Asymptotics of the mean Minkowski functionals of Gaussian excursions

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1 Introduction

Let $X = \{X(t), t \in M\}$ be a real–valued Gaussian random field with a.s. smooth paths. The set M is a compact in $\mathbb{R}^d$. The *Euler-Poincaré heuristic* states that

$$\left| \mathbb{P}\left(\sup_{t\in M} X(t) > u \right) - \mathbb{E}\ V_0\left(A_u(X;M)\right) \right| \leqslant c_0 \exp\{-u^2(1+\alpha)/2\}, \qquad u \to \infty \tag{1}$$

for some positive constants c_0 and α, where $V_0\left(A_u(X;M)\right)$ is the Euler-Poincaré characteristic of the excursion set of X over the level u which is a topological invariant of the excursion set. The main problem is to identify the class of Gaussian fields that satisfy this relation. It is known that Gaussian fields with constant variance (cf. [6] and [1, Theorem 14.3.3]) on stratified manifolds M belong to that class. In [2, Theorem 8.4], a more general bound than (1) is proven for stationary Gaussian fields and all levels u. For non–stationary Gaussian X with a unique point of maximum variance attained in the interior of the set M and $u \to \infty$ see [2, Theorem 8.10].

2 Preliminaries

Consider a parallelepiped

$$F = \prod_{j=1}^{d} [0, a_j]$$

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in $\mathbb{R}^d$, $d \geqslant 1$, where $a_1, \dots, a_d$ are some positive real numbers. L et $X = \{X(t), \, t \in F\}$ be a centered Gaussian random field with variance $\sigma^2(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For $X \in \mathcal{C}^2(F)$ a.s., let X'_i, σ'_i denote partial derivatives of X, σ with respect to ith variable. We also use a notation $X''_{ij} = \partial^2 X / \partial t_i \partial t_j$ and denote by X'' the Hessian matrix of X:

$$X'' = \left(X''_{ij}\right)_{1 \leqslant i,j \leqslant d}.$$

Put $\nabla X = (X'_1, \dots, X'_d)^\top$ and $Z = (X, \nabla X)^\top$. Evidently, $\{Z(t), \, t \in F\}$ is a centered Gaussian $(d+1)$-vector field. Let $\Sigma(t)$ denote its covariation matrix.

Denote by Φ the cumulative distribution function of the standard normal distribution and put $\bar{\Phi} = 1 - \Phi$. For an absolutely continuously distributed ξ let p_ξ denote its distribution density.

By $A_u = A_u(X, F)$ denote the excursion set of X in F over the level u:

$$A_u = \{t \in F \, : \, X(t) \geqslant u\}.$$

Let V_0, V_{d-1} and V_d stands for the Euler characteristic, the boundary area and the volume of a set. In this paper we would like to find the asymptotic of $\mathbb{E}\, V_0(A_u), \mathbb{E}\, V_{d-1}(A_u)$ and $\mathbb{E}\, V_d(A_u)$ as $u \to +\infty$.

3 Main results

Theorem 3.1. *Suppose that*

(a) $X \in \mathcal{C}^1(F)$ *a.s.,*

(b) σ *has a unique global maximum at the origin and* $\sigma'_i(0) < 0$ *for* $i = 1, \dots, d$.

Then

$$\mathbb{P}\left(\sup_{t \in F} X(t) > u\right) = \bar{\Phi}\left(\frac{u}{\sigma(0)}\right) \cdot [1 + o(1)], \quad u \to +\infty.$$

Decompose F into the union of open sets J of dimensions from 0 to d which are the *k–faces* of F of all dimensions $k = 0, \dots, d$. So, the vertices of F are its 0–faces, the edges of F are its 1–faces, etc. The interior of F is its unique d–face. Let $X_J = \{X(t), \, t \in J\}$ be the restriction of X onto the face J. Let $\nabla_J X$ denote the vector of partial derivatives of $X(t)$ with respect to all coordinates t_j, $j \neq d$ of $t \in J$ that vary in J. Let X''_J be the matrix with elements X''_{ij} for all coordinates t_i, t_j that vary in J, $i, j \neq d$.

The random field X is called *s*uitably regular if the following conditions hold a.s. for any level $u > 0$ and any face J of F where the coordinate t_d of t is not constant:

- $X \in \mathcal{C}^2(\tilde{F})$ in an open neighborhood $\tilde{F}$ of F,
- X_J has no critical points $t \in J$ such that $X(t) = u$,
- $\mathbb{P}\left(\exists\, t \in J \,:\, X(t) = u, \nabla_J X(t) = 0, \det X''_J(t) = 0\right) = 0.$

By [1, Theorem 6.2.2], the excursion sets A_u of a suitably regular random field X are basic complexes, so $V_0(A_u)$ is well–defined. Sufficient conditions on the covariance function of a Gaussian random field X to be suitably regular are given in [1, Theorem 11.3.3].

Theorem 3.2. *Suppose that*

(a) X *is suitably regular,*

(b) Z(t) has a nondegenerate distribution for all $t \in F$ *(i.e.,* $\det \Sigma(t) \neq 0$*),*

(c) $\mathbb{P}\left(\exists\, u > 0, t \in F \,:\, X(t) = u, \nabla X(t) = 0, \det X''(t) = 0\right) = 0,$

(d) σ *has a unique global maximum at the origin and* $\sigma'_i(0) < 0$ *for* $i = 1, \ldots, d$.

Then

$$\mathbb{E}\, V_0(A_u) = \bar{\Phi}\left(\frac{u}{\sigma(0)}\right) \cdot [1 + O(u^{-1})], \quad u \to +\infty. \tag{2}$$

Theorem 3.3. *Suppose that*

(a) $\sigma \in \mathcal{C}(F)$*,*

(b) $\sigma \in \mathcal{C}^2$ *in some neighborhood of the origin,*

(c) σ *has a unique global maximum at the origin and* $\sigma'_i(0) < 0$ *for* $i = 1, \ldots, d$.

Then

$$\mathbb{E}\, V_d(A_u) = \frac{C}{u^{2d}} \bar{\Phi}\left(\frac{u}{\sigma(0)}\right) \cdot [1 + o(1)], \quad u \to +\infty, \tag{3}$$

where

$$C = \frac{(-1)^d \sigma^{3d}(0)}{\prod_{j=1}^d \sigma'_j(0)}.$$

Theorem 3.4. *Suppose that*

(a) $X \in \mathcal{C}^1(F)$ *a.s.,*

(b) $\sigma \in \mathcal{C}^2$ *in some neighborhood of the origin,*

(c) σ *has a unique global maximum at the origin and* $\sigma_i'(0) < 0$ *for* $i = 1, \ldots, d$,

(d) $\sigma(t) > 0$ *for all* $t \in F$.

Then

$$\mathbb{E}\, V_{d-1}(A_u) = \frac{C}{u^{2d-1}} \bar{\Phi}\left(\frac{u}{\sigma(0)}\right) \cdot [1 + o(1)], \quad u \to +\infty, \tag{4}$$

where

$$C = \frac{(-1)^d}{2} \mathbb{E} \left\| \nabla \frac{X}{\sigma}(0) \right\| \frac{\sigma^{3d-1}(0)}{\prod_{j=1}^d \sigma_j'(0)}$$

and $\|\cdot\|$ *is the Euclidian norm in* $\mathbb{R}^d$.

4 Proofs

4.1 Proof of Theorem 3.1

To proof Theorem 3.1, we use essentially the follows result of Talagrand [5]:

Lemma 4.1 (Talagrand)**.** *For some compact metric space* T *let* $Y = \{Y(t), t \in T\}$ *be a centered separable a.s. bounded Gaussian process with continuous variance* $\sigma^2(t)$. *Put* $\sigma_0 = \sup_{t \in T} \sigma(t)$. *Then*

$$\mathbb{P}\,(\sup_{t \in T} Y(t) > u) = \bar{\Phi}\left(\frac{u}{\sigma_0}\right) \cdot [1 + o(1)], \quad u \to +\infty,$$

if and only if there exists a unique $t_0 \in T$ *such that* $\sigma(t_0) = \sigma$ *and*

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E} \sup_{t \in T_h} [Y(t) - Y(t_0)] = 0,$$

where

$$T_h = \{t \in T \,:\, \mathbb{E}\, Y(t) Y(t_0) \geqslant \sigma_0^2 - h^2\}.$$

Now let us proceed to the proof of Theorem 3.1. As in Lemma 4.1, consider the set

$$F_h = \{t \in F \,:\, \mathbb{E}\, X(t)X(0) \geqslant \sigma^2(0) - h^2\}.$$

Since

$$2\mathbb{E}\, X(t)X(0) \leqslant \sigma^2(t) + \sigma^2(0),$$

we get $F_h \subset \tilde{F}_h$, where

$$\tilde{F}_h = \{t \in F \,:\, \sigma^2(t) \geqslant \sigma^2(0) - 2h^2\}.$$

Thus it suffices to show that

$$\lim_{h\to 0} \frac{1}{h} \mathbb{E} \sup_{t \in \tilde{F}_h} [X(t) - X(0)] = 0. \tag{5}$$

By Taylor's formula,

$$X(t) = X(0) + \sum_{i=1}^{d} t_i X_i'(\theta_t t) \tag{6}$$

and

$$\sigma(t) = \sigma(0) + \sum_{i=1}^{d} t_i \sigma_i'(\tilde{\theta}_t t) \tag{7}$$

for some $\theta_t, \tilde{\theta}_t \in [0, 1]$.

Assumption (b) of Theorem 3.1 implies that there exists $\varepsilon > 0$ such that

$$K_1 = \min_{i=1,\dots,d} \inf_{\|t\|<\varepsilon} [-\sigma_i'(t)] > 0$$

and

$$K_2 = \inf_{\|t\|<\varepsilon} \sigma(t) > 0.$$

Therefore it follows from (7) that for $\|t\| < \varepsilon$

$$\sigma^2(0) - \sigma^2(t) = [\sigma(0) + \sigma(t)][\sigma(0) - \sigma(t)] \geqslant 2K_2K_1 \sum_{i=1}^{d} t_i \geqslant 2K_1K_2\|t\|$$

Thus, if $h^2/(K_1K_2) < \|t\| < \varepsilon$, then $t \notin \tilde{F}_h$. On the other hand, for sufficiently small $h > 0$ it holds $t \notin \tilde{F}_h$ for $\|t\| \geqslant \varepsilon$ as well (it follows from the fact that σ is continuous and has the unique maximum at the origin). Thus, for sufficiently small h we have

$$\sup_{t\in\tilde{F}_h} \|t\| \leqslant \frac{h^2}{K_1 K_2}. \tag{8}$$

Furthermore, by (6),

$$|X(t) - X(0)| \leqslant \eta \sum_{i=1}^{d} t_i \leqslant \sqrt{d}\eta\|t\|, \tag{9}$$

where

$$\eta = \max_{i=1,\dots,d} \sup_{t\in F} |X_i'(t)|.$$

Combining (8) and (9), we obtain for sufficiently small h

$$\mathbb{E} \sup_{\tilde{F}_h} |X(t) - X(0)| \leqslant \frac{\sqrt{d}\mathbb{E}\,\eta}{K_1 K_2} h^2. \tag{10}$$

By [1, Theorem 2.1.2], it holds $\mathbb{E}\,\eta < \infty$, therefore (10) implies (5), and the proof is complete.

4.2 Proof of Theorem 3.2

Within this section, we assume that the conditions of Theorem 3.2 hold.

Denote by $\mathcal{J}_k$ the collection of faces of dimension k of the parallelepiped F. In particular, $\mathcal{J}_0$ is the set of all vertices of F. To each face $J \in \mathcal{J}_k$ there corresponds a subset $\delta(J)$ of $\{1, \dots, d\}$, of size k, and a sequence $\varepsilon(J) = \{\varepsilon_j\}_{j\notin\delta(J)}$ of $d-k$ zeros and ones so that

$$J = \{x \in F \,|\, x_j = \varepsilon_j a_j, \text{if}\, j \notin \delta(J), \quad 0 < x_j < a_j, \text{if}\, j \in \delta(J)\}.$$

It follows from Morse's Theorem (see [1, p. 210] for details) that

$$V_0(A_u) = \sum_{k=0}^{d} \sum_{J\in\mathcal{J}_k} \sum_{i=0}^{k} (-1)^i \mu_i(J),$$

where $\mu_i(J)$ is the number of points $t \in J$ satisfying the following four conditions:

$$X(t) \geqslant u, \tag{11}$$

$$X_j'(t) = 0, \quad j \in \delta(J), \tag{12}$$

$$(2\varepsilon_j - 1)X_j'(t) > 0, \quad j \notin \delta(J), \tag{13}$$

$$\text{index}[X''_{mn}(t)]_{m,n\in\delta(J)} = k - i. \tag{14}$$

The index of a matrix is the number of its negative eigenvalues. Therefore,

$$\mathbb{E}\, V_0(A_u) = \mathbb{E}\,\mu_0(\{0\}) + \sum_{\{t\}\in\mathcal{J}_0\setminus\{0\}} \mathbb{E}\,\mu_0(\{t\}) + \sum_{k=1}^{d}\sum_{J\in\mathcal{J}_k}\sum_{i=0}^{k}(-1)^i\mathbb{E}\,\mu_i(J).$$

Thus to obtain (2) it is sufficient to prove the following three asymptotic relations:

$$\mathbb{E}\,\mu_0(\{0\}) = \bar{\Phi}(u/\sigma(0)) \cdot (1 + O(u^{-1})), \quad u \to +\infty, \tag{15}$$

$$\mathbb{E}\,\mu_0(\{t\}) = \bar{\Phi}(u/\sigma(0)) \cdot O(u^{-1}), \quad u \to +\infty, \tag{16}$$

for $\{t\} \in \mathcal{J}_0 \setminus \{0\}$ and

$$\sum_{i=0}^{k}(-1)^i\mathbb{E}\,\mu_i(J) = \bar{\Phi}(u/\sigma(0)) \cdot O(u^{-1}), \quad u \to +\infty, \tag{17}$$

for $J \in \mathcal{J}_k, k = 1, \dots, d$.

Before proving them we need to formulate and prove several auxiliary results.

Lemma 4.2. *It holds $\sigma \in \mathcal{C}^1(F)$.*

Proof. Fix $t \in F$ and consider some small vector $\triangle_i t$ parallel to the i-th coordinate axis such that $t + \triangle_i t \in F$. By Taylor's formula, for some $\theta(t) \in [0, 1]$

$$\mathbb{E}\left|\frac{X^2(t+\triangle_i t) - X^2(t)}{\triangle_i t}\right| = \mathbb{E}\left(|X(t+\triangle_i t) - X(t)|\left|\frac{X(t+\triangle_i t) - X(t)}{\triangle_i t}\right|\right)$$

$$= \mathbb{E}\left(|X(t+\triangle_i t) - X(t)|\left|X'(t+\theta(t)\triangle_i t)\right|\right) \leqslant 2\mathbb{E}\left(\sup_{t\in F}|X(t)|\sup_{t\in F}|X'(t)|\right)$$

$$\leqslant 2\left(\mathbb{E}\sup_{t\in F}{}^2|X(t)|\,\mathbb{E}\sup_{t\in F}{}^2|X'(t)|\right)^{\frac{1}{2}}. \tag{18}$$

The processes $X(t)$ and $X_i'(t)$ are continuous on the compact F a.s., which implies their boundedness a.s. Therefore it follows from a corollary of the famous Tsirelson-Ibragimov-Sudakov inequality [1, Theorem 2.1.2] that the moment generating functions of $\sup^2_{t\in F}|X(t)|$, $\sup^2_{t\in F}|X'(t)|$ (and, hence, their moments) together with the right-hand side of (18) are finite. Therefore, by Lebesgue's dominated convergence theorem,

$$\lim_{\triangle_i t\to 0}\frac{\sigma^2(t+\triangle_i t)-\sigma^2(t)}{\triangle_i t}=\lim_{\triangle_i t\to 0}\mathbb{E}\,\frac{X^2(t+\triangle_i t)-X^2(t)}{\triangle_i t}$$
$$=\mathbb{E}\lim_{\triangle_i t\to 0}\frac{X^2(t+\triangle_i t)-X^2(t)}{\triangle_i t}=\mathbb{E}\,\frac{\partial X^2}{\partial t_i}(t)=2\mathbb{E}\,X(t)X_i'(t). \quad (19)$$

To prove the lemma, it remains to show that $\mathbb{E}\,X(t)X_i'(t)$ is continuous. Again, by Lebesgue's dominated convergence theorem,

$$\lim_{\triangle_i t\to 0}[\mathbb{E}\,X(t+\triangle_i t)X_i'(t+\triangle_i t)-\mathbb{E}\,X(t)X_i'(t)]$$
$$=\mathbb{E}\lim_{\triangle_i t\to 0}[X(t+\triangle_i t)X_i'(t+\triangle_i t)-X(t)X_i'(t)]=\mathbb{E}\,0=0.$$

□

Lemma 4.3. *The correlation coefficient between $X(0)$ and $X_i'(0)$ is negative.*

Proof. Relation (19)) yields

$$\frac{\partial\sigma^2}{\partial t_i}(t)=2\mathbb{E}\,X(t)X_i'(t),$$

which implies $\mathbb{E}\,X(t)X_i'(t)=\sigma(t)\sigma_i'(t)$. To conclude the proof, it remains to note that $\sigma(0)>0$ and $\sigma_i'(0)<0$. □

Lemma 4.4. *Let ξ,η be centered Gaussian random variables. If the correlation coefficient ρ between ξ and η is negative, then*

$$\mathbb{P}\{\xi>u,\eta>0\}=O\left(u^{-1}\bar\Phi\left(u/\sigma_\xi\right)\right),\quad u\to+\infty.$$

Proof. If we normalize ξ and η, the correlation coefficient does not change. Hence we may assume that $\operatorname{Var}\xi=\operatorname{Var}\eta=1$. Using the formula for the density of bivariate normal distribution, we get

$$\mathbb{P}\{\xi>u,\eta>0\}=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_u^\infty dx\int_0^\infty \exp\left[-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right]dy$$
$$=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_u^\infty dx\int_0^\infty \exp\left[-\frac{(-\rho x+y)^2+(1-\rho^2)x^2}{2(1-\rho^2)}\right]dy$$
$$=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_u^\infty e^{-x^2/2}\,dx\int_{-\rho x}^\infty e^{-y^2/[2(1-\rho^2)]}\,dy.$$

It is known (see, e.g., [4]) that for any $t > 0$ it holds

$$\left(\frac{1}{t} - \frac{1}{t^3}\right) e^{-t^2/2} \leqslant \int_t^\infty e^{-s^2/2}\, ds \leqslant \frac{1}{t} e^{-t^2/2}. \tag{20}$$

Therefore, making a substitute $y' = y/\sqrt{1-\rho^2}$, we obtain

$$\mathbb{P}\{\xi > u, \eta > 0\} = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_u^\infty e^{-x^2/2}\, dx \int_{-\rho x}^\infty e^{-y^2/[2(1-\rho^2)]}\, dy$$

$$= \frac{1}{2\pi} \int_u^\infty e^{-x^2/2}\, dx \int_{-\rho x/\sqrt{1-\rho^2}}^\infty e^{-y^2/2}\, dy \leqslant \frac{1}{2\pi} \frac{\sqrt{1-\rho^2}}{|\rho|} \int_u^\infty \frac{1}{x} e^{-x^2/2}\, dx$$

$$\leqslant \frac{1}{2\pi} \frac{\sqrt{1-\rho^2}}{|\rho|} \frac{1}{u} \int_u^\infty e^{-x^2/2}\, dx = O(u^{-1}\bar{\Phi}(u)), \quad u \to +\infty.$$

□

The proof of the next lemma can be found in [2, Theorem 6.2].

Lemma 4.5. *Let $Y = \{Y(t),\, t \in U\}$ be a vector-valued Gaussian random field, $Y(t) \in \mathbb{R}^d$ a.s., U an open subset of $\mathbb{R}^d$, and $y \in \mathbb{R}^d$ a fixed point. Denote by N_y the number of points $t \in U$ such that $Y(t) = y$. Suppose that*

(a) $Y \in \mathcal{C}^1(U)$ a.s.,

(b) For each $t \in U$, $Y(t)$ has a nondegenerate distribution,

(c) $\mathbb{P}\{\exists t \in U : Y(t) = y, \det Y'(t) = 0\} = 0$,

where Y' stands for the Jacobian matrix of Y. Then

$$\mathbb{E}\, N_y = \int_U \mathbb{E}\, \big(|\det Y'(t)| \,\big|\, Y(t) = y\big)\, p_{Y(t)}(y)\, dt.$$

Lemma 4.6. *Under the conditions of Theorem 3.2, there exist positive constants A, B such that*

$$\sup_{t\in F} \mathbb{E}\, \left(|X''_{ij}(t)|^d \,\big|\, Z(x) = (u, 0, \ldots, 0)^\top\right) \leqslant (A + B|u|)^d$$

for all $i, j \in \{1, \ldots, d\}$.

Proof. Under the expectation of a random vector (matrix) we understand the vector (matrix) of the expectations of its components. For $i, j \in \{1, \ldots, d\}$ put

$$\xi_{ij} = X''_{ij} - \mathbb{E}\,(X''_{ij} Z^\top)\Sigma^{-1} Z.$$

Denote by $\mathbb{I}_{d+1}$ the identity matrix of the size $(d+1) \times (d+1)$. Since $\Sigma = \mathbb{E}\,(ZZ^\top)$, we have

$$\begin{aligned}\mathbb{E}\,(\xi_{ij} Z^\top) &= \mathbb{E}\,(X''_{ij} Z^\top) - \mathbb{E}\,(X''_{ij} Z^\top)\Sigma^{-1}\mathbb{E}\,(ZZ^\top) \\ &= \mathbb{E}\,(X''_{ij} Z^\top) - \mathbb{E}\,(X''_{ij} Z^\top)\mathbb{I}_{d+1} = (0, 0, \ldots, 0)^\top,\end{aligned}$$

which means that (ξ_{ij}) and Z are independent. Therefore,

$$\begin{aligned}&\mathbb{E}\left(|X''_{ij}|^d \,\middle|\, Z(t) = (u, 0, \ldots, 0)^\top\right) \\ &\quad = \mathbb{E}\left(\left|\xi_{ij} + \mathbb{E}\,(X''_{ij} Z^\top)\Sigma^{-1} Z\right|^d \,\middle|\, Z(t) = (u, 0, \ldots, 0)^\top\right) = \mathbb{E}\left|\xi_{ij} + a_u\right|^d,\end{aligned}$$

where $a_u = \mathbb{E}\,(X''_{ij} Z^\top)\Sigma^{-1}(u, 0, \ldots, 0)^\top$. Furthermore, by Minkowski inequality,

$$\mathbb{E}\,|\xi_{ij} + a_u|^d \leqslant \left((\mathbb{E}\,|\xi_{ij}|^d)^{\frac{1}{d}} + |a_u|\right)^d = \left(b_d^{\frac{1}{d}}(\mathbb{E}\,\xi_{ij}^2)^{\frac{1}{2}} + |a_u|\right)^d,$$

where b_d denotes the d-th absolute moment of the standard Gaussian distribution. To conclude the proof, let us estimate $\mathbb{E}\,\xi_{ij}^2$ and $|a_u|$. We have

$$\begin{aligned}\sup_{t\in F} \mathbb{E}\,\xi_{ij}^2(t) &\leqslant 2\sup_{t\in F} \mathbb{E}\,X''^2_{ij}(t) + 2\sup_{t\in F} \mathbb{E}\,\left[\mathbb{E}\left(X''_{ij}(t) Z^\top(t)\right)\Sigma^{-1}(t) Z(t)\right]^2 \\ &= 2\sup_{t\in F} \mathbb{E}\,X''^2_{ij}(t) + 2\sup_{t\in F}\left[\mathbb{E}\,Z^\top(t) D(t) \mathbb{E}\,Z(t) + \mathrm{tr}\,(D(t)\Sigma(t))\right],\end{aligned}$$

where $D(t) = \Sigma^{-1}(t)\mathbb{E}\left(X''_{ij}(t) Z(t)\right)\mathbb{E}\left(X''_{ij}(t) Z^\top(t)\right)\Sigma^{-1}(t)$ and $\mathrm{tr}(A)$ is the trace of a quadratic matrix A. Since all considered functions are continuous with respect to t over the compact set F and $\det \Sigma(t) \neq 0$ in F, all suprema in the right-hand side are finite. Similarly,

$$\begin{aligned}\sup_{t\in F} |a_u(t)| &= \left|\mathbb{E}\,\left(X''_{ij}(t) Z^\top(t)\right)\Sigma^{-1}(t)(u, 0, \ldots, 0)^\top\right| \\ &\leqslant \sup_{t\in F}\left|\left(\mathbb{E}\,\left(X''_{ij}(t) Z^\top(t)\right)\Sigma^{-1}(t)\right)_1\right| |u|,\end{aligned}$$

where $(b)_1$ is the first coordinate of a vector b. $\square$

Let us turn back to the proof of Theorem 3.2 . It suffices to show that relations (15)-(17) hold. Let us start with (15).

If $\{t\} \in \mathcal{J}_0$ then $\delta(\{t\})$ is empty. Hence in calculating $\mu_0(\{t\})$ we take into account only the conditions (11) and (13), i.e., $X(t) > u$ and $(2\varepsilon_j - 1)X'_j(t) > 0$, $j = 1, \dots, d$. Therefore,

$$\mathbb{E}\,\mu_0(\{t\}) = \mathbb{P}\,\{X(t) > u, (2\varepsilon_j - 1)X'_j(t) > 0, j = 1, \dots, d\}.$$

If $t = 0$, then $\varepsilon_j = 0$ for all $j = 1, \dots, d$, and

$$\begin{aligned}&\big|\bar{\Phi}(u/\sigma(0)) - \mathbb{E}\,\mu_0(\{0\})\big| \\ &\quad = \big|\mathbb{P}\,\{X(0) > u\} - \mathbb{P}\,\{X(0) > u, X'_j(0) < 0, j = 1, \dots, d\}\big| \\ &\quad\quad \leqslant \sum_{j=1}^{d} \mathbb{P}\,\{X(0) > u, X'_j(0) > 0\}.\end{aligned}$$

If we combine this with Lemma 4.3 and Lemma 4.4, we get (15).

If $\{t\} \in \mathcal{J}_0 \setminus \{0\}$, then

$$\begin{aligned}\mathbb{E}\,\mu_0(\{t\}) &= \mathbb{P}\,\{X(t) > u, (2\varepsilon_j - 1)X'_j(t) > 0, j = 1, \dots, d\} \\ &\leqslant \mathbb{P}\,\{X(t) > u\} = \bar{\Phi}(u/\sigma(t)),\end{aligned}$$

which together with $\sigma(t) < \sigma(0)$ yields (16).

To prove (17), denote by $\tilde{\mu}(J)$ the number of points $t \in J$ satisfying only conditions (11) and (12). Clearly,

$$\left|\sum_{i=0}^{k} (-1)^i \mu_i(J)\right| \leqslant \tilde{\mu}(J). \tag{21}$$

For simplicity, we assume that $k = d$, so J is the interior of the cube: $J = \operatorname{int} F$. For other faces of F, the proof is analogous.

Consider the random vector field of $d + 1$ variables

$$Z_v(t) = \big(X(t) - v, X'_1(t), \dots, X'_d(t)\big)^{\top}, \quad (v, t) \in \mathbb{R} \times \operatorname{int} F.$$

Denote by Z'_v the Jacobian matrix of Z:

$$Z'_v(t) = \begin{pmatrix} -1 & X'_1(t) & \dots & X'_d(t) \\ 0 & X''_{11}(t) & \dots & X''_{1d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & X''_{d1}(t) & \dots & X''_{dd}(t) \end{pmatrix}.$$

It is not difficult to see that $\tilde{\mu}(J)$ coincides with the number of zeros of Z in the domain $(u,\infty)\times\operatorname{int}F$. To count them, we apply Lemma 4.5 with

$$U=(u,\infty)\times\operatorname{int}F,\quad Y=Z_v,\quad y=(0,0,\dots,0).$$

Evidently, the covariance matrix of Z_v coincides with Σ and $\det Z'_v=-\det X''$. Therefore, assumptions (a)-(c) of Theorem 3.2 imply that Z'_v satisfies the conditions of Lemma 4.5. Thus we have

$$\mathbb{E}\,\tilde{\mu}(J)=\int_F\int_u^\infty \mathbb{E}\,\big(|\det Z'_v(t)|\,\big|\,Z_v(t)=0\big)\,p_{Z_v(t)}(0)\,dv\,dt. \tag{22}$$

By Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}\,\big(|\det Z'_v(t)|\,\big|\,Z_v(t)=0\big)&=\mathbb{E}\,\Big(\big|\det\big(X''_{ij}(t)\big)_{1\leqslant i,j\leqslant d}\big|\,\Big|\,Z_v(t)=0\Big)\\ &\leqslant\sum_{(i_1,\dots,i_d)\in\mathbf{S}_d}\mathbb{E}\,\big(|X''_{1i_1}(t)\dots X''_{di_d}(t)|\,\big|\,Z_v(t)=0\big)\\ &\leqslant\sum_{(i_1,\dots,i_d)\in\mathbf{S}_d}\left[\prod_{j=1}^d\mathbb{E}\,\Big(|X''_{ji_j}(t)|^d\,\Big|\,Z_v(t)=0\Big)\right]^{1/d},\end{aligned}$$

where $\mathbf{S}_d$ is the set of all permutations of $(1,\dots,d)$. By Lemma 4.6 there exist positive constants A,B such that

$$\mathbb{E}\,\Big(|X''_{ji_j}(t)|^d\,\Big|\,Z_v(t)=0\Big)\leqslant(A+B|v|)^d.$$

Combining the last two inequalities we obtain

$$\mathbb{E}\,\big(|\det Z'_v(t)|\,\big|\,Z_v(t)=0\big)\leqslant d!(A+B|v|)^d. \tag{23}$$

Furthermore, let us estimate $p_{Z_v(t)}(0)$. Since only the first component of $Z_v(t)$ is non-centered,

$$p_{Z_v(t)}(0)=\frac{1}{(2\pi)^{\frac{d+1}{2}}\sqrt{\det\Sigma(t)}}\exp\Big(-\frac{v^2}{2}q_{11}(t)\Big),$$

where $q_{11}(t)$ is the element in the upper left corner of the matrix $\Sigma^{-1}(t)$. It is known (see, e.g., [3, Th. 5, p. 86]) that $q_{11}(t)$ can be expressed in terms of the conditional variance:

$$q_{11}(t)=\frac{1}{\operatorname{Var}\big(X(t)-v\,\big|\,X'_1(t),\dots,X'_d(t)\big)}=\frac{1}{\operatorname{Var}\big(X(t)\,\big|\,\nabla X(t)\big)}.$$

We have

$$\operatorname{Var}\big(X(t)\,\big|\,\nabla X(t)\big) \leqslant \operatorname{Var} X(t),$$

and

$$\operatorname{Var}\big(X(t)\,\big|\,\nabla X(t)\big) = \operatorname{Var} X(t)$$

if and only if $X(t)$ and $\nabla X(t)$ were independent (see, e.g., [3, p. 72]). Since $X(0)$ and $\nabla X(0)$ are not independent (see Lemma 4.3), we get $1/q_{11}(0) < \operatorname{Var} X(0) = \sigma^2(0)$. The origin is a point of the global maximum of $\sigma(t)$, therefore $1/q_{11}(t) < \sigma^2(0)$ for all $t \in F$. Since $q_{11}(t)$ is continuous with respect to t (by the formula for a matrix inversion), we get

$$c_0 = \inf_{t \in F} \left[\sigma^2(t) - \frac{1}{q_{11}(t)}\right] > 0.$$

Thus,

$$p_{Z_v(t)}(0) \leqslant \frac{1}{(2\pi)^{(d+1)/2}\sqrt{\det \Sigma(t)}} \exp\Big(-\frac{v^2}{2}\frac{1}{\sigma^2(0) - c_0}\Big). \tag{24}$$

Finally, (21), (22), (23) and (24) yield (17), and the proof is complete.

5 Proofs of Theorems 3.3 and 3.4

The both proofs use the next asymptotic relation which is obtained using Laplace's method.

Lemma 5.1. *Let $H = \prod_{j=1}^m [0, a_j]$ be a parallelepiped in $\mathbb{R}^m$, $m \geqslant 1$, where $a_1, \ldots, a_m$ are some positive real numbers. Consider functions $f, S : H \to (0, \infty]$. Put $S_i' = \partial S/\partial t_i$, $f_i' = \partial f/\partial t_i$, and $S_{ij}'' = \partial^2 S/\partial t_i \partial t_j$ whenever they exist. Suppose that*

(a) *S has a unique global minimum at the origin,*

(b) *$f, S \in C(H)$,*

(c) *$f \in C^1, S \in C^2$ in some neighborhood of the origin and $\sum_{i=1}^m S_i'(0) > 0$,*

(d) $\sup_{t \in H} f(t) e^{-S(t)} < \infty$.

Then

$$\int_H f(t) e^{-\lambda S(t)}\, dt = \frac{f(0)}{\prod_{i=1}^m S_i'(0)} \lambda^{-m} e^{-\lambda S(0)} [1 + o(1)], \quad \lambda \to +\infty. \tag{25}$$

Proof. Fix some small $\varepsilon > 0$, large λ such that $\ln^2 \lambda/\lambda < \varepsilon$ and divide H into three sets:

$$H_1 = \left[0, \frac{\ln^2 \lambda}{\lambda}\right]^m, \quad H_2 = [0,\varepsilon]^m \setminus H_1, \quad H_3 = H \setminus [0,\varepsilon]^m.$$

Put

$$A_k(\lambda) = \int_{H_k} f(t) e^{-\lambda S(t)}\, dt, \quad k = 1, 2, 3.$$

By Taylor's formula,

$$S(t) = S(0) + \sum_{i=1}^m t_i S_i'(0) + \frac{1}{2} \sum_{i,j=1}^m t_i t_j S_{ij}'(\theta_t t) \tag{26}$$

and

$$f(t) = f(0) + \sum_{i=1}^m t_i f_{t_i}'(\tilde\theta_t t) \tag{27}$$

for some $\theta_t, \tilde\theta_t \in [0,1]$. Therefore,

$$\begin{aligned} A_k(\lambda) = \int_{H_k} &\left(f(0) + \sum_{i=1}^m t_i f_{t_i}'(\tilde\theta_t t) \right) \\ &\times \exp\left[-\lambda S(0) - \lambda \sum_{i=1}^m t_i S_i'(0) - \frac{1}{2} \lambda \sum_{i,j=1}^m t_i t_j S_{ij}''(\theta_t t) \right] dt. \end{aligned} \tag{28}$$

First, let us estimate $A_1(\lambda)$. We have

$$\begin{aligned} &A_1(\lambda) \\ &= f(0)\left[1 + O(\lambda^{-1}\ln^2 \lambda)\right] \exp\left[-\lambda S(0) + O(\lambda^{-1}\ln^4 \lambda)\right] \int_{H_1} \exp\left[-\lambda \sum_{i=1}^m t_i S_i'(0)\right] dt \\ &= f(0) e^{-\lambda S(0)} \left[1 + O(\lambda^{-1}\ln^4 \lambda)\right] \prod_{i=1}^m \int_0^{\lambda^{-1}\ln^2\lambda} e^{-\lambda S_i'(0) t_i} dt_i \\ &= f(0) e^{-\lambda S(0)} \left[1 + O(\lambda^{-1}\ln^4 \lambda)\right] \lambda^{-m} \prod_{i=1}^m \frac{1 - e^{-\lambda S_i'(0) \lambda^{-1} \ln^2 \lambda}}{S_i'(0)} \\ &= \frac{f(0)}{\prod_{i=1}^m S_i'(0)} \lambda^{-m} e^{-\lambda S(0)} [1 + o(1)], \quad \lambda \to \infty. \end{aligned} \tag{29}$$

Furthermore, put

$$K_1 = \sup_{t\in H_2} \Big| f(0) + \sum_{i=1}^m t_i f'_{t_i}(\tilde\theta_t t) \Big|$$

and

$$K_2 = \frac{1}{2}\sum_{i,j=1}^m \sup_{t\in H_2} \Big| S''_{ij}(\theta_t t) \Big|.$$

Since assumption (a) holds, we may chose ε such that if $t = (t_1, \dots, t_m)^\top \in H_2, s = (s_1, \dots, s_m)^\top \in H_2$ and $t_j \geqslant s_j$ for $j = 1, \dots, m$ then $S(t) \geqslant S(s)$. Combining this with (26) and putting $s = (\lambda^{-2}\ln\lambda, \dots, \lambda^{-2}\ln\lambda)$, we get

$$S(t) \geqslant S(s) \geqslant S(0) + \frac{\ln^2\lambda}{\lambda}\sum_{i=1}^m S'_i(0) - K_2\frac{\ln^4\lambda}{\lambda^2}$$

for all $t \in H_2$, which implies

$$A_2(\lambda) \leqslant V_m(H_2) K_1 \exp\left[-\lambda S(0) - \ln^2\lambda \sum_{i=1}^m S'_i(0) + K_2 \frac{\ln^4\lambda}{\lambda} \right] = o(A_1(\lambda)) \tag{30}$$

as $\lambda \to \infty$.

Finally, it follows from (a) and (b) that there exists $\delta > 0$ such that $S(t) \geqslant S(0) + \delta$ for $t \in H_3$. Therefore,

$$A_3(\lambda) \leqslant V_m(H_3) K_1 \exp\left[-\lambda S(0) - \lambda\delta\right] = o(A_1(\lambda)), \quad \lambda \to \infty. \tag{31}$$

Combining (29)-(31) concludes the proof. □

Now we return to the proofs of the theorems.

Proof of Theorem 3.3. By Fubini's theorem,

$$\begin{aligned}\mathbb{E}\, V_d(A_u) &= \mathbb{E}\int_F \mathbb{I}(X(t) \geqslant u)\, dt = \int_F \mathbb{P}\left(X(t) \geqslant u\right) dt \\ &= \int_F \Phi\Big(\frac{u}{\sigma(t)}\Big)\, dt = \frac{1}{(2\pi)^{1/2}}\int_F dt \int_1^\infty \frac{u}{\sigma(t)} \exp\left[-\frac{u^2x^2}{2\sigma^2(t)}\right] dx.\end{aligned}$$

Making a substitute $x = 1/(1-x')$, we get

$$\mathbb{E}\, V_d(A_u) = \frac{u}{(2\pi)^{1/2}}\int_F dt \int_0^1 \frac{1}{(1-x)^2\sigma(t)} \exp\left[-\frac{u^2}{2(1-x)^2\sigma^2(t)}\right] dx.$$

To finish the proof, it suffices to apply Lemma 5.1 with

$$m = d + 1, \quad \lambda = u^2, \quad H = F \times [0,1],$$

$$f(t,x) = \frac{1}{(1-x)^2\sigma(t)}, \quad S(t,x) = \frac{1}{2(1-x)^2\sigma^2(t)}.$$

In this case we have

$$f(0) = \frac{1}{\sigma(0)}, \quad S(0) = \frac{1}{2\sigma^2(0)},$$

$$\frac{\partial S}{\partial t_i}(0) = -\frac{\sigma_i'(0)}{\sigma^3(0)}, \quad \frac{\partial S}{\partial x}(0) = \frac{1}{\sigma^2(0)},$$

and (3) follows from (25) and (20). □

To prove Theorem 3.4 we also need the following result.

Lemma 5.2 (Ibragimov, Zaporozhets). *Let $Y = \{Y(t),\, t \in K\}$ be a Gaussian random field, and K be a compact subset of $\mathbb{R}^d$ with a non–empty interior and a finite Hausdorff measure of the boundary. Put $m(t) = \mathbb{E}\, Y(t)$ and $\sigma^2(t) = \operatorname{Var} Y(t)$. Consider a zero set of Y*

$$Y^{-1}(0) = \{t \in K \,:\, Y(t) = 0\}$$

and a zero set of the gradient of the normalized field Y/σ

$$\nabla_Y^{-1}(0) = \{t \in K \,:\, \nabla\big(Y/\sigma\big)(t) = 0\}.$$

Suppose that

(a) $Y \in \mathcal{C}^1(K)$ *a.s.,*

(b) $\mathbb{E}\, V_{d-1}(\nabla_Y^{-1}(0)) < \infty$,

(c) $\sigma(t) > 0$ *for all* $t \in K$.

Then

$$\mathbb{E}\, V_{d-1}(Y^{-1}(0)) = \frac{1}{2\sqrt{2\pi}} \int_K \exp\left[-\frac{m^2(t)}{2\sigma^2(t)}\right] \mathbb{E}\left\|\nabla \frac{Y(t)}{\sigma(t)}\right\| dt.$$

Proof. See [7]. □

Proof of Theorem 3.4. The boundary of A_u coincides with a zero set of the translated field $Y(t) = X(t) - u, t \in F$. Therefore, using Lemma 5.2, we get

$$\mathbb{E}\, V_{d-1}(A_u) = \frac{1}{2\sqrt{2\pi}} \int_F \exp\left[-\frac{u^2}{2\sigma^2(t)}\right] \mathbb{E} \left\| \nabla \frac{Y(t)}{\sigma(t)} \right\| dt.$$

To conclude the proof it remains to apply Lemma 5.1 with

$$m = d, \quad \lambda = u^2, \quad H = F,$$

$$f(t) = \mathbb{E} \left\| \nabla \frac{X(t)}{\sigma(t)} \right\|, \quad S(t) = \frac{1}{2\sigma^2(t)}.$$

In this case we have

$$f(0) = \mathbb{E} \left\| \nabla \frac{X}{\sigma}(0) \right\|, \quad S(0) = \frac{1}{2\sigma^2(0)}, \quad \frac{\partial S}{\partial t_i}(0) = -\frac{\sigma_i'(0)}{\sigma^3(0)},$$

and (4) follows from (25) and (20). □

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