum will be found. Although they are not sufficient and do not yield the solution, they unify the above variety of results whose form depends heavily on the dimension k. Thus, the common structure of extremal measures becomes clear: for any directional distribution θ from $\mathbf{L_0}$ its rose of intersections with k–flats $T_{kk}\theta$ is θ–almost surely constant.

3.3 Variational calculus on the space of signed measures

In what follows we make use of papers [17] – [19] to state results that will be helpful in obtaining the necessary conditions of extremum in Section 3.4.

Let E be a locally compact Polish space and let $\mathbf{M}$ be the cone of all nonnegative finite measures on E equipped with convergence in total variation. Introduce the Banach space $\widetilde{\mathbf{M}}$ of all signed measures on E with finite total variation norm:

$$\|\mu\| \;=\; \sup_{|\phi(\omega)|\le 1}\left|\int_E \phi(\omega)\,\mu(d\omega)\right| \;<\; \infty\,, \quad \mu\in\widetilde{\mathbf{M}}\,.$$

Let the functionals $F:\widetilde{\mathbf{M}}\to\mathbb{R}$, $H:\widetilde{\mathbf{M}}\to\mathbb{R}$ be continuous and Fréchet differentiable (cf. [8]) on a closed convex subset A of $\mathbf{M}$. We shall tackle the following optimization problem with equality constraints:

$$\begin{cases} F(\mu) \;\longrightarrow\; \inf\,, \\ \mu\in A\,, \\ H(\mu) \;=\; 0\,. \end{cases} \tag{3.4}$$

The equality constraint $H(\mu)=0$ will be considered with particular functionals H whose first Fréchet derivative $H'(\mu)$ is independent of μ and admits the integral representation

$$H'(\mu)[\eta] \;=\; \int_E h\,d\eta\,, \quad \eta\in\widetilde{\mathbf{M}} \tag{3.5}$$

for some measurable function $h: E\to\mathbb{R}$.

Now let us cite the necessary conditions for a minimum in the problem (3.4) with one equality constraint $H(\mu)=0$ (cf. Theorem 3.5 and Remark 3.3 [17]):

Theorem 3.1 *Let the functionals F and H be twice Fréchet differentiable at any measure μ satisfying* (3.4). *Assume that there exists a measurable function $f: E\to\mathbb{R}$ such that*

$$F'(\mu)[\eta] \;=\; \int_E f\,d\eta\,, \quad \eta\in\widetilde{\mathbf{M}}\,. \tag{3.6}$$

Let μ_0 be a local minimum in the optimization problem (3.4). *If there exists a positive number ε such that*

(i) $(1 \pm \varepsilon)\mu_0 \in A$,
(ii) $\mu_0 + t\delta_x \in A$ *for all* $x \in E$ *and* $0 < t \le \varepsilon$,
then there exists a real u *such that* $f(x) \ge u \cdot h(x)$ *for all* $x \in E$ *and* $f(x) = u \cdot h(x)$ μ_0*–a.e.*

Necessary conditions for a maximum can be deduced from Theorem 3.1 if F is replaced by $-F$.

3.4 Necessary conditions of maximum

Express now the isoperimetric problem (3.2) in terms of variational calculus. In this case $E = G(k,d)$ (which is a compact Polish space), $\widetilde{\mathbf{M}} = \widetilde{\mathcal{L}}$, and $A = \mathbf{M} = \mathcal{L}$ where $\mathcal{L}$ is the subset of all nonnegative measures of $\widetilde{\mathcal{L}}$ on $G(k,d)$. According to (3.4), we shall write $F(\theta) = C(1,\theta)$ (the intensity λ is supposed to be fixed, we put $\lambda = 1$ without loss of generality), $H(\theta) = \theta\,(G(k,d)) - 1$. Thus, problem (3.2) rewrites in the following optimization setting:

$$\begin{cases} F(\theta) \;=\; \dfrac{1}{2} \displaystyle\int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi_1,\xi_2]\,\theta(d\xi_1)\,\theta(d\xi_2) \;\longrightarrow\; \max\,, \\ \theta \in \mathcal{L}\,, \\ H(\theta) \;=\; \theta\,(G(k,d)) - 1 \;=\; 0\,. \end{cases} \tag{3.7}$$

The continuous functional F clearly attains its maximal value on the compact subset $\mathcal{L}$ of the space $\widetilde{\mathcal{L}}$.

Now we are ready to prove the following result.

Theorem 3.2 *Let* θ *be a directional distribution on* $G(k,d)$ *that maximizes the intersection density of order* 2 *of the stationary* k*–flat process* Φ_k^d. *Let* $c_{\max}$ *be the maximum considered in* (3.3). *Then, the rose of intersections* $T_{kk}\theta$ *satisfies the following necessary conditions:*
(i) $(T_{kk}\theta)\,(\eta) = \int\limits_{G(k,d)} [\xi,\eta]\,\theta(d\xi) \;=\; 2c_{\max}$ θ*–a.e.;*
(ii) $(T_{kk}\theta)\,(\eta) \le 2c_{\max}$ *for all* $\eta \in G(k,d)$.

P r o o f. First we check the conditions of Theorem 3.1. Due to the fact that $A = \mathcal{L}$, any finite positive measure $\theta \in A$ satisfies assumptions (i) and (ii) of Theorem 3.1. We shall prove now that the functionals F and H are twice Fréchet differentiable at any $\mu \in \mathcal{L}$ and

$$\begin{aligned} F'(\mu)[\nu] &= \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\,\mu(d\xi)\,\nu(d\eta)\,, \qquad (3.8)\\ F''(\mu)[\nu,\nu] &= 2F(\nu)\,,\\ H'(\mu)[\nu] &= \nu\,(G(k,d))\,,\\ H''(\mu)[\nu,\nu] &\equiv 0\,. \end{aligned}$$

Consider the difference

$$F(\mu+\nu) - F(\mu) \;=\; \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\,\mu(d\xi)\,\nu(d\eta) + \frac{1}{2} \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\,\nu(d\xi)\,\nu(d\eta) \tag{3.9}$$

for an arbitrary $\nu \in \widetilde{\mathcal{L}}$. It can be easily seen that the second term in the right–hand side of (3.9) is $o\,(\|\nu\|)$ as $\|\nu\| \to 0$: due to the estimate $[\xi,\eta] \le 1$ one can prove that

$$\left| \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\,\nu(d\xi)\,\nu(d\eta) \right| \;\le\; \|\nu\|^2\,.$$

The first term in the right–hand side of (3.9) is a linear functional in ν, it is also bounded: its operator norm is not greater than $|\mu\,(G(k,d))| < \infty$. Then it is continuous, and the first Fréchet derivative of F exists and is equal to (3.8).

Analogously to the considerations above, we have

$$F'(\mu+\nu)[\tau] - F'(\mu)[\tau] \;=\; \int\limits_{G(k,d)} \int\limits_{G(k,d)} [\xi,\eta]\,\nu(d\xi)\,\tau(d\eta) \;=\; F''(\mu)[\nu,\tau]$$

for all $\tau \in \widetilde{\mathcal{L}}$. This bilinear form does not depend on μ.

Then we find the derivative of H: it is

$$H(\mu+\nu) - H(\mu) \;=\; \nu\,(G(k,d)) \;=\; H'(\mu)[\nu]\,.$$

The difference does not depend on μ, which yields $H''(\mu)[\cdot,\cdot] = 0$.

Furthermore, $F'(\mu)$ has the representation (3.6) with $f(\eta) = (T_{kk}\mu)\,(\eta) = \int\limits_{G(k,d)} [\xi,\eta]\,\mu(d\xi)$ (see (3.8)). The functional H satisfies (3.5) with $h(\cdot) \equiv 1$. Take a probability measure θ on $G(k,d)$ to be a local maximum of (3.7). By Theorem 3.1, there exists a constant u such that

$$(T_{kk}\theta)(\eta) \;=\; \int\limits_{G(k,d)} [\xi,\eta]\,\theta(d\xi) \;=\; u \quad \theta\text{–a.e.}\,,$$

$$(T_{kk}\theta)(\eta) \;\le\; u\,, \quad \eta \in G(k,d)\,,$$

and since $c_{\max} = \frac{1}{2} \int\limits_{G(k,d)} (T_{kk}\theta)(\eta)\,\theta(d\eta)$ we conclude that $u = 2c_{\max}$. □

The necessary conditions for a maximum are satisfied by all known extremal measures for problem (3.2). Let us illustrate the above theorem by showing this for some interesting particular cases.

Examples 3.3 1) Hyperplane case $k = d-1$: it can be easily shown that the integral $\int\limits_{G(k,d)} [\xi,\eta]\,\gamma(d\xi)$ is constant for all η due to the rotation invariance of γ and $[\xi,\eta]$.

2) Suppose that the directional distribution θ is discrete:

$$\theta \;=\; p_1\delta_{\xi_1} + \ldots + p_n\delta_{\xi_n}\,,$$

$p_1 + \ldots + p_n = 1$, for some $\{\xi_i\}_{i=1}^n \subset G(k,d)$. Due to condition (i) of Theorem 3.2, one gets

$$\sum_{j=1}^n p_j\,[\xi_i,\xi_j] \;=\; 2c_{\max}$$

for each i. In case $d-k \mid d$ we have $\xi_i^\perp \perp \xi_j^\perp$, $p_i = 1/n$ with $n = d/(d-k)$, which yields $c_{\max} = k/(2d)$.

Now consider problem (3.2) of maximizing the intersection density of Φ_k^d in the class $\mathbf{M}_\gamma$ of all directional distributions that are absolutely continuous with respect to the uniform distribution γ. In terms of the optimization setting (3.4), the set A is equal to $\mathbf{M}_\gamma$. For $\mu \in \mathbf{M}_\gamma$ denote by $\frac{d\mu}{d\gamma}$ the Radon–Nikodym density of μ with respect to γ.

Proposition 3.4 *Let θ be a directional distribution on $G(k,d)$ that maximizes the intersection density of order 2 of the stationary k–flat process Φ_k^d. Let $c_{\max}$ be the corresponding maximum value. If θ is absolutely continuous with respect to the Haar measure γ on $G(k,d)$ and $\frac{d\theta}{d\gamma} = g$, then the following necessary conditions hold for the rose of intersections $T_{kk}\theta$:*

(i) $(T_{kk}\theta)(\eta) = \int\limits_{G(k,d)} [\xi,\eta]g(\xi)\,\gamma(d\xi) = 2c_{\max}$ *γ–a.e.;*

(ii) $\inf\limits_{E:\,\gamma(E)=0} \; \sup\limits_{G(k,d)\setminus E} (T_{kk}\theta)(\eta) \le 2c_{\max}$.

P r o o f. Here we have an optimization problem of type (3.7) with $A = \mathbf{M}_\gamma$. The proof is provided analogously to that of Theorem 3.2 using Theorem 4.1 of [17]. □

Remark Although the above necessary conditions for a maximum are obviously not sufficient and, thus, do not lead directly to the solution of problem (3.2), they unify the different approaches described in Section 3.2. We also hope that they might simplify the better understanding of the nature of extremal measures. Thus, we conjecture that in the open cases all (or at least some) extremal directional distributions are discrete with atoms in some regular directions in $\mathbb{R}^d$. Once these directions have been determined, the corresponding weights of the distribution can be found from the linear system of equations resulting from the necessary condition 3.2 of Theorem 3.2.

4 Roses of neighborhood

4.1 Representation by roses of intersections of dual processes

In Section 3.4, necessary conditions for a maximum of the intersection density of Φ_k^d are stated in terms of the rose of intersections $T_{kk}\theta$. Therefore, the problem of retrieving the directional distribution θ of Φ_k^d from $T_{kk}\theta$ arises naturally from this setting. In [23], a complete answer was given and appropriate formulae were proved for the case when Φ_k^d intersects with an r–flat η, $k+r \geq d$, and the correspondence $f \leftrightarrow \theta$ turned out to be one–to–one.

Now suppose that the test flat η is lower dimensional, that is, $k+r<d$. Then, for all r–flats η, the intersection process $\Phi_k^d \cap \eta$ is empty with probability 1. Hence, the difficulties in defining the rose of intersections $T_{kr}\theta(\eta)$ are obvious. In practice, it might be the case if the material or structure under observation cannot be cut by a flat, but indirect measurements along a test line can be performed instead, for instance, by means of the radar emission, X–rays or laser beams.

In order to tackle this problem, we shall introduce the *process of neighborhood* $\Phi_k^d \odot \eta$ by considering intersections of Φ_k^d with the cylinder of radius $a>0$ and axis at η. A similar approach was developed in [6] and [21]. Janson and Kallenberg [6] considered the process of thick cylindric fibers in $\mathbb{R}^d$ and its intersection density of order 2. In [21], Schneider introduced the notion of the *proximity* of Φ_k^d with $k<d/2$ that generalizes the usual intersection density of order 2 for $k \geq d/2$.

Let φ be a realization of a stationary Φ_k^d and $\eta \in F(r,d)$, $k+r<d$. For almost all η, there exists a unique point $x_\xi \in \eta$ given by

$$\operatorname{dist}(\xi,\eta) \;=\; \inf_{y\in\xi,\; x\in\eta} \rho(x,y) \;=\; \inf_{y\in\xi} \rho(x_\xi,y)$$

for any $\xi \in \varphi$, where $\rho(x,y)$ is the Euclidean distance in $\mathbb{R}^d$. Clearly, the collection of all points

$$\left\{x_\xi \in \eta : \operatorname{dist}(\xi,\eta) < a,\ \xi \in \Phi_k^d\right\}$$

for some $a>0$ forms a stationary point process $\Phi_k^d \odot \eta$ in η for almost all η, which will be called the *a–process of neighborhood* (we suppress a in the notation). Its intensity $N_{kr}(a,\eta)$ will be called the *a–rose of neighborhood* of Φ_k^d. According to Corollary 4.3 below, all possible choices of radius a are essentially equivalent. For the sake of convenience, we sometimes choose $a=1$ and then write $N_{kr}(\eta)$ instead of $N_{kr}(1,\eta)$ calling $N_{kr}(\cdot)$ the *rose of neighborhood* of Φ_k^d. Due to the stationarity of Φ_k^d, it suffices to consider only those flats $\eta \in F(r,d)$ that contain the origin, i.e. $\eta \in G(r,d)$.

For any stationary k–flat process Φ_k^d with intensity λ and directional distribution θ, introduce the family of *dual processes* $D(\lambda,\theta)$: a stationary $(d-k)$–flat process Φ_{d-k}^d belongs to $D(\lambda,\theta)$ iff its intensity is equal to λ and its directional distribution is $\theta^\perp(d\zeta)=\theta\big(d\zeta^\perp\big)$ for $\zeta \in G(d-k,d)$. The following result connects the rose of neighborhood with the rose of intersections of a dual process.

Theorem 4.1 *For $k+r<d$, the following relationship holds for the a–rose of neighborhood of the stationary k–flat process Φ_k^d:*

$$N_{kr}(a,\eta) \;=\; \kappa_{d-k-r}a^{d-k-r}\left(T_{d-k,d-r}\theta^\perp\right)\left(\eta^\perp\right), \quad \eta \in G(r,d) \tag{4.1}$$

where $T_{d-k,d-r}\theta^\perp$ is the rose of intersections of the dual process $\Phi_{d-k}^d \in D(\lambda,\theta)$ with $(d-r)$–flats. By (2.3), $T_{d-k,d-r}\theta^\perp$ is the same for all processes $\Phi_{d-k}^d \in D(\lambda,\theta)$.

Proof. For any nonparallel $(d-k)$–flat ζ and $(d-r)$–flat β, their intersection is not empty since $d-k+d-r>d$. Therefore, the usual rose of intersections of $\Phi^d_{d-k}\in D(\lambda,\theta)$ with $(d-r)$–flats is well–defined. The intensity of $\Phi^d_k\odot\eta$ is given by

$$N_{kr}(a,\eta) \;=\; \frac{1}{\kappa_r}E\left(\sum_{\zeta\in\Phi^d_k} I_a(\zeta)\right) \tag{4.2}$$

where $I_a(\zeta)=I_{\{\zeta:\,\mathrm{dist}(\zeta,\eta)<a,\,x_\zeta\in B_1(0)\subset\eta\}}(\zeta)$, $B_m(0)$ is the ball in the appropriate ambient subspace of $\mathbb{R}^d$ with radius m and the center in the origin. Determining the expectation in (4.2) by means of the Campbell–Mecke Theorem and using (2.2) for the intensity measure $\Lambda(\cdot)$ of Φ^d_k, one gets

$$\begin{aligned} N_{kr}(a,\eta) &= \frac{1}{\kappa_r}\int\limits_{F(k,d)} I_a(\zeta)\,\Lambda(d\zeta)\\ &= \frac{\lambda}{\kappa_r}\int\limits_{G(k,d)}\int\limits_{\xi^\perp} I_a(y+\xi)\,\nu^{\xi^\perp}_{d-k}(dy)\,\theta(d\xi)\\ &= \frac{\lambda}{\kappa_r}\int\limits_{G(d-k,d)}\int\limits_{\xi^\perp} I_a(y+\xi)\,\nu^{\xi^\perp}_{d-k}(dy)\,\theta^\perp\big(d\xi^\perp\big)\,. \end{aligned}$$

Now prove that

$$\int\limits_{\xi^\perp} I_a(y+\xi)\,\nu^{\xi^\perp}_{d-k}(dy) \;=\; \kappa_{d-k-r}\kappa_r a^{d-k-r}\big[\xi^\perp,\eta^\perp\big]$$

for any $\xi\in G(k,d)$ that is not parallel to η. Using a reasoning similar to that of [21], formulae (7), (8) we get

$$\int\limits_{\xi^\perp} I_a(y+\xi)\,\nu^{\xi^\perp}_{d-k}(dy) \;=\; \big[\xi^\perp,\eta^\perp\big]\int\limits_{H} I_a(z+\xi)\,\nu^{H}_{d-k}(dz)$$

where $H=(\xi+\eta)^\perp+\eta$ in the sense of Minkowski addition. We show that the integral

$$J_a(H,\xi) \;=\; \int\limits_H I_a(z+\xi)\,\nu^H_{d-k}(dz)$$

is equal to $\kappa_{d-k-r}\kappa_r a^{d-k-r}$: as $H=\eta\oplus(\eta)^\perp_H$, a direct orthogonal sum, where $(\eta)^\perp_H$ stands for the orthogonal complement of η in H, we have $z=x_{y+\xi}+l$ for any $z\in H$, where $x_{y+\xi}\in\eta$, $l\in(\eta)^\perp_H$. Then, the indicator $I_a(z+\xi)$ rewrites

$$I_a(z+\xi) \;=\; \begin{cases} 1, & x_{y+\xi}\in B_1(0)\subset\eta,\;\; l\in B_a(0)\subset(\eta)^\perp_H\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$$

This means that the integral $J_a(H,\xi)$ is equal to the desired expression. Hence, we have

$$\begin{aligned} N_{kr}(a,\eta) &= \frac{\lambda}{\kappa_r}\,\kappa_r\kappa_{d-k-r}a^{d-k-r}\int\limits_{G(d-k,d)}\big[\xi^\perp,\eta^\perp\big]\,\theta^\perp\big(d\xi^\perp\big)\\ &= \kappa_{d-k-r}a^{d-k-r}\big(T_{d-k,d-r}\theta^\perp\big)\big(\eta^\perp\big)\,. \end{aligned}$$

□

Corollary 4.2 *For $a=1$, formula (4.1) for the rose of neighborhood simplifies to* $N_{kr}(\eta)=\kappa_{d-k-r}\big(T_{d-k,d-r}\theta^\perp\big)\big(\eta^\perp\big)$.

The following corollary shows that test cylinders of any radius a_1 can be used for the computation of $N_{kr}(a_2,\cdot)$ since a_2/a_1 is just a scaling factor. Hence, there is no loss of information if we use $N_{kr}(\cdot)$ instead of $N_{kr}(a,\cdot)$.

Corollary 4.3 *For any positive radii a_1 and a_2,*

$$N_{kr}(a_2,\eta) \;=\; \left(\frac{a_2}{a_1}\right)^{d-k-r} N_{kr}(a_1,\eta)\,, \quad \eta \in G(r,d)\,.$$

In particular, $N_{kr}(a,\cdot) = a^{d-k-r} N_{kr}(\cdot)$ for any positive a.

Theorem 4.1 shows that the problem of restoring the directional distribution θ of a stationary process Φ_k^d from its rose of neighborhood $N_{kr}(\eta)$ constructed for r–flats η, $k+r<d$, can be reduced to the dual problem for any $\Phi_{d-k}^d \in D(\lambda,\theta)$ intersected with $(d-r)$–flats. A partial solution of this problem was given in [23]. In particular, the directional distribution θ of a line process Φ_1^d can be retrieved in this way from its rose of neighborhood with r–flats, $1 \le r < d-1$. From the stereological and statistical point of view, this means that in order to estimate θ, we can use lines as test objects instead of flats.

Example 4.4 [$d=3$, $k=r=1$.] Consider a stationary line process Φ_1^3 with intensity λ and directional distribution θ. Let $N_{11}(v)$ be its rose of neighborhood with lines $\big(v \in \mathbf{S}^2$ is the direction vector of a test line$\big)$. By Theorem 4.1, we can write

$$N_{11}(v) \;=\; 2\left(T_{22}\theta^{\perp}\right)\left(v^{\perp}\right) \;=\; 2\lambda \int\limits_{\mathbf{S}^2} \sqrt{1-\langle u,v\rangle^2}\,\theta(du)\,.$$

Applying Theorem 7.1 of [23], we have

$$\int\limits_{\mathbf{S}^2} g(v)\,\theta(dv) \;=\; \frac{1}{16\pi^4\lambda} \times \int\limits_{\mathbf{S}^2} W(u)\,(\Delta_0+2)\left(\frac{d}{d(\mu^2)}\right)\left[\;\int\limits_{\langle u,v\rangle^2>\mu^2} \frac{g(v)\,|\,\langle u,v\rangle\,|\,\omega_3(dv)}{\sqrt{\langle u,v\rangle^2-\mu^2}}\right]\Bigg|_{\mu=0} \omega_3(du)$$

for any even four times continuously differentiable function $g \in C^4\left(\mathbf{S}^2\right)$, where

$$W(u) \;=\; \left(\frac{d}{d(\mu^2)}\right)\left[\;\int\limits_{\langle u,t\rangle^2>\mu^2} \frac{N_{11}(t)|\,\langle u,t\rangle\,|\,\omega_3(dt)}{\sqrt{\langle u,t\rangle^2-\mu^2}}\right]\Bigg|_{\mu=0}\,,$$

ω_3 denotes the surface area measure on the unit sphere in $\mathbb{R}^3$, and Δ_0 stands for the Beltrami–Laplace operator.

4.2 An estimator for the rose of neighborhood

In this section, an estimator for the a–rose of neighborhood of a line process Φ_1^3 in the three–dimensional space will be given. The choice of dimension $d=3$ is due to its importance for applications (see also the remark at the end of this section). In practice, it is impossible to count the intersections of a line process with an infinite test cylinder. Thus, we propose an estimator for $N_{11}(a,\eta)$ by supposing the cylinder with the axis at η to have finite length $2b$. This estimator appears to be asymptotically unbiased as the length $2b$ tends to infinity or the radius of the cylinder a becomes arbitrarily small. Due to stationarity of Φ_1^3, it suffices to consider only test lines going through the origin.

Count all intersections of Φ_1^3 with the cylinder $B_b^{\eta}(0)\times B_a^{\eta^{\perp}}(0)$ where $B_c^{\zeta}(0)$ denotes the c–neighborhood of the origin in a linear subspace ζ. Then, the estimator $\widehat{N}_{11}(a,\eta)$ for the a–rose of neighborhood is given by

$$\widehat{N}_{11}(a,\eta) \;=\; \frac{\#\left\{\xi\in\Phi_1^3:\xi\cap\left(B_b^\eta(0)\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\right\}}{2b} \tag{4.3}$$

(we suppress b in the notation). The following theorem yields the asymptotical unbiasedness of $\widehat{N}_{11}(a,\eta)$ and gives the corresponding rate of convergence.

Theorem 4.5 *For a stationary line process Φ_1^3 with directional distribution θ and a–rose of neighborhood $N_{11}(a,\eta)$ with lines in $\mathbb{R}^3$,*

$$0 \;\le\; E\,\widehat{N}_{11}(a,\eta)-N_{11}(a,\eta) \;=\; \frac{\pi a^2}{2b}\,(T_{12}\theta)(\eta^\perp) \;\le\; \frac{\lambda\pi a^2}{2b}\,,\qquad \eta\in G(1,3)\,. \tag{4.4}$$

P r o o f. By the Campbell–Mecke Theorem, we can write

$$\begin{aligned} &E\,\#\left\{\zeta\in\Phi_1^3:\zeta\cap\left(B_b^\eta(0)\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\right\}\\ &\quad= E\sum_{\zeta\in\Phi_1^3} I_{\left\{\zeta\cap\left(B_b^\eta\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\right\}}(\zeta)\\ &\quad= \lambda\int\limits_{G(1,3)}\int\limits_{\xi^\perp} I_{\left\{(y+\xi)\cap\left(B_b^\eta\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\right\}}(y,\xi)\,\nu_2^{\xi^\perp}(dy)\,\theta(d\xi)\,. \end{aligned} \tag{4.5}$$

With the substitution $y=Pr_{\xi^\perp}(x)$, $x\in H$ (cf. the proof of Theorem 4.1) where $Pr_\varphi(\cdot)$ stands for the orthogonal projection operator onto the linear subspace φ, the inner integral in (4.5) rewrites

$$[\xi^\perp,\eta^\perp]\int\limits_H I_{\left\{\left(Pr_{\xi^\perp}x+\xi\right)\cap\left(B_b^\eta\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\right\}}(y,\xi)\,\nu_2^H(dx)\,. \tag{4.6}$$

We compute this integral for lines η such that $\left[\xi^\perp,\eta^\perp\right]>0$. Since $x-Pr_{\xi^\perp}(x)$ is parallel to ξ, we can write $Pr_{\xi^\perp}(x)+\xi=x+\xi$, and the integral in (4.6) takes the form

$$\int\limits_H I_{\left\{(x+\xi)\cap\left(B_b^\eta\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\right\}}(y,\xi)\,\nu_2^H(dx)\,. \tag{4.7}$$

Due to the obvious relationship

$$I_{\left\{(x+\xi)\cap\left(B_b^\eta\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\right\}} \;=\; I_{\left\{x\in B_b^\eta\times B_a^{\eta^\perp}(0)\right\}}+I_{\left\{x\notin B_b^\eta\times B_a^{\eta^\perp}(0),\;(x+\xi)\cap\left(B_b^\eta\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\right\}}\,,$$

the integral in (4.7) is equal to

$$\begin{aligned} &\nu_2^H\Big(H\cap\left(B_b^\eta(0)\times B_a^{\eta^\perp}(0)\right)\Big)\\ &\qquad+\;\nu_2^H\Big(x\in H:x\notin B_b^\eta(0)\times B_a^{\eta^\perp}(0),\;(x+\xi)\cap\left(B_b^\eta\times B_a^{\eta^\perp}(0)\right)\neq\emptyset\Big)\\ &\quad= 4ab+\nu_2^H\left(Pr_H^\xi(B_a(o))\right)\\ &\quad= 4ab+4\cot\alpha\int\limits_0^a\sqrt{a^2-t^2}\,dt \end{aligned}$$

where α is the angle between the lines ξ and η, and $Pr_H^{\xi}(\cdot)$ denotes the projection onto H in direction ξ. Then, (4.5) rewrites

$$\begin{aligned}
&E\,\#\left\{\zeta \in \Phi_1^3 : \zeta \cap \left(B_b^{\eta}(0) \times B_a^{\eta^\perp}(0)\right) \neq \emptyset\right\} \\
&= 4ab\lambda \int\limits_{G(1,3)} \left[\xi^\perp, \eta^\perp\right] \theta(d\xi) + \lambda\pi a^2 \int\limits_{G(1,3)} \left[\xi, \eta^\perp\right] \theta(d\xi) \\
&= 4ab\left(T_{22}\theta^\perp\right)\left(\eta^\perp\right) + \pi a^2 \left(T_{12}\theta\right)\left(\eta^\perp\right) \\
&= 2bN_{11}(a,\eta) + \pi a^2 \left(T_{12}\theta\right)\left(\eta^\perp\right).
\end{aligned}$$

Since θ is a probability measure, the upper bound in (4.4) is obvious. □

The following result is an immediate consequence of Theorem 4.5.

Corollary 4.6 *For each $a > 0$, the estimator $\widehat{N}_{11}(a,\eta)$ given in* (4.3) *is asymptotically unbiased as $b \to \infty$:*

$$\widehat{N}_{11}(a,\eta) - N_{11}(a,\eta) = O(1/b).$$

Moreover, $\widehat{N}_{11}(a,\eta)/a$ is an asymptotically unbiased estimator for $N_{11}(\eta)$ as $a \to 0$ or $b \to \infty$, where

$$E\,\frac{\widehat{N}_{11}(a,\eta)}{a} - N_{11}(\eta) = O(a/b).$$

Remark 4.7 It is clear how the estimator (4.3) can be generalized in order to estimate the a–rose of neighborhood $N_{kr}(a,\eta)$ for the process Φ_k^d of k–dimensional flats in the higher dimensional space $\mathbb{R}^d$ with $d > 3$, $k + r < d$:

$$\widehat{N}_{kr}(a,\eta) = \frac{\#\left\{\xi \in \Phi_k^d : \xi \cap \left(B_b^{\eta}(0) \times B_a^{\eta^\perp}(0)\right) \neq \emptyset\right\}}{\kappa_r b^r}.$$

However, in this section, we considered just the three–dimensional case in order to avoid unnecessary technical difficulties in the proof of Theorem 4.5.

4.3 Estimating the directional distribution density

Identifying any line through the origin with the pair of its unit direction vectors, each function (measure) on $G(1,3)$ can be thought of as an even function (measure) on the sphere $\mathbf{S}^2$. Suppose the directional distribution θ has a density ψ with respect to the uniform directional distribution γ on $\mathbf{S}^2$, $\gamma(\cdot) = \omega_3(\cdot)/\omega_3(\mathbf{S}^2)$. Suppose ψ to be twice continuously differentiable on $\mathbf{S}^2$: $\psi \in C_e^2(\mathbf{S}^2)$ (subindex e means "even"). By the example considered in Section 4.1, the rose of neighborhood $N_{11}(u)$, $u \in \mathbf{S}^2$ is proportional to the rose of intersections $\left(T_{22}\psi^\perp\right)\left(u^\perp\right)$. By Proposition 8.1 of [23], the density ψ can be restored from $T_{22}\psi^\perp$ using its expansion in spherical harmonics (cf. [14]). Combining both assertions with that of Corollary 4.3, one can write

$$\psi(u) = \psi^\perp\left(u^\perp\right) = \sum_{k=0}^{\infty} \sum_{j=1}^{4k+1} \frac{\int_{\mathbf{S}^2} N_{11}(a,v) S_{2k,j}(v)\, \omega_3(dv)}{ac_k} S_{2k,j}(u), \quad u \in \mathbf{S}^2 \tag{4.8}$$

where

$$c_k = -\pi \frac{\Gamma(k+1/2)\,\Gamma(k-1/2)}{(k+1)!\,k!}, \quad k \in \mathbb{N}$$

and S_{nj} is a spherical harmonic of order n. Note that in accordance with Corollary 4.3, both sides of equation (4.8) do not depend on a. For instance, we can set $a = 1$.

To estimate ψ, take a finite sum in (4.8) and substitute the value $N_{11}(a, v)$ by $\widehat{N}_{11}(a, v)$:

$$\widehat{\psi}_{N,b}(u) = \sum_{k=0}^{N} \sum_{j=1}^{4k+1} \frac{\int\limits_{\mathbf{S}^2} \widehat{N}_{11}(a, v) S_{2k,j}(v)\, \omega_3(dv)}{ac_k} S_{2k,j}(u)\,, \quad u \in \mathbf{S}^2. \tag{4.9}$$

To evaluate the integral $\int\limits_{\mathbf{S}^2} \widehat{N}_{11}(a, v) S_{2k,j}(v)\, \omega_3(dv)$ in (4.9) numerically, values of the estimator $\widehat{N}_{11}(a, v)$ in a finite number of directions v are required. These directions can be chosen at random, in order to use Monte–Carlo methods of numerical integration.

Theorem 4.8 *Let the density ψ of the directional distribution of the stationary line process Φ_1^3 be twice continuously differentiable on $\mathbf{S}^2$. Then the estimator $\widehat{\psi}_{N,b}(u)$ given in* (4.9) *is asymptotically unbiased as $b \to \infty$, $N \to \infty$ (or $a \to 0$, $N \to \infty$): for all $u \in \mathbf{S}^2$*

$$\lim_{N\to\infty} \lim_{b\to\infty} E\,\widehat{\psi}_{N,b}(u) = \lim_{N\to\infty} \lim_{a\to 0} E\,\widehat{\psi}_{N,b}(u) = \psi(u)\,.$$

Proof. By Fubini's Theorem, the mean value of $\widehat{\psi}_{N,b}(u)$ is given by

$$E\,\widehat{\psi}_{N,b}(u) = \sum_{k=0}^{N} \sum_{j=1}^{4k+1} \frac{\int\limits_{\mathbf{S}^2} E\,\widehat{N}_{11}(a, v) S_{2k,j}(v)\, \omega_3(dv)}{ac_k} S_{2k,j}(u)\,.$$

By Theorem 4.5, one can write

$$\begin{aligned} E\,\widehat{\psi}_{N,b}(u) &= \sum_{k=0}^{N} \sum_{j=1}^{4k+1} \frac{\int\limits_{\mathbf{S}^2} N_{11}(a, v) S_{2k,j}(v)\, \omega_3(dv)}{ac_k} S_{2k,j}(u) \\ &\quad + \frac{\pi a}{2b} \sum_{k=0}^{N} \sum_{j=1}^{4k+1} \frac{\int\limits_{\mathbf{S}^2} (T_{12}\psi)(v^{\perp}) S_{2k,j}(v)\, \omega_3(dv)}{c_k} S_{2k,j}(u)\,. \end{aligned} \tag{4.10}$$

Since the first sum does not depend on a and b, for fixed N the limit of $E\,\widehat{\psi}_{N,b}(u)$ as $b \to \infty$ or $a \to 0$ is equal to

$$\sum_{k=0}^{N} \sum_{j=1}^{4k+1} \frac{\int\limits_{\mathbf{S}^2} N_{11}(v) S_{2k,j}(v)\, \omega_3(dv)}{c_k} S_{2k,j}(u)\,.$$

Taking the limit as $N \to \infty$ completes the proof. □

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