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ON THE ROSE OF INTERSECTIONS OF STATIONARY FLAT PROCESSES

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Abstract

The paper yields retrieval formulae of the directional distribution of a stationary k–flat process in $\mathbb{R}^d$ if its rose of intersections with all r–flats is known. Cases $k = d - 1$, $1 \leq r \leq d - 1$ for arbitrary d and $d = 4$, $k = 2$, $r = 2$ are considered. Some generalisations to manifold processes in $\mathbb{R}^d$ are made. The proofs use the methods of harmonic analysis on higher Grassmannians (spherical harmonics, integral transforms).

GRASSMANN MANIFOLDS; STATIONARY FLAT (POISSON) PROCESSES; MANIFOLD PROCESSES; SPHERICAL HARMONICS; (SPHERICAL) RADON TRANSFORM; SINE TRANSFORM; COSINE TRANSFORM; ROSE OF INTERSECTIONS; STOCHASTIC GEOMETRY

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1. Introduction

Consider a stationary k–flat process Φ_k^d in $\mathbb{R}^d$, i.e. Φ_k^d is a random point process on the phase space of all k–flats in d–dimensional space, each realisation of which is an at most countable "locally finite" collection of k–planes (cf. section 2 for exact definitions). Its stationarity means stability of its distribution law with respect to shifts in $\mathbb{R}^d$. If Φ_k^d is Poisson (cf. [22], [32]) then it is completely determined by its intensity measure $\Lambda(\cdot)$, i.e. by its intensity λ and directional distribution $\theta(\cdot)$ (see equation (2.2)). For arbitrary stationary processes Φ_k^d this is evidently false, but nevertheless the knowledge of the intensity measure allows us to make some general conclusions about the behaviour of the process. In view of that it is sometimes necessary to find $\theta(\cdot)$ possessing only partial information about Φ_k^d, for instance, when one knows something about various intersections of Φ_k^d with r–flats. To be more precise, suppose one intersects Φ_k^d with an r–flat η, $r = d - k + j$. Then $\Phi_k^d \cap \eta$ is a j-flat stationary process in η with intensity $f(\eta)$ which is called the rose of intersections of Φ_k^d. The possibility of obtaining the properties of Φ_k^d from $f(\cdot)$ is itself an interesting mathematical problem. In this regard one can pose the following two questions:
1) Does there exist a one–to–one correspondence between $f(\cdot)$ and $\theta(\cdot)$?
2) How can $\theta(\cdot)$ be restored from $f(\cdot)$ (exact formulae)?

The complete answer to the first question was obtained by G. Matheron (1975), P. Goodey, R. Howard and M. Reeder (1990, 1996) (cf. [17], [4] – [6], respectively).

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It appears that uniqueness of retrieval holds only for particular k and r (see section 3).

The partial answer to the second question (fiber processes in dimensions 2 and 3) could be found in the papers by J. Mecke and W. Nagel (1979, 1980) (cf. [18], [21] and others).

The main results of the present paper yield the retrieval formulae for the directional distribution $\theta(\cdot)$ of any stationary process of hyperplanes in $\mathbb{R}^d$ from its rose of intersections when the intersecting plane η has dimension r, $1 \leq r \leq d-1$. These results are generalized to hold for stationary manifold processes in $\mathbb{R}^d$.

The proofs involve the ideas of harmonic analysis on Grassmann manifolds.

In section 2 all necessary definitions and notions concerning stationary k–flat processes are introduced. Section 3 contains both the exact mathematical statement of the problem of obtaining $\theta(\cdot)$ from $f(\cdot)$ and the history of the question.

The main integral formulae are given in section 6 for the case when the process Φ^d_{d-1} is intersected with lines. In section 7 the more general situation of arbitrary $r \in \{2, \dots, d-1\}$ is considered.

Some notions and results that we use intensively in sections 6, 7, such as Grassmann manifolds and various integral transforms (sine, cosine and spherical Radon transforms) are introduced in sections 4 and 5.

Section 8 is devoted to the deduction of the formulae in the form of expansions in spherical harmonics. It contains the appropriate expansion formulae for the cases when Φ^d_{d-1} is intersected with lines and hyperplanes in d dimensions and when the 2-flat process Φ^4_2 intersects with 2-flats in dimension 4, respectively. In the case of Φ^4_2 there are infinitely many directional distributions θ that yield the same rose of intersections $f(\cdot)$; their description in terms of Fourier coefficients of θ is given.

Finally a possible generalization of all these results to the wider class of manifold processes is made.

2. Stationary k–flat processes

In this section we shall follow the guidelines of [23] in introducing the basic notions of k-flat processes (cf. [32], [22] for other constructions).

Let $F(d,k)$ be the set of all k–flats in $\mathbb{R}^d$, $d \geq 2$, $1 \leq k \leq d-1$. Let $G(d,k)$ be the *Grassmann manifold* of all non–oriented k–dimensional linear subspaces of $\mathbb{R}^d$ (for more detailed information see section 4). Say $B \subset F(d,k)$ is *bounded* if $\sup\limits_{\xi \in B} \rho(0,\xi) < \infty$ where $\rho(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R}^d$. Let $\mathfrak{G}$, $\mathfrak{F}$ be the σ–algebras of Borel subsets of $G(d,k)$, $F(d,k)$ in their usual topologies. One calls $\varphi \subset F(d,k)$ a *flat field* if any bounded set $B \subset \mathbb{R}^d$ is intersected by a finite number of k–flats of φ. Let $\mathcal{M}$ be the set of all flat fields and $\mathfrak{M}$ — the usual σ–algebra on $\mathcal{M}$.

Definition 2.1 Φ^d_k *is called a k–flat process iff* $\Phi^d_k : (\Omega, \Gamma, P) \to (\mathcal{M}, \mathfrak{M})$ *is a random element. Its distribution is a measure* $\kappa(\cdot) = P\{\Phi^d_k \in \cdot\}$ *on* $\mathfrak{M}$ *(*$k = 0$ *— ordinary point process,* $k = d-1$ *— hyperplane process in* $\mathbb{R}^d$*).*

A k–flat process Φ^d_k is called *stationary* if its distribution is invariant with respect to all shifts in $\mathbb{R}^d$. Denote by $\nu_d(\cdot)$ the d–dimensional Lebesgue measure in $\mathbb{R}^d$. We shall call λ the *intensity* of a stationary process Φ_k^d if $\lambda = \frac{E\,\nu_k(\Phi_k^d \cap B)}{\nu_d(B)}$ for every B — bounded subset of $\mathbb{R}^d$ with $\nu_d(B) > 0$. The definition of λ does not depend on the choice of B. Suppose $\lambda \in (0;\infty)$. The *rose of directions (directional distribution)* of Φ_k^d is a probability measure on $G(d,k)$:

$$\theta(\mathcal{C}) = \frac{E \mid \{\xi \in \Phi_k^d : \xi \cap \mathbf{S}^{d-1} \neq \emptyset,\ r(\xi) \in \mathcal{C}\} \mid}{\lambda k_{d-k}}, \quad \forall \mathcal{C} \in \mathfrak{G} \tag{2.1}$$

where $\mid A \mid$ denotes the cardinal number of a set A, $r(\xi)$ is the direction of a k–flat ξ, i.e. the unique $\bar{\xi} \in G(d,k)$ that is parallel to ξ, $k_d = \frac{2\pi^{\frac{d}{2}}}{d\,\Gamma(\frac{d}{2})}$ is the volume of a unit ball, and $\mathbf{S}^{d-1}$ is a unit sphere in $\mathbb{R}^d$.

Let $\Phi_k^d(\mathcal{B})$ denote the number of k-flats of Φ_k^d that belong to a set $\mathcal{B} \in \mathfrak{F}$ (it can also get infinite values if $\mathcal{B}$ is not bounded). The measure $\Lambda(\mathcal{B}) = E\left(\Phi_k^d(\mathcal{B})\right)$, $\mathcal{B} \in \mathfrak{F}$ is called *the intensity measure* of Φ_k^d.

If Φ_k^d is stationary the following factorization of its intensity measure takes place (cf. e.g. [32], [22]):

$$\Lambda(\mathcal{B}) = \lambda \int\limits_{G(d,k)} \int\limits_{\xi^\perp} I_{\mathcal{B}}(y+\xi)\nu_{d-k}^{\xi^\perp}(dy)\theta(d\xi), \quad \forall \mathcal{B} \in \mathfrak{F} \tag{2.2}$$

where $\nu_{d-k}^{\xi^\perp}(\cdot)$ is the Lebesgue measure on $\xi^\perp$.

For almost all $\eta \in F(d, d-k+j)$ the intersection $\Phi_k^d \cap \eta$ is a stationary j–flat process on η. Let $f(\eta) = \lambda_{\Phi_k^d \cap \eta}$ be the intensity of $\Phi_k^d \cap \eta$. Due to the stationarity of Φ_k^d it is sufficient to consider only those affine flats η that contain zero, i.e. $\eta \in G(d, d-k+j)$. Then (cf. [17], [4])

$$f(\eta) = \lambda \int\limits_{G(d,k)} [\xi, \eta]\,\theta(d\xi) \tag{2.3}$$

where $[\xi,\eta]$ is the $d-j$ –volume of the unit parallelepiped spanned over orthonormal bases in $\xi^\perp$ and $\eta^\perp$: if $\xi^\perp = < a_1, \dots, a_{d-k} >$, $\eta^\perp = < b_1, \dots, b_{k-j} >$ then

$$[\xi,\eta] = Vol(a_1, \dots, a_{d-k}, b_1, \dots, b_{k-j}).$$

It is independent of the choice of the orthonormal bases in $\xi^\perp$ and $\eta^\perp$. The function $f(\eta)$ is called *the rose of intersections* of Φ_k^d. Later on assume for simplicity $\lambda = 1$.

3. The rose of intersections and the directional distribution

Suppose Φ_k^d is a stationary k–flat process intersected with any r–flat η, $r = d-k+j$. The following natural question is now to be answered: does its rose of intersections f determine $\theta(\cdot)$ uniquely? Introduce the set $V_f(d,k,j)$ of all probability measures $\theta_0(\cdot)$ on $\mathfrak{G}$ such that $\int\limits_{G(d,k)} [\xi,\eta]\,\theta_0(d\xi) = f(\eta)$ for all η. Then the uniqueness would imply $\mid V_f(d,k,j) \mid = 1$. One can distinguish the following particular cases for any dimension d:

1. $j = 0$ ($\Phi_k^d \cap \eta$ is an ordinary point process):
   - $k = d-1$ or 1: $\mid V_f(d,k,j) \mid = 1$ (G. Matheron, 1975, cf. [17]).
   - $2 \leq k \leq d-2$: $V_f(d,k,j)$ is infinite dimensional (P. Goodey, R. Howard, 1990, cf. [4], [5]).
2. $1 \leq j < k \leq d-1$ (P. Goodey, R. Howard, 1990):
   - $k < d-1$: $V_f(d,k,j)$ is infinite dimensional.
   - $k = d-1$: $\mid V_f(d,k,j) \mid = 1$.

As $\theta(A) = \int\limits_{G(d,k)} I_A(\xi)\, \theta(d\xi)$ for any $A \in \mathfrak{G}$ and $I_A(\cdot)$ can be approximated by smooth functions one can easily see that any measure $\theta(\cdot)$ is uniquely determined by all integrals

$$\int\limits_{G(d,k)} g(\xi)\, \theta(d\xi) \tag{3.1}$$

where g belongs to some sufficiently large class of functions, say $C^p(G(d,k))$, $p \in \mathbb{N} \cup \{\infty\}$ or $C(G(d,k))$. J. Mecke [18] provides an easy integral retrieval formula for fibre processes on the plane ($d = 2$, $k = r = 1$) that expresses (3.1) through the integrals of $f(\cdot)$, while J. Mecke and W. Nagel [21] obtain a sort of expansion formula in spherical harmonics for the case $d = 3$, $k = 1$, $r = 2$. In both cases the uniqueness of retrieval of $\theta(\cdot)$ from f takes place.

In what follows we shall generalize these results ($k = d-1$) for arbitrary dimensions d and r and also consider one case of non–uniqueness $d = 4$, $k = r = 2$. Here the whole set $V_f(4,2,0)$ will be described. The choice of parameters k and r could be explained by the fact that only in these cases the appropriate Grassmannian $G(d,k)$ is isomorphic to a sphere (or a product of spheres) (see section 4), and by this means the standard methods of harmonic analysis on the sphere are applicable. All other cases of non–uniqueness remain still open.

4. Grassmannians

Assume $d \geq 3$, $1 \leq k \leq d-1$. Let Φ_k^d be a stationary k-flat process in $\mathbb{R}^d$ with intensity 1, directional distribution $\theta(\cdot)$ and the rose of intersections $f(\cdot)$. Before we proceed to get the retrieval formulae for $\theta(\cdot)$ let us investigate the structure of manifolds $G(d,k)$ on which all our measures and functions are defined.

For any d and $k \leq d$ the Grassmannian $G(d,k)$ is a compact analytic manifold of dimension $k(d-k)$ (cf. [16]). Moreover, it is symmetric and separable. We shall be interested in specific cases $k = 1$, $k = d-1$ for arbitrary $d \geq 2$ as well as $d = 4$, $k = 2$. It is clear that $G(d,1) \cong \mathbf{S}_+^{d-1} \stackrel{def}{=} \{u \in \mathbf{S}^{d-1} : u \equiv -u\}$ where $\mathbf{S}_+^{d-1}$ is a sphere in $\mathbb{R}^d$ with each diameter having its end points identified ($\mathbf{S}_+^{d-1}$ is obviously topologically equivalent to projective space $\mathbb{RP}^{\,d-1}$). Mapping any hyperplane ξ to its orthogonal complement $\xi^\perp \in G(d,1)$ we get $G(d,d-1) \cong \mathbf{S}_+^{d-1}$. It can be proved that

$$G(4,2) \cong \{(u,v) \in \mathbf{S}^2 \times \mathbf{S}^2 : (u,v) \equiv (-u,-v)\}, \tag{4.1}$$

J. Mecke [20] shows it by means of a special type of Grassmann coordinates. However other approaches are also possible (cf. [4] for references). Due to the above facts we shall consider later on just even functions and measures defined on $\mathbf{S}^{d-1}$ and $\mathbf{S}^2 \times \mathbf{S}^2$.

The structure of $G(d,k)$ as a quotient space allows the introduction of the unique left and right invariant normalized Haar measure $\mu(\cdot)$ on $G(d,k)$ (cf. [1]). In the cases of hyperplanes and lines $\mu(\cdot) \equiv \omega_d(\cdot)/\omega_d$ – the normed surface area measure on $\mathbf{S}^{d-1}$ in $\mathbb{R}^d$
($\omega_d = \omega_d(\mathbf{S}^{d-1}) = d\, k_d$).

5. Integral transforms

In this section we outline some facts about sine, cosine and Radon integral transforms on the sphere (cf. [3] p. 377-386 for properties, [7] and [8] for applications, [14] for general Radon transforms on symmetric spaces).

Let $<g,h>_{\mathbf{S}^{d-1}} = \int\limits_{\mathbf{S}^{d-1}} g(\xi)h(\xi)\,\omega_d(d\xi)$ be the scalar product of any real functions g and h from $L_2(\mathbf{S}^{d-1})$. Denote by $C_e^p(\mathbf{S}^{d-1})$, $p \in \mathbb{N} \cup \{\infty\}$ the space of all p times continuously differentiable even functions on $\mathbf{S}^{d-1}$; let $C_e(\mathbf{S}^{d-1})$ be the space of all continuous even functions on $\mathbf{S}^{d-1}$.

For all $\eta \in \mathbf{S}^{d-1}$ and any integrable g introduce the so–called *cosine transform* Tg and *sine transform* Kg:

$$Tg(\eta) = \int\limits_{\mathbf{S}^{d-1}} |<\xi,\eta>|\; g(\xi)\,\omega_d(d\xi) \tag{5.1}$$

$$Kg(\eta) = \int\limits_{\mathbf{S}^{d-1}} \sqrt{1- <\xi,\eta>^2}\, g(\xi)\,\omega_d(d\xi) \tag{5.2}$$

Let $\eta \in G(d,r)$, $1 \le r \le d-1$. Denote by $\mathbf{S}_\eta^{r-1}$ the totally geodesic submanifold $\mathbf{S}^{d-1} \cap \eta$ of $\mathbf{S}^{d-1}$, $dim(\mathbf{S}_\eta^{r-1}) = r-1$. For any integrable function g on $\mathbf{S}^{d-1}$ introduce *the spherical Radon transform of order* r:

$$(R_r g)\,(\eta) = \frac{1}{\omega_r} \int\limits_{\mathbf{S}_\eta^{r-1}} g(\xi)\,\omega_r^\eta(d\xi) \tag{5.3}$$

where $\omega_r^\eta(\cdot)$ is the surface area measure on the subsphere $\mathbf{S}_\eta^{r-1}$. If $r = d-1$ we shall write $R = R_{d-1}$ and call it simply *the spherical Radon transform:* identifying $\eta \in G(d,d-1)$ with the direction unit vector of the line $\eta^\perp$ we have

$$Rg(\eta) = \frac{1}{\omega_{d-1}} \int\limits_{<\xi,\eta>=0} g(\xi)\,\omega_{d-1}^{\eta^\perp}(d\xi) \tag{5.4}$$

It is known that cosine and Radon transforms R_r (and consequently sine transform, see (5.7)) are injective on $C_e^\infty(\mathbf{S}^{d-1})$. Introduce the differential operator

$$\Box = \frac{1}{2\omega_{d-1}}(\triangle_0 + d - 1) \tag{5.5}$$

where $\triangle_0$ is a Beltrami – Laplace operator on $\mathbf{S}^{d-1}$. It can be shown (cf. [8]) that

$$\Box T = R, \tag{5.6}$$

$$K = \frac{\omega_{d-1}}{2k_{d-2}} RT \tag{5.7}$$

on $C_e(\mathbf{S}^{d-1})$, or equivalently to (5.7),

$$T = \frac{2k_{d-2}}{\omega_{d-1}} R^{-1} K, \tag{5.8}$$

as R_r is invertible for all r.

As for relation (5.7) we shall prove it in more general form (for the transforms on signed measures, r arbitrary) in lemma 5.2.

The following inversion formulae being true for all $d \geq 3$ follow from general results of S. Helgason [13], [14] p. 54 (we applied them for $k = d-1$, $1 < r \leq d-1$): for any $g \in C_e(\mathbf{S}^{d-1})$

$$g(\xi) = c_R \left(\frac{d}{d(x^2)}\right)^{r-1} \left[\int\limits_0^x y^{r-1}(x^2-y^2)^{\frac{r-3}{2}} \int\limits_{d(\mathbf{S}_\eta^{r-1},\,\xi)=\arccos\, y} (R_r g)(\eta)\, \mu(d\eta)\, dy \right] \Bigg|_{x=1} \tag{5.9}$$

where $d(\cdot,\cdot)$ is the geodesic distance on $\mathbf{S}^{d-1}$, $\mu(\cdot)$ is the invariant measure on $\{\eta \in G(d,r) : d(\mathbf{S}_\eta^{r-1},\, \xi) = \arccos\, y\}$ normed so that its total mass is 1, and

$$c_R = \frac{2^r}{(r-2)!}; \tag{5.10}$$

$$g(\xi) = c_R^1 \left(\frac{d}{d(x^2)}\right)^{d-2} \left[\int\limits_0^x (x^2-y^2)^{\frac{d}{2}-2} \int\limits_{<\xi,\eta>^2=1-y^2} Rg(\eta)\, \omega_{d-1}^{\xi^\perp,y}(d\eta)\, dy \right] \Bigg|_{x=1} \tag{5.11}$$

where $\omega_{d-1}^{\xi^\perp,y}(\cdot)$ is a surface area measure on the pair of subspheres $<\xi,\eta>^2 = 1-y^2$ of $\mathbf{S}^{d-1}$ and

$$c_R^1 = \frac{2^{d-3}}{(d-3)!\, \omega_{d-1}}. \tag{5.12}$$

We shall find now a more compact form of (5.11) and will use it in the proofs later on; nevertheless, (5.11) can be also used instead. In our opinion, formula (5.13) below with its integration over the part of a sphere gives a reader more geometrical clearness and agrees with the earlier results of Pogorelov (see remark 5.1).

Lemma 5.1 (Modified inversion formula for the Radon transform)
For any $g \in C_e(\mathbf{S}^{d-1})$ and $d \geq 3$ the following inversion formula holds:

$$g(\xi) = c_R^2 \left(\frac{d}{d(\mu^2)}\right)^{d-2} \left[\int\limits_{<\xi,\eta>^2>\mu^2} \frac{Rg(\eta)\,|<\xi,\eta>|}{(<\xi,\eta>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(d\eta)\right]\Bigg|_{\mu=0} \tag{5.13}$$

where

$$c_R^2 = \frac{(-1)^{d-2}2^{d-3}}{(d-3)!\,\omega_{d-1}}. \tag{5.14}$$

Proof. Applying to (5.11) subsequently the following changes of variables:

$$y^2 = 1-t^2, \quad \mu^2 = 1-x^2$$

we get

$$g(\xi) = c_R^1(-1)^{d-2}\left(\frac{d}{d(\mu^2)}\right)^{d-2}\left[\int\limits_\mu^1 \frac{t(t^2-\mu^2)^{\frac{d}{2}-2}}{\sqrt{1-t^2}} \int\limits_{<\xi,\eta>^2=t^2} Rg(\eta)\,\omega_{d-1}^{\xi^\perp;t}(d\eta)\,dt\right]\Bigg|_{\mu=0}.$$

Then using $<\xi,\eta>=t$ the result follows from Fubini's theorem.

Remark 5.1 A. V. Pogorelov gives another proof of (5.13) for the case $d=3$ (cf. [26]) that does not depend on inversion formula (5.11). See the detailed discussion in [31].

Regarding $g(\cdot)$ as a density of an absolutely continuous measure with respect to $\omega_d(\cdot)$ one can extend transforms (5.1) and (5.2) to act on the space $\mathcal{L}$ of finite signed measures on $\mathbf{S}^{d-1}$:

$$[T\theta](\eta) = \int\limits_{\mathbf{S}^{d-1}} |<\xi,\eta>|\;\theta(d\xi), \tag{5.15}$$

$$[K\theta](\eta) = \int\limits_{\mathbf{S}^{d-1}} \sqrt{1-<\xi,\eta>^2}\,\theta(d\xi). \tag{5.16}$$

For any $\theta \in \mathcal{L}$, $1 < r \leq d-1$ put

$$f(\eta) \stackrel{def}{=} \int\limits_{\mathbf{S}^{d-1}} [\eta,\nu^\perp]\,\theta(d\nu), \quad \eta \in G(d,r).$$

It is a rose of intersections of some stationary process Φ_{d-1}^d only if $\theta(\cdot)$ is a directional distribution, i.e. an even probability measure on $\mathbf{S}^{d-1}$.

Lemma 5.2 For any $d \geq 3$, $1 < r \leq d-1$, $\eta \in G(d,r)$, and finite non–negative measure $\theta \in \mathcal{L}$ the following integral relation holds:

$$f(\eta) = \frac{\omega_r}{2k_{r-1}} \left(R_r[T\theta]\right)(\eta). \tag{5.17}$$

Proof. Choose arbitrary $\eta \in G(d,r)$. Then $f(\eta) = \int\limits_{\mathbf{S}^{d-1}} [\nu^\perp, \eta]\, \theta(d\nu)$,

$$R_r[T\theta](\eta) = \frac{1}{\omega_r} \int\limits_{\mathbf{S}_\eta^{r-1}} \int\limits_{\mathbf{S}^{d-1}} |{<}\xi,\nu{>}|\ \theta(d\nu)\, \omega_r^\eta(d\xi) = \int\limits_{\mathbf{S}^{d-1}} \left(\frac{1}{\omega_r} \int\limits_{\mathbf{S}_\eta^{r-1}} |{<}\xi,\nu{>}|\ \omega_r^\eta(d\xi) \right) \theta(d\nu)$$

by Fubini's theorem. Let us take arbitrary ν. Then $\nu = \nu_\eta + \nu_{\eta^\perp}$ where ν_ζ is the orthogonal projection of ν on ζ. It is clear that

$$[\nu^\perp, \eta] = |\ \nu_\eta\ | = |{<}\nu, \xi_0{>}|$$

where $\xi_0 = \frac{\nu_\eta}{|\nu_\eta|}$. Then one should prove that

$$\int\limits_{\mathbf{S}_\eta^{r-1}} |{<}\xi,\nu{>}|\ \omega_r^\eta(d\xi) = 2k_{r-1}\ |{<}\nu,\xi_0{>}| \tag{5.18}$$

for any $\eta \in G(d,r)$, $\nu \in \mathbf{S}^{d-1}$. So for $\xi \in \mathbf{S}_\eta^{r-1}$

$$|{<}\xi,\nu{>}| = |{<}\xi,\nu_\eta{>}| = |\ \nu_\eta\ | \cdot |{<}\xi,\xi_0{>}| = |{<}\nu,\xi_0{>}| \cdot |{<}\xi,\xi_0{>}|\ .$$

Then

$$\int\limits_{\mathbf{S}_\eta^{r-1}} |{<}\xi,\nu{>}|\ \omega_r^\eta(d\xi) = |{<}\nu,\xi_0{>}| \int\limits_{\mathbf{S}^{r-1}} |{<}\xi,\xi_0{>}|\ \omega_r(d\xi)$$

(the right–hand side integral does not depend on η and ξ_0). Suppose $\xi_0 = e_1$ — a basis unit vector. Then by [25] p. 1

$$\int\limits_{\mathbf{S}^{r-1}} |{<}\xi,e_1{>}|\ \omega_r(d\xi) = \omega_{r-1} \int\limits_{-1}^{1} |\ t\ |\ (1-t^2)^{\frac{r-3}{2}}\, dt. \tag{5.19}$$

One can calculate that the right-hand side of (5.19) is equal to

$$\omega_{r-1} \int\limits_0^1 (1-u)^{\frac{r-3}{2}}\, du = \frac{2\omega_{r-1}}{r-1} = 2k_{r-1}.$$

Thus we have shown that relation (5.18) holds, and therefore the proof is complete.

Corollary 5.1 If $r = d-1$ then equation (5.17) is simply

$$[K\theta] = \frac{\omega_{d-1}}{2k_{d-2}} R[T\theta]. \tag{5.20}$$

As functions $[K\theta](\cdot)$ and $[T\theta](\cdot)$ belong to $C_e(\mathbf{S}^{d-1})$ one can apply the inverse of R_r to (5.17) and get the analogue of formula (5.8) for any finite measure θ:

$$[T\theta](\xi) = \frac{2k_{r-1}}{\omega_r} \left(R_r^{-1} f\right)(\xi), \tag{5.21}$$

$$[T\theta] = \frac{2k_{d-2}}{\omega_{d-1}} R^{-1}[K\theta]. \tag{5.22}$$

Going back to k–flat processes we note that if $k = d-1$, $r = 1$ the rose of intersections $f(\eta)$ is nothing else but the *cosine transform* of measure $\theta(\cdot)$ (cf. [4]) if we identify functions and measures on $G(d, d-1)$ with even functions and measures on $\mathbf{S}^{d-1}$ (cf. section 4). And when $k = d-1$, $r = k$ one can easily see that $f(\eta) = [K\theta](\eta)$. These relations together with (5.6)–(5.13) allow us now to prove the retrieval formulae for the rose of directions.

6. Inversion formulae for $k = d-1$, $r = 1$

Let Φ^d_{d-1} be a stationary hyperplane process with directional distribution $\theta(\cdot)$ and rose of intersections with lines $f(\cdot)$.

Proposition 6.1 If $\theta(\cdot)$ is absolutely continuous with respect to $\omega_d(\cdot)$ with density $\gamma \in C_e(\mathbf{S}^{d-1})$ then

$$\gamma(\xi) = c_R^2 \left(\frac{d}{d(\mu^2)}\right)^{d-2} \left[\int\limits_{<\xi,\eta>^2>\mu^2} \frac{\Box f(\eta)\ |<\xi,\eta>|}{(<\xi,\eta>^2 - \mu^2)^{2-\frac{d}{2}}}\, \omega_d(d\eta) \right]\Bigg|_{\mu=0} \tag{6.1}$$

where c_R^2 is given by (5.14).

Proof. We apply formula (5.13) for $g = \gamma$ and then note that in view of (5.6) and provided that $f(\eta) = T\gamma(\eta)$ we get the above result.

Relation (6.1) in three dimensions can be found in the book of A. V. Pogorelov [26].

Theorem 6.1 Let Φ^d_{d-1} be a stationary hyperplane process with arbitrary directional distribution measure $\theta(\cdot)$ and rose of intersections with lines $f(\cdot)$. Then for any $g \in C_e^m\left(\mathbf{S}^{d-1}\right)$, $m \geq (d+5)/2$, and dimension $d \geq 3$ the following formula holds:

$$\int\limits_{\mathbf{S}^{d-1}} g(\xi)\, \theta(d\xi) = c_R^2 \times$$

$$\times \int\limits_{\mathbf{S}^{d-1}} f(\eta)\, \Box\left(\left(\frac{d}{d(\mu^2)}\right)^{d-2} \left[\int\limits_{<\xi,\eta>^2>\mu^2} \frac{g(\xi)\ |<\xi,\eta>|}{(<\xi,\eta>^2 - \mu^2)^{2-\frac{d}{2}}}\, \omega_d(d\xi) \right]\Bigg|_{\mu=0} \right) \omega_d(d\eta)$$

where c_R^2 is given by (5.14).

Proof. By theorem 4.1 in [7], for any $g \in C_e^m\left(\mathbf{S}^{d-1}\right)$ there exists an integrable function $h(\cdot)$ on $\mathbf{S}^{d-1}$ such that

$$g(\xi) = Th(\xi). \tag{6.2}$$

Then by Fubini's theorem

$$\int\limits_{\mathbf{S}^{d-1}} g(\xi)\,\theta(d\xi) = \int\limits_{\mathbf{S}^{d-1}} Th(\xi)\,\theta(d\xi) = \int\limits_{\mathbf{S}^{d-1}}\int\limits_{\mathbf{S}^{d-1}} |<\xi,\eta>|h(\eta)\,\theta(d\xi)\omega_d(d\eta) =$$
$$\int\limits_{\mathbf{S}^{d-1}} h(\eta)T\theta(\eta)\,\omega_d(d\eta) = \int\limits_{\mathbf{S}^{d-1}} h(\eta)f(\eta)\,\omega_d(d\eta).$$

Then as $\Box$ and R^{-1} commute and by (5.6) we have from (6.2) that

$$h = T^{-1}g = R^{-1}\Box g = \Box R^{-1} g.$$

Then using lemma 5.1 one gets

$$\int\limits_{\mathbf{S}^{d-1}} g(\xi)\,\theta(d\xi) = \int\limits_{\mathbf{S}^{d-1}} f(\eta)\Box R^{-1}g(\eta)\,\omega_d(d\eta) =$$
$$c_R^2 \int\limits_{\mathbf{S}^{d-1}} f(\eta)\,\Box\left(\left(\frac{d}{d(\mu^2)}\right)^{d-2}\left[\int\limits_{<\xi,\eta>^2>\mu^2} \frac{g(\xi)\,|<\xi,\eta>|}{(<\xi,\eta>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(d\xi)\right]\Bigg|_{\mu=0}\right)\omega_d(d\eta),$$

and we are done.

Remark 6.1 A result similar to theorem 6.1 was obtained by W. Weil (cf. [33], [8]) in the setting of distributions. Namely, the support function of a generalized zonoid is the cosine transform of a finite signed measure. Weil generalized this idea and introduced the generating distribution for any centrally symmetric compact convex body in $\mathbb{R}^d$. It was shown in [33] how this distribution can be restored from its cosine transform.

Remark 6.2 (Case $k = 1$, $r = d-1$) All results of this section could be applied directly to the dual case of a stationary line process Φ_1^d intersected with hyperplanes: $r = d-1$.

Remark 6.3 One can also use other inversion formulae for the spherical Radon transform (cf. [11], [12] p. 186–187) in order to prove relations similar to those of proposition 6.1 and theorem 6.1 (see [31]). These inversion formulae involve certain polynomials of the Beltrami–Laplace operator, if d is even, and for odd dimensions they can be written in terms of fractional integrals and wavelets (cf. [27]–[29]).

7. Inversion formulae for $k = d-1$, $1 < r \le d-1$

Consider a stationary hyperplane process Φ^d_{d-1} with directional distribution θ and rose of intersections with r–planes f. The results of the previous section allow us to get θ (or its density γ) if we know $[T\theta](\eta)$ ($T\gamma(\eta)$, respectively). Now let us rewrite these expressions in terms of f. Using relation (5.21) and applying inversion formula (5.9) for the spherical Radon transform of order r one gets

$$(7.1)\quad [T\theta](\xi) = \frac{2k_{r-1}c_R}{\omega_r}\left(\frac{d}{d(x^2)}\right)^{r-1}\left[\int_0^x y^{r-1}(x^2-y^2)^{\frac{r-3}{2}}\int_{d(\mathbf{S}^{r-1}_\eta,\,\xi)=\arccos y} f(\eta)\,\mu(d\eta)\,dy\right]\Bigg|_{x=1}.$$

Now substituting $[T\theta]$ from the above relation for f in the results of section 6 we get the same scope of retrieval formulae for the case $1 < r \le d-1$:

Theorem 7.1 Let Φ^d_{d-1} be a stationary hyperplane process with directional distribution $\theta(\cdot)$ and rose of intersections with r–planes $f(\cdot)$, $1 < r \le d-1$, $d \ge 3$. Then $[T\theta]$ can be determined using (7.1) and we have

1. *If $\theta(\cdot)$ is absolutely continuous with respect to $\omega_d(\cdot)$ with density $\gamma \in C_e(\mathbf{S}^{d-1})$ then*

$$\gamma(\xi) = c_R^2\left(\frac{d}{d(\mu^2)}\right)^{d-2}\left[\int_{<\xi,\eta>^2>\mu^2}\frac{\Box\,[T\theta](\eta)\;|<\xi,\eta>|}{(<\xi,\eta>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(d\eta)\right]\Bigg|_{\mu=0};$$

2. *For arbitrary measure θ and any $g \in C^m_e\left(\mathbf{S}^{d-1}\right)$, $m \ge (d+5)/2$*

$$\int_{\mathbf{S}^{d-1}} g(\xi)\,\theta(d\xi) = c_R^2\times$$

$$\times\int_{\mathbf{S}^{d-1}}[T\theta](\eta)\,\Box\left(\frac{d}{d(\mu^2)}\right)^{d-2}\left[\int_{<\xi,\eta>^2>\mu^2}\frac{g(\xi)\;|<\xi,\eta>|}{(<\xi,\eta>^2-\mu^2)^{2-\frac{d}{2}}}\,\omega_d(d\xi)\right]\Bigg|_{\mu=0}\omega_d(d\eta)$$

where c_R^2 is constant (5.14).

Remark 7.1 In the case $k = r = d-1$ the fact that $G(d,r) \cong \mathbf{S}^{d-1}_+$ and the Haar measure is just the surface area measure on the sphere makes relation 7.1 much more simple (cf. [31]).

8. Case $G(4,2)$

The following section will be devoted to obtaining retrieval formulae for directional distributions by means of expansions in spherical harmonics. At the beginning we mention already known results about the eigenvalues of cosine and sine transforms (cf. [10]), while the results in the case $G(4,2)$ are new.

Let $\{S_{n,j} : n \in \mathbb{Z}_+ \quad j = 1, \ldots, N(d,n)\}$ be an orthonormal basis of spherical harmonics on $\mathbf{S}^{d-1}$ in the usual norm $\|\cdot\|_{\mathbf{S}^{d-1}}$ in $L_2(\mathbf{S}^{d-1})$ (see [25], [2], [9], [10]),

$$N(d,n) = \begin{cases} \frac{(2n+d-2)\Gamma(n+d-2)}{\Gamma(n+1)\Gamma(d-1)}, & n \geq 1 \\ 1, & n = 0. \end{cases} \tag{8.1}$$

For any integrable function g: $\int\limits_{\mathbf{S}^{d-1}} |g(\xi)|\, \omega_d(d\xi) < \infty$ there exists its expansion in spherical harmonics:

$$g(\xi) \sim \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} c_{nj}\, S_{n,j}(\xi) \tag{8.2}$$

where

$$c_{nj} = < g, S_{n,j} >_{\mathbf{S}^{d-1}} \tag{8.3}$$

We know that any measure θ on $\mathbf{S}^{d-1}$ is defined by the values of its integrals $\int\limits_{\mathbf{S}^{d-1}} g(\xi)\, \theta(d\xi)$ for all $g \in C_e^{\infty}(\mathbf{S}^{d-1})$. Let g have expansion (8.2). Then because of the uniform convergence of this series to g (cf. [10]) we have

$$\int\limits_{\mathbf{S}^{d-1}} g(\xi)\, \theta(d\xi) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} c_{nj}\, B_{nj}$$

where

$$B_{nj} = \int\limits_{\mathbf{S}^{d-1}} S_{n,j}(\xi)\, \theta(d\xi). \tag{8.4}$$

Therefore is it sufficient to know all B_{nj} to get a complete description of θ. If θ is a directional distribution measure of a hyperplane process and f its rose of intersections with lines or hyperplanes then B_{nj} can be determined from f, i.e. from its expansion coefficients $b_{nj} = < f, S_{nj} >_{\mathbf{S}^{d-1}}$.

The following result is a direct corollary from lemmas 3.4.5, 3.4.7 [10]:

Proposition 8.1 (Intersections with lines and hyperplanes) Let Φ^d_{d-1} be a stationary hyperplane process with directional distribution θ and rose of intersections with r–planes f ($r \in \{1, d-1\}$). Then for any $g \in C_e^{\infty}(\mathbf{S}^{d-1})$ with expansion in spherical harmonics (8.2)

$$\int\limits_{\mathbf{S}^{d-1}} g(\xi)\, \theta(d\xi) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(d,n)} c_{nj}\, B_{nj}$$

where

$$B_{nj} = \left\{ \begin{array}{ll} 0, & n \text{ odd} \\ \frac{<f, S_{nj}>_{\mathbf{S}^{d-1}}}{\omega_{d-1}\, a_n}, & n \text{ even} \end{array} \right| , \quad j = 1 \ldots N(d,n).$$

The value of a_n is

1. *in case of lines* ($r = 1$):

$$a_n = \begin{cases} \frac{2}{d-1}, & n = 0 \\ \frac{2}{(d-1)(d+1)}, & n = 2 \\ 2(-1)^{\frac{n-2}{2}} \frac{1\cdot 3 \cdot \ldots \cdot (n-3)}{(d-1)(d+1)\ldots(d+n-1)}, & \textit{for even } n \geq 4 \end{cases},$$

2. *in case of hyperplanes* ($r = d - 1$):

$$a_n = -\frac{(d-2)!}{2} \frac{\Gamma\left(\frac{n+d}{2}\right)\Gamma\left(\frac{n-1}{2}\right) n!}{\Gamma\left(\frac{n+d+1}{2}\right)(n+d-2)!\,(n/2)!} \quad \textit{for even } n \geq 0.$$

Let us proceed now to the case $G(4,2)$. Suppose our stationary process Φ_2^4 is intersected by a 2–plane $\eta \in G(4,2)$. We know that

$$G(4,2) \cong \{(u,v) \in \mathbf{S}^2 \times \mathbf{S}^2 : (u,v) \equiv (-u,-v)\}.$$

Then for all $\eta,\, \xi \in G(4,2)$ $\xi \mapsto (u,v)$, $\eta \mapsto (\tilde{u},\tilde{v})$, where $u,\, v,\, \tilde{u},\, \tilde{v} \in \mathbf{S}^2$. J. Mecke [20] has shown that

$$[\xi,\eta] = \frac{1}{2}\, |{<}u,\tilde{u}{>} - {<}v,\tilde{v}{>}|$$

(see also [4], p. 98). If θ is a directional distribution of Φ_2^4 (i.e. a probability measure on $G(4,2)$) then one can prove that its image under the isomorphism (4.1) is again a probability measure $\tilde{\theta}$ on $\mathbf{S}^2 \times \mathbf{S}^2$. Thus without abuse of notation we shall use in the sequel θ instead of $\tilde{\theta}$. Then the rose of intersections of Φ_2^4 is

$$f(\eta) = f(\tilde{u},\tilde{v}) = \frac{1}{2} \int\limits_{\mathbf{S}^2\times\mathbf{S}^2} |{<}u,\tilde{u}{>} - {<}v,\tilde{v}{>}|\ \theta\big(d(u,v)\big).$$

Consider $\{S_{n,\,j} : n \in \mathbb{Z}_+ \quad j = 1,\ldots,N(3,n)\}$ — an orthonormal basis of spherical harmonics on $\mathbf{S}^2$, $N(3,n) = 2n+1$. $f(\tilde{u},\tilde{v})$ can be expanded in this system of $S_{n,\,j}$ as a function of 2 independent variables:

$$f(\tilde{u},\tilde{v}) \sim \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} b_{njki}\, S_{n,\,j}(\tilde{u}) S_{k,\,i}(\tilde{v}) \tag{8.5}$$

where

$$b_{njki} = \int\limits_{\mathbf{S}^2}\int\limits_{\mathbf{S}^2} f(\tilde{u},\tilde{v}) S_{n,\,j}(\tilde{u}) S_{k,\,i}(\tilde{v})\, \omega_3(d\tilde{u})\omega_3(d\tilde{v}). \tag{8.6}$$

One can show that $\{S_{n,\,j} \cdot S_{k,\,i}\}_{n,k,j,i}$ constitute a basis in $L_2(\mathbf{S}^2 \times \mathbf{S}^2)$. Again, we are looking for integrals

$$\int\limits_{\mathbf{S}^2\times\mathbf{S}^2} g(u,v)\,\theta\big(d(u,v)\big) \quad \forall g \in C_e^\infty(\mathbf{S}^2 \times \mathbf{S}^2).$$

If

$$g(u,v) \sim \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} c_{njki}\, S_{n,\,j}(u) S_{k,\,i}(v) \tag{8.7}$$

then

$$\int_{\mathbf{S}^2\times\mathbf{S}^2} g(u,v)\,\theta\big(d(u,v)\big) = \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} c_{njki}\, B_{njki}$$

where

$$B_{njki} = \int_{\mathbf{S}^2\times\mathbf{S}^2} S_{n,\,j}(u) S_{k,\,i}(v)\,\theta\big(d(u,v)\big). \tag{8.8}$$

Hence the coefficients B_{njki} define θ completely. By Funk – Hecke theorem

$$|{<}u,\tilde{u}{>} - {<}v,\tilde{v}{>}| \sim \sum_{n,k=0}^{\infty} 4\pi^2 a_{nk} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} S_{n,\,j}(u) S_{n,\,j}(\tilde{u}) S_{k,\,i}(v) S_{k,\,i}(\tilde{v}), \tag{8.9}$$

$$a_{nk} = \int_{-1}^{1}\int_{-1}^{1} |\, t - x \,|\, P_n(x) P_k(t)\, dt\, dx \tag{8.10}$$

where

$$P_n(t) = \frac{1}{2^n\, n!}\left(\frac{d}{dt}\right)^n (t^2-1)^n \tag{8.11}$$

are Legendre polynomials in 3 dimensions (cf. Rodrigues formula in [25]). Namely, $P_0(t) = 1$, $P_1(t) = t$, $P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$, etc. Function $|{<}u,\tilde{u}{>} - {<}v,\tilde{v}{>}|$ is continuous on (u,v). Therefore its expansion (8.9) converges to it in the sense of Abel summation (cf. [25]). Moreover, taking into account the explicit expressions for a_{nk} (cf. theorem 8.1) one can show that the uniform convergence in (8.9) takes place. Then integrating (8.9) with respect to θ and interchanging integration and summation one gets

$$f(\tilde{u},\tilde{v}) \sim \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} 2\pi^2 a_{nk}\, B_{njki}\, S_{n,\,j}(\tilde{u}) S_{k,\,i}(\tilde{v}).$$

Ergo if $a_{nk} \neq 0$ then for all $i = 1, \dots, 2k+1$, $j = 1, \dots, 2n+1$

$$B_{njki} = \frac{b_{njki}}{2\pi^2 a_{nk}}. \tag{8.12}$$

As the unique retrieval of θ from f is here impossible (see section 3) we shall not be able to find all B_{njki}. As proved in [4], p. 102–103, all a_{nk}, $|n-k| > 2$ are equal to zero that explains the situation.

Theorem 8.1 Assume that $f: G(4,2) \to \mathbb{R}$ is the rose of intersections of a stationary 2–flat process in $\mathbb{R}^4$. Let $V_f(2,4,0)$ be the set of all directional distributions of 2–flat processes with rose of intersections f. Then, the probability measure θ is an element of $V_f(2,4,0)$ iff its coefficients (8.8) satisfy the following conditions:

$$B_{njki} = B_{kinj} \quad \forall n,\, k, \tag{8.13}$$

$$B_{njki} = \begin{cases} \dfrac{\int\limits_{\mathbf{S}^2}\int\limits_{\mathbf{S}^2} f(\tilde{u},\tilde{v}) S_{n,\,j}(\tilde{u}) S_{k,\,i}(\tilde{v})\, \omega_3(d\tilde{u})\omega_3(d\tilde{v})}{2\pi^2 a_{nk}}, & |n-k| \in \{0,2\}, \\ 0, & |n-k| \textit{ is odd} \end{cases}$$

where coefficients a_{nk} defined in (8.10) have the following properties:

$$a_{nk} = a_{kn} \quad \forall n,\, k,$$

$$a_{nk} = \begin{cases} \frac{8}{3}, & n = k = 0, \\ -\frac{2\,(2n)!}{2^{2n}\, n!\,(n+1+1/2)!\,(n-1/2)}, & n = k \geq 1, \\ -\frac{1}{2} a_{mm}, & n = m-1,\ k = m+1, \\ -\frac{1}{2} a_{mm}, & n = m+1,\ k = m-1, \\ 0, & \textit{otherwise}, \end{cases}$$

$(m+\frac{1}{2})! \overset{def}{=} (1+\frac{1}{2})(2+\frac{1}{2})\cdot\ldots\cdot(m+\frac{1}{2})$. *Then for any $g \in C_e^\infty(\mathbf{S}^2 \times \mathbf{S}^2)$ with expansion in spherical harmonics (8.7)*

$$\int\limits_{\mathbf{S}^2 \times \mathbf{S}^2} g(u,v)\, \theta\big(d(u,v)\big) = \sum_{n,k=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=1}^{2k+1} c_{njki}\, B_{njki}. \tag{8.14}$$

Proof. The symmetry of B_{njki} on k, i and n, j is evident from relations (8.8) and (4.1): $\theta(A,B) = \theta(B,A)$ for all spherical Borel sets A and B. Let $n+k$ be odd; making the change of variables $u \mapsto -u$, $v \mapsto -v$ in (8.8) we get

$$B_{njki} = (-1)^{n+k} B_{njki} = -B_{njki}$$

by the homogeneity of spherical harmonics. Consequently $B_{njki} = 0$. Now calculate a_{nk} for even $n+k$. Clearly $a_{nk} = a_{kn}$ because of the symmetry of (8.10). One can show that $a_{nk} \neq 0$ iff $|n-k| \in \{0,2\}$ (cf. [6], p. 267). It is also a consequence of the lemmas below: due to (8.10), (8.15), the symmetry of a_{nk} on n and k and lemma 8.2 $a_{nk} \neq 0$ iff $n+k$ is even and

$$\begin{cases} n+k-2 \leq 2k \\ n+k-2 \leq 2n \end{cases}$$

which yields $n-2 \leq k \leq n+2$, and in view of the fact that $n+k$ is even one gets $k = n-2, n, n+2$. One should note in this regard that the values of B_{njki} are undetermined for those n, k ($|n-k|$ even) that $a_{nk} = 0$ (cf. (8.12)), i.e. if $|n-k|$ is even and not in $\{0,2\}$. These B_{njki} can be chosen freely as long as their symmetry condition (8.13) holds and by (8.14) a probability measure is obtained. Furthermore, calculating directly we have by (8.10) and (8.11)

$$a_{00} = \int_{-1}^{1}\int_{-1}^{1} |\,t - x\,|\ \,dt\,dx = \frac{8}{3}, \quad a_{11} = \int_{-1}^{1}\int_{-1}^{1} |\,t - x\,|\ tx\,dt\,dx = -\frac{8}{15};$$

the rest of a_{nk} will be obtained in lemmae 8.1, 8.2.

Lemma 8.1 For any $n, k \geq 2$

$$a_{nk} = \frac{(-1)^n}{2^{n+k-1}\, n!\, k!} \int_{-1}^{1} (x^2-1)^n \left(\frac{d}{dx}\right)^{n+k-2} (x^2-1)^k dx. \tag{8.15}$$

Proof. First one should prove that for $n, k \geq 2$

$$a_{nk} = \frac{1}{2^{k-1}\, k!} \int_{-1}^{1} P_n(x) \left(\frac{d}{dx}\right)^{k-2} (x^2-1)^k dx. \tag{8.16}$$

Then we use lemma 11 p. 17 [25]: for all $f \in C^n[-1,1]$

$$\int_{-1}^{1} f(x) P_n(x)\,dx = \frac{(-1)^n}{2^n\, n!} \int_{-1}^{1} (x^2-1)^n \left(\frac{d}{dx}\right)^{n} f(x) dx$$

to get the desired result. Let us now prove (8.16):

$$a_{nk} = \int_{-1}^{1} \left[\int_{-1}^{x} (x-t) P_k(t)\,dt - \int_{x}^{1} (x-t) P_k(t)\,dt \right] P_n(x)\,dx = \int_{-1}^{1} h(x) P_n(x)\,dx \tag{8.17}$$

where $h(x)$ can be rewritten as

$$h(x) = x \left(\int_{-1}^{x} P_k(t)\,dt + (-1)^{k+1} \int_{-1}^{-x} P_k(t)\,dt \right) - \left(\int_{-1}^{x} tP_k(t)\,dt + (-1)^k \int_{-1}^{-x} tP_k(t)\,dt \right).$$

One can easily see that $h(x)$ is the solution of the following Cauchy problem on $[-1,1]$ with initial conditions due to the orthogonality of $P_k(t)$, $k \geq 2$ to $P_0(t) = 1$, $P_1(t) = t$:

$$\begin{cases} h''(x) &= 2P_k(x), \quad x \in [-1,1] \\ h'(1) &= \int_{-1}^{1} P_k(t)\,dt = 0 \\ h(1) &= \int_{-1}^{1} (1-t) P_k(t)\,dt = 0 \end{cases} \tag{8.18}$$

Then solving (8.18) we get

$$h(x) = \frac{1}{2^{k-1}\,k!}\left(\frac{d}{dx}\right)^{k-2}(x^2-1)^k, \quad k \geq 2.$$

Substituting this representation of h into (8.17) we prove (8.16).

The following lemma enables us to get all a_{nk} other than zero $\forall n$, k:

Lemma 8.2 For all $n \geq 1$

$$a_{nn} = -\frac{2\,(2n)!}{2^{2n}\,n!\,(n+1+1/2)!\,(n-1/2)},$$

$$a_{n\,n+2} = \frac{(2n)!}{2^{2n}\,n!\,(n+2+1/2)!}.$$

Proof. Case $n = 1$ could be verified by direct calculation. Suppose now $n \geq 2$. Then by lemma 8.1

$$a_{nn} = \frac{(-1)^n}{2^{2n-1}\,(n!)^2}\int_{-1}^{1}(t^2-1)^n\left(\frac{d}{dt}\right)^{2n-2}(t^2-1)^n\,dt =$$

$$\frac{2(-1)^n}{2^{2n}\,(n!)^2}\int_{-1}^{1}(t^2-1)^n\left(\frac{(2n)!}{2}t^2 - n(2n-2)!\right)dt =$$

$$\frac{2(-1)^n}{2^{2n}\,(n!)^2}\left(\frac{(2n)!}{2}\int_{-1}^{1}t^2(t^2-1)^n\,dt - n(2n-2)!\int_{-1}^{1}(t^2-1)^n\,dt\right) =$$

$$\frac{2}{2^{2n}\,(n!)^2}\left(\frac{(2n)!\,n!}{2(n+1+1/2)!} - n(2n-2)!\frac{2n!}{(n+1/2)!}\right) =$$

$$-\frac{2\,(2n)!}{2^{2n}\,n!\,(n+1+1/2)!\,(n-1/2)},$$

$$a_{n\,n+2} = a_{n+2\,n} = \frac{(-1)^n\int_{-1}^{1}(t^2-1)^{n+2}\left(\frac{d}{dt}\right)^{2n}(t^2-1)^n\,dt}{2^{2n+1}\,n!\,(n+2)!} =$$

$$\frac{(-1)^n\,(2n)!\int_{-1}^{1}(t^2-1)^{n+2}\,dt}{2^{2n+1}\,n!\,(n+2)!} = \frac{(2n)!}{2^{2n}\,n!\,(n+2+1/2)!}.$$

Using the formulae of lemma 8.2 one can easily show that

$$a_{nn} = -2a_{n-1\,n+1}, \quad n \geq 2.$$

It could be verified by direct calculations that the above relation holds also for $n = 1, 2$: $a_{02} = a_{20} = \frac{4}{15}$, $a_{13} = a_{31} = \frac{4}{105}$, $a_{11} = -\frac{8}{15}$, $a_{22} = -\frac{8}{105}$.

9. One concluding remark

Inversion formulae of sections 6 – 8 could be generalized to a wider class of point processes on abstract spaces, namely, to fiber and hypersurface processes in $\mathbb{R}^d$ or, more generally, to processes of manifolds in $\mathbb{R}^d$. This kind of processes was extensively studied in a large number of papers by J. Mecke, W. Nagel, I. Molchanov, D. Stoyan, M. Zähle, etc. ([18], [21], [19], [24], see [32], chapter 9 for more references). In this case the rose of intersections $f(\eta)$ of a stationary process Σ_k^d of k–dimensional manifolds with an r–flat η is the mean total surface area of $\Sigma_k^d \cap B$ for a unit test window $B \subset \eta$. It can be shown that $f(\eta)$ is well–defined and equal to (2.3) for almost all $\eta \in G(d,r)$. Thus the inversion formulae of this paper can be applied directly to $f(\cdot)$.

10. Note added in proof

After the submission of this paper the preprint [28] appeared, where the main problem considered in sections 6–7 was solved via the inversion of a certain analytic family of functional operators in L^p–spaces that contains both the spherical Radon and the generalized cosine transforms.

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