Random Sets and Random Fields

Evgueni Spodarev
Overview

- Motivation
- Basics of mathematical morphology
- Introduction to the theory of random sets
- Comparison of grey scale images
- Random fields
- Examples of random fields
- Extrapolation of spatial data sets
- Applications
- Literature
Motivation

- Geometrical random structures in nature

Alcaline zinc–nickel layer on steel  
Nickel foam
Motivation

Distribution of galaxies in space (Hammer-Aitoff projection)
Motivation

- Modelling the structure of materials

Polyurethane foam

Edges of a 3D Voronoi tessellation
Motivation

- Estimation of the image characteristics

Microscopic structure of Cu powder: the form and the size of particles

Porous structure of sand stone: percolation
Mathematical morphology

Basic notation

\( \mathcal{K} \) family of all compact convex sets (bodies) in \( \mathbb{R}^d \)

\( \mathcal{R} = \left\{ \bigcup_{i=1}^{n} K_i : K_i \in \mathcal{K}, \ i = 1, \ldots, n, \ \forall n \right\} \) convex ring

\( S = \left\{ K : K \cap W \in \mathcal{R}, \ \forall W \in \mathcal{K} \right\} \) extended convex ring

\( B_r(a) \) ball with center in \( a \) and radius \( r \)

\( \kappa_j \) volume of \( B_1(o) \) in \( \mathbb{R}^j, \ j = 0, \ldots, d \)

\( K_1 \oplus K_2 = \bigcup_{x \in K_2} (K_1 + x) \) Minkowski addition

\( K_1 \ominus K_2 = \bigcap_{x \in K_2} (K_1 + x) \) Minkowski subtraction
**Mathematical morphology**

- **Morphological operations**

  - **Dilation** \( K \mapsto K \oplus (-B) \)
  - **Erosion** \( K \mapsto K \ominus (-B) \)
  - **Threshold filter** \( f(x) \mapsto 1(x : f(x) \geq a), \ x \in W \)

Notice that \( K \ominus (-B) \ominus B \subseteq K \subseteq K \ominus (-B) \ominus B \)
Intrinsic volumes

Steiner formula in $\mathbb{R}^2$

For any $K \in \mathcal{K}$ and $r > 0$

$$A(K \oplus B_r(o)) = A(K) + rS(K) + \pi r^2 \chi(K),$$

where

- $A(K) =$ the area of $K$
- $S(K) =$ the boundary length of $K$
- $\chi(K) = 1$ the Euler number of $K$ (“porosity”)
Intrinsic volumes

Steiner formula in $\mathbb{R}^d$

There exist functionals $V_j, W_j : \mathcal{K} \to [0, \infty), \ j = 0, \ldots, d,$ (Minkowski functionals, quermassintegrals or intrinsic volumes) such that for any $r > 0$ and $K \in \mathcal{K}$ it holds

$$V_d(K \oplus B_r(o)) = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K) = \sum_{j=0}^{d} r^j \binom{d}{j} W_j(K)$$

where $W_j(K) = \frac{\kappa_j}{\binom{d}{j}} V_{d-j}(K), \ \forall K \in \mathcal{K},$ and

the functionals $V_0, \ldots, V_d$ are additive, motion invariant, monotone with respect to inclusion, and locally bounded.
In $\mathbb{R}^3$: For any $K \in \mathcal{K}$, $\partial K \in C^2$, it holds

\begin{itemize}
  \item $V_3(K) = |K|$ (volume of $K$)
  \item $2V_2(K) = S(K)$ (surface area of $K$)
  \item $\pi V_1(K) = (1/2) \int_{\partial K} (1/R_1 + 1/R_2) \, d\sigma$ (integral of mean curvature of $\partial K$ or, equivalently, $2\pi \times$ mean breadth of $K$)
  \item $4\pi V_0(K) = 4\pi = \int_{\partial K} (1/R_1 \cdot 1/R_2) \, d\sigma$ ($4\pi \times$ Euler number = integral of Gaussian curvature of $\partial K$),
\end{itemize}

where $R_1$ and $R_2$ are the principal radii of curvature of $\partial K$. 
Theorem 1 (Hadwiger (1957))

Let $F : \mathcal{K} \rightarrow \mathbb{R}$ be any additive, motion invariant and continuous functional. Then, $F$ can be represented in the form

$$F = \sum_{j=0}^{d} a_j V_j$$

for some constants $a_0, \ldots, a_d \in \mathbb{R}$.

Thus, the intrinsic volumes $V_0, \ldots, V_d$ form a basis!
Intrinsic volumes

Additive extension to the convex ring $\mathcal{R}$

For each $j = 0, \ldots, d$, there exists a unique additive extension of $V_j : \mathcal{K} \rightarrow [0, \infty)$ to $\mathcal{R}$ given by the inclusion–exclusion formula:

$$V_j(K_1 \cup \ldots \cup K_n) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{j_1 < \ldots < j_i} V_j(K_{j_1} \cap \ldots \cap K_{j_i}), \quad K_1, \ldots, K_n \in \mathcal{K}$$
Intrinsic volumes

Geometrical interpretation: For any $K \in \mathcal{R}$ with $K \neq \emptyset$,

\[
\begin{align*}
V_d(K) &= |K| \quad \text{(volume)} \\
2V_{d-1}(K) &= S(K) \quad \text{(surface area)} \\
V_0(K) &= \chi(K) \quad \text{(Euler number)}
\end{align*}
\]

In $\mathbb{R}^2$: $\chi(K) = \# \{\text{clumps}\} - \# \{\text{holes}\}$

\[
\chi(K) = 3 - 1 = 2
\]
Intrinsic volumes

Steiner formula on \( \mathcal{R} \) (Schneider (1980))

- Let the functional \( \rho_r : \mathcal{R} \rightarrow \mathbb{R} \) be given by

\[
\rho_r(K) = \sum_{j=0}^{d-1} r^{d-j} \kappa_{d-j} V_j(K), \quad K \in \mathcal{R}
\]

- For \( K \in \mathcal{K} \), it holds

\[
\rho_r(K) = V_d((K \oplus B_r(o)) \setminus K)
\]

- Geometrical interpretation of \( \rho_r(K) \) for arbitrary \( K \in \mathcal{R} \)?
Intrinsic volumes

**Index function** $J$

For any $q, x \in \mathbb{R}^d$

- put $J(\emptyset, q, x) = 0$

- For $K \in \mathcal{R}$, $K \neq \emptyset$, let

$$J(K, q, x) = \begin{cases} 1 - \lim_{\delta \to +0} \lim_{\varepsilon \to +0} V_0 \left( K \cap B_{|x - q| - \varepsilon}(x) \cap B_{\delta}(q) \right), & q \in K, \\ 0, & q \notin K. \end{cases}$$
Intrinsic volumes

For any $r > 0$ and $K \in \mathcal{R}$, it holds

$$\rho_r(K) = \int_{\mathbb{R}^d} I_r(K, x) \, dx,$$

where

$$I_r(K, x) = \sum_{q \neq x} J(K \cap B_r(x), q, x)$$

$$\rho_r(K) = \text{volume of } (K \oplus B_r(o)) \setminus K \text{ weighted with multiplicities}$$
Computation of intrinsic volumes

Simultaneous computation of all $V_0(K), \ldots, V_d(K)$

Let $F_i : \mathcal{R} \to \mathbb{R}$, $i = 0, \ldots, d$ be additive, motion invariant and continuous functionals. Then, by Hadwiger’s theorem,

$$F_i(K) = \sum_{j=0}^{d} a_{ij} V_j(K), \quad \forall K \in \mathcal{R}$$

If $F = (F_0(K), \ldots, F_d(K))^\top$ can be easily computed and the matrix $A = (a_{ij})_{i,j=0}^{d}$ is regular,

then $V = (V_0(K), \ldots, V_d(K))^\top$ can be computed as the (uniquely determined) solution $V = A^{-1} F$ of the system of linear equations $F = AV$
Computation of intrinsic volumes

**Example:** Steiner’s formula on $\mathcal{R}$

- $F_i(K) = \rho_{r_i}(K)$, $r_i > 0$, $r_i \neq r_j$ for $i = 0, \ldots, d - 1$
- $F_d(K) = V_d(K)$
- $A = A_{r_0 \ldots r_{d-1}} =$

\[
\begin{pmatrix}
    r_0^d \kappa_d & r_0^{d-1} \kappa_{d-1} & \cdots & r_0^2 \kappa_2 & r_0 \kappa_1 & 0 \\
    r_1^d \kappa_d & r_1^{d-1} \kappa_{d-1} & \cdots & r_1^2 \kappa_2 & r_1 \kappa_1 & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    r_{d-1}^d \kappa_d & r_{d-1}^{d-1} \kappa_{d-1} & \cdots & r_{d-1}^2 \kappa_2 & r_{d-1} \kappa_1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
Computation of intrinsic volumes

In $\mathbb{R}^2$

For any $r_0, r_1 > 0$ with $r_0 \neq r_1$, it holds

$$\begin{pmatrix} V_0(K) \\ V_1(K) \end{pmatrix} = \begin{pmatrix} \pi r_0^2 & 2r_0 \\ \pi r_1^2 & 2r_1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_{r_0}(K) \\ \rho_{r_1}(K) \end{pmatrix}$$

$$= \frac{1}{2\pi r_0 r_1 (r_0 - r_1)} \begin{pmatrix} 2r_1 & -2r_0 \\ -\pi r_1^2 & \pi r_0^2 \end{pmatrix} \begin{pmatrix} \rho_{r_0}(K) \\ \rho_{r_1}(K) \end{pmatrix}$$

$V_2(K) = A(K)$ has to be computed separately
Random sets

- Let $(\Omega, \mathcal{F}, P)$ be an arbitrary probability space
- $\mathcal{C} =$ family of all compact sets in $\mathbb{R}^d$
- $\mathcal{F} =$ family of all closed sets in $\mathbb{R}^d$
- $\sigma(\mathcal{F}) =$ $\sigma$–algebra in $\mathcal{F}$, generated by the sets $F_C = \{F \in \mathcal{F} : F \cap C \neq \emptyset\}$ for any $C \in \mathcal{C}$

An $(\mathcal{F}, \sigma(\mathcal{F}))$–measurable mapping $\Xi : \Omega \rightarrow \mathcal{F}$ is called a random closed set (RACS). Its distribution is uniquely determined by the capacity functional $T_\Xi(C) = P(\Xi \cap C \neq \emptyset)$, $C \in \mathcal{C}$
Random sets

Characteristics of the capacity functional:

1. $0 \leq T_{\Xi} \leq 1$, $T_{\Xi}(\emptyset) = 0$.

2. From $C_n \downarrow C$ follows $T_{\Xi}(C_n) \downarrow T_{\Xi}(C)$.

3. $S_n(C_0; C_1, \ldots, C_n) \geq 0$ for all $C_0, \ldots, C_n \in \mathcal{C}$ and $n \in \mathbb{N}$, where

$$S_0(C_0) = 1 - T_{\Xi}(C_0),$$

$$S_1(C_0; C_1) = T_{\Xi}(C_0 \cup C_1) - T_{\Xi}(C_0),$$

$$\ldots$$

$$S_n(C_0; C_1, \ldots, C_n) =$$

$$S_{n-1}(C_0; C_1, \ldots, C_{n-1}) - S_{n-1}(C_0 \cup C_n; C_1, \ldots, C_{n-1}).$$
Theorem 2 (Uniqueness)

- If $\Xi_1$ and $\Xi_2$ are two random sets with $T_{\Xi_1} = T_{\Xi_2}$ then $\Xi_1 \overset{d}{=} \Xi_2$.

- If $T$ is a functional on $\mathcal{C}$ satisfying the properties 1–3 then there exists a random set $\Xi$ with $T_{\Xi} = T$. 
Stationarity and isotropy

A RACS $\Xi$ is called stationary if $\Xi \overset{d}{=} \Xi + x, \forall x \in \mathbb{R}^d$, and isotropic if $\Xi \overset{d}{=} g\Xi, \forall g \in SO(d)$

**Theorem 3** (Matheron (1975))

- The RACS $\Xi$ is stationary (isotropic) $\iff T_\Xi(C + x) = T_\Xi(C)$
  $\forall x \in \mathbb{R}^d$ and $T_\Xi(gC) = T_\Xi(C)$ $\forall g \in SO(d)$, respectively

- Each stationary RACS $\Xi \neq \emptyset$ is a.s. unbounded

- For any stationary convex RACS $\Xi$, it holds $\Xi \in \{\emptyset, \mathbb{R}^d\}$ a.s.
Characteristics of random sets

**Volume fraction:** $p_\Xi = P(x \in \Xi), \forall x \in \mathbb{R}^d$.

It holds $p_\Xi = T_\Xi(\{o\}) = E|\Xi \cap W|/|W|$ where $|W|$ is the volume of the observation window $W$.

**Covariance function:** $C_\Xi(x) = P(\{o, x\} \in \Xi), x \in \mathbb{R}^d$.

It holds $C_\Xi(x) = 2p_\Xi - T_\Xi(\{o, x\}) = E|\Xi \cap (\Xi - x) \cap W|/|W|$.

**Centered covariance function:** $Cov_\Xi(x) = C_\Xi(x) - p_\Xi^2$. It holds $Cov_\Xi(x) = E[1(o \in \Xi)1(x \in \Xi)] - E 1(o \in \Xi)E 1(x \in \Xi)$.

**Contact distribution function:** $H_K(r) = P(d_K(o, \Xi) \leq r | o \notin \Xi), r \geq 0$, where $K$ is a convex body with $o \in K$ and $d_K(x, F) = \min\{r \geq 0 : (x + rK) \cap F \neq \emptyset\}, F \in \mathcal{F}, x \in \mathbb{R}^d$. 
Examples

- Germ–grain models: \( \Xi = \bigcup_{i=1}^{\infty} (\Xi_i + X_i) \), where
  \( \{X_1, X_2, \ldots\} \) = point process (of germs) and
  \( \{\Xi_1, \Xi_2, \ldots\} \) = sequence of nonempty compact RACS (random grains)

Theorem 4

*Each RACS \( \Xi \) can be represented as a germ–grain model*

\[ \Xi = \bigcup_{i=1}^{\infty} (\Xi_i + X_i) \]
Stationary germ–grain models in $\mathbb{R}^2$

Realizations of germ–grain models: Boolean model with spherical and polygonal grains, respectively; cluster process of segments
Examples

Boolean model

The germ–grain model \( \Xi = \bigcup_{i=1}^{\infty} (\Xi_i + X_i) \) is called a **Boolean model** if

- the point process of germs \( \{X_1, X_2, \ldots\} \) is a stationary Poisson process in \( \mathbb{R}^d \) (with intensity \( \lambda \))
- the grains \( \Xi_1, \Xi_2, \ldots \) are i.i.d. and independent of \( \{X_1, X_2, \ldots\}\);

\( \Xi_i \overset{d}{=} \Xi_0 \)

- \( E |\Xi_0 \oplus K| < \infty, \quad \forall K \in \mathcal{K} \).

**Capacity functional:** \( T_{\Xi}(C) = 1 - e^{-\lambda E |(-\Xi_0) \oplus C|}, \quad \forall C \in \mathcal{C} \)
Examples

Boolean models in $\mathbb{R}^3$
Random sets made from 3D Voronoi tessellations
Specific intrinsic volumes

- **Model assumptions**
  - Let $\Xi$ be stationary, $\Xi \in S$ a.s.
  - $E 2^{N(\Xi \cap [0,1]^d)} < \infty$, where $N(\emptyset) = 0$ and
    \[
    N(K) = \min \{ m \in \mathbb{N} : K = \bigcup_{i=1}^{m} K_i, \ K_i \in \mathcal{K} \} \text{ for } K \in \mathcal{R} \setminus \{ \emptyset \}
    \]

- **Specific intrinsic volumes:** Let
  \[
  \overline{V}_j(\Xi) = \lim_{n \to \infty} \frac{E V_j(\Xi \cap W_n)}{|W_n|}
  \]
  for $j = 0, \ldots, d$, where $\{W_n\} = \text{sequence of monotonously increasing sampling windows } W_n = nW$ with $W \in \mathcal{K}$ and $|W| > 0$

In particular,
\[
\overline{V}_d(\Xi) = P(o \in \Xi) = \frac{E|\Xi \cap W|}{|W|}
\]
Estimation of \((\overline{V}_0(\Xi), \ldots \overline{V}_d(\Xi))\)^T

**Problem:** Estimate \(\overline{V}(\Xi) = (\overline{V}_0(\Xi), \ldots \overline{V}_d(\Xi))\)^T on the basis of a single sample from \(\Xi \cap W\)

**Solution:** For each \(i = 0, \ldots, d\), consider a random field \(Y_i = \{Y_i(x), x \in \mathbb{R}^d\}\) such that

- \(Y_i\) is stationary of second order, i.e. \(EY_i(x) = \mu_i\) and \(Cov(Y_i(x), Y_i(x+h)) = Cov_{Y_i}(h) \quad \forall \ x, h \in \mathbb{R}^d\)

- \(\mu_i = EY_i(o) = \sum_{j=0}^{d} a_{ij} \overline{V}_j(\Xi)\), where the matrix \(A = (a_{ij})_{i,j=0}^{d}\) is regular

Then, it holds \(\overline{V}(\Xi) = A^{-1} \mu\), where \(\mu = (\mu_0, \ldots, \mu_d)^\top\)
Estimation of \((\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))\)^\top

- **Steiner formula:**

\[
\lim_{n \to \infty} \frac{E \rho_r(\Xi \cap W_n)}{V_d(W_n)} = \sum_{j=0}^{d-1} r^{d-j} k_{d-j} \overline{V}_j(\Xi), \quad r > 0.
\]

- Writing the above formula for radii \(r_0, \ldots, r_{d-1}, r_i \neq r_j, i \neq j\) together with \(E \frac{V_d(\Xi \cap W_n)}{V_d(W_n)} = \overline{V}_d(\Xi)\), we get a system of \(d + 1\) linear equations with variables \(\overline{V}_j(\Xi), j = 0, \ldots, d\).
Estimation of \((\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))^{\top}\)

In matrix form: \(A_{r_0 \ldots r_{d-1}} v = c\), where \(v = (\overline{V}_0(\Xi), \ldots, \overline{V}_d(\Xi))^{\top}\),

\[
c = \left(\lim_{n \to \infty} \frac{E \rho_{r_0}(\Xi \cap W_n)}{V_d(W_n)}, \ldots, \lim_{n \to \infty} \frac{E \rho_{r_{d-1}}(\Xi \cap W_n)}{V_d(W_n)}, \frac{E V_d(\Xi \cap W_n)}{V_d(W_n)}\right)^{\top}
\]

\[
A_{r_0 \ldots r_{d-1}} = \begin{pmatrix}
r_0^d k_d & r_0^{d-1} k_{d-1} & \ldots & r_0^2 k_2 & r_0 k_1 & 0 \\
r_1^d k_d & r_1^{d-1} k_{d-1} & \ldots & r_1^2 k_2 & r_1 k_1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
r_{d-1}^d k_d & r_{d-1}^{d-1} k_{d-1} & \ldots & r_{d-1}^2 k_2 & r_{d-1} k_1 & 0 \\
0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}
\]
Comparison of grey scale images

Grey scale image $\rightarrow$ family of binary images

Individual grey scales can be represented by binary images that are analyzed in the sequel.
Comparison of grey scale images

Comparison of particular grey scales

Histological section of prostate tissue: cancer diagnostics
Comparison of grey scale images

- **Binary images**: comparison of their estimated (specific) intrinsic volumes

Bone structure: calcium phase of the healthy and deceased bone (osteoporosis)
Random fields: motivation

Motor car insurance (Bavaria): significant changes of the number of cancellations of insurance policies

Centers of regions with the same postal code
Extrapolated numbers of cancellations in 1998
Random fields: motivation

Quality of ground water in Baden–Württemberg

Bores

Concentration of nitrate in the water (1994)
Random fields

- Random field \( \{ Z(x, \omega) : x \in \mathbb{R}^d, \omega \in \Omega \} \) is a family of random variables:
  - \( Z(x, \cdot) \) is a random variable (briefly: \( Z(x) \))
  - \( Z(\cdot, \omega) \) is a realization of the random field \( Z \) (briefly: \( z(x) \))

- \( Z(x) = m(x) + Y(x) \) where \( m(x) = E Z(x) \) is the mean (drift) and \( Y(x) = Z(x) - m(x) \) is the deviation from the mean (residual). It holds \( E Y(x) = 0 \ \forall x. \)

- Let \( W \subset \mathbb{R}^d \) be an observation window (normally a rectangle)

- Let \( x_1, \ldots, x_n \in W \) be measurement locations placed arbitrarily within \( W \). Let \( z(x_1), \ldots, z(x_n) \) be the measured values at these locations.
A realization \( \{z(x)\}_{x \in \mathbb{R}^2} \) of the random field: \( z(x) = \) wheat harvest in pounds at \( x \)

Four simulated realizations of a random field
Stationary random fields

A random field $Z$ is called (strictly) stationary if all its finite dimensional distributions are translation invariant: for all $h \in \mathbb{R}^d$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{R}^d$ holds

$$(Z(x_1 + h), \ldots, Z(x_n + h)) \overset{d}{=} (Z(x_1), \ldots, Z(x_n)).$$

A random field $Z$ is called stationary of second order if

1. $E(Z(x)) = m(x) = z = \text{const} \quad \forall x$
2. $\text{cov}(Z(x), Z(x + h)) = E[Z(x)Z(x + h)] - z^2 = C(h)$, the covariance function $C$ exists and depends only on the difference $h$. 

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Stationary random fields

- Strict stationarity $\iff$ stationarity of second order
- A second order stationary random field is called isotropic if $C(h) = C(|h|)$, $h \in \mathbb{R}^d$.
- A random field $Z$ is mean square continuous (m.s.c.) if $E(Z(x) - Z(x_0))^2 \to 0$, $x \to x_0$ for all $x_0 \in \mathbb{R}^d$.
- A second order stationary random field is m.s.c. $\iff C(h)$ is continuous at $h = 0$. 
Stationary random fields of second order

Covariance function

- **C is positive definite:** \( \forall n \in \mathbb{N}, w_i \in \mathbb{R}, x_i \in \mathbb{R}^d \)

\[
\sum_{i,j=1}^{n} w_i w_j C(x_i - x_j) = \text{Var} \left( \sum_{i=1}^{n} w_i Z(x_i) \right) \geq 0
\]

- \( |C(h)| \leq C(0) = \text{Var}Z \)
Stationary random fields of second order

Examples of covariance functions

- **Nugget effect (white noise):** $C(h) = c > 0$ for $|h| = 0$ and $C(h) = 0$, $|h| > 0$.

- **Exponential model:** $C(h) = be^{-|h|/a}$, where $b > 0$ is the sill and $a > 0$ is the range.

- **Spherical model:** for positive $a$ and $b$

$$
C(h) = \begin{cases} 
  b \left(1 - 3/2|h|/a + 1/2|h|^3/a^3\right), & 0 \leq |h| \leq a, \\
  0, & |h| > a.
\end{cases}
$$
Stationary random fields of second order

Variogram

\[ \gamma(h) \overset{\text{def}}{=} \frac{1}{2} E (Z(x + h) - Z(x))^2 \]

It holds \( \gamma(h) = C(0) - C(h) = E[Z(x)^2] - E[Z(x)Z(x + h)] \), \( \gamma(0) = 0 \).

\( \gamma \) is conditionally negative definite: for \( n \in \mathbb{N} \), \( w_i \in \mathbb{R} \) with
\[ \sum_{i=1}^{n} w_i = 0 \text{ and } x_i \in \mathbb{R}^d \] it holds \[ \sum_{i,j=1}^{n} w_i w_j \gamma(x_i - x_j) \leq 0. \]

\( \gamma \) is a variogram \( \iff \forall \lambda \ e^{-\lambda \gamma} \) is a covariance function.

If \( \gamma(h) \leq \gamma(\infty) < \infty \) for all \( h \) then \( C(h) = \gamma(\infty) - \gamma(h) \) is a valid covariance function.
Stationary random fields of second order

Variogram

- If $\gamma_1$ and $\gamma_2$ are variograms then $\gamma = \gamma_1 + \gamma_2$ is a variogram as well.

- If $Z$ is stationary and isotropic then $\gamma(h) = \gamma(|h|)$, $h \in \mathbb{R}^d$.

- Many isotropic variogram models can be constructed using models for covariance functions. But not all of them:
  $\gamma(h) = b|h|^\alpha$, $b > 0$, $0 < \alpha < 2$.

- Anisotropic variogram models? e.g., geometrically anisotropic...
Stationary random fields of second order

Exponential geometrically anisotropic variogram

\[ \gamma(h) = \begin{cases} 
0, & h = 0, \\
\ a + \ b(1 - e^{-\sqrt{h^\top K h/c}}), & h \neq 0,
\end{cases} \]

- **Nugget effect** \( a \): discontinuity of the data at the microscopic scale
- **Sill** \( b \): variability of the data at large distances \( h \)
- **Range** \( c \): the correlation range of random variables \( Z(x) \) and \( Z(x + h) \)
- **\( K \)** is the matrix of the composition of a rotation and a scaling.
Examples of random fields

Random sets as random fields

Binary image: $Z(x) = I\{x \in \Xi\}, \ x \in \mathbb{R}^d$ for a random set $\Xi$.

Difference map of nitrate concentrations in the ground water of Baden–Württemberg, 1993–1994
Examples of random fields

Random fields induced by random sets

\[ Z(x) = V_j(\Xi \cap (W + x)), \quad x \in \mathbb{R}^d, \quad j = 0, \ldots, d \]
where \( V_j \) are the intrinsic volumes

\[ Z_r(x) = \sum_{q \in \partial \Xi \cap B_r(x) \setminus \{x\}} J(\Xi \cap B_r(x), q, x), \quad x \in \mathbb{R}^d \]

Gaussian random fields: \( Z \) is a Gaussian random field if all its finite dimensional distributions are normal: for all \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in \mathbb{R}^d \) and \( w_1, \ldots, w_n \in \mathbb{R} \) holds

\[ w_1 Z(x_1) + \ldots + w_n Z(x_n) \sim N(\cdot, \cdot). \]
Examples of random fields

Gaussian random fields are characterized uniquely by their drift $m(x) = E Z(x)$ and their covariance function $C$. Hence: strict stationarity $\iff$ stationarity of second order.
Examples of random fields

Brownian field: \( m(x) = 0, \ C(x, y) = 1/2(|x| + |y| - |x - y|), \)
\( \gamma(h) = 1/2|h| \)

A realization of the Brownian field
Extrapolation of spatial data

Let $Z$ be a stationary random field of second order with mean $\mu$, covariance function $C$ and variogram $\gamma$.

**Goal:** Extrapolation of the field $Z$ from the measured data $z(x_1), \ldots, z(x_n)$.

Estimation of $Z(x) \forall x \Rightarrow$ ordinary kriging
Ordinary kriging

Construct the best linear estimator \( \hat{Z}(x) = \sum_{i=1}^{n} \lambda_i Z(x_i), x \in W \)
where the weights \( \lambda_i \) satisfy the following requirements:

- \( \hat{Z}(x) \) is unbiased: \( E \hat{Z}(x) = z \implies \sum_{i=1}^{n} \lambda_i = 1 \)

- The variance of the estimation error is minimal:
  \[
  E[(\hat{Z}(x) - Z(x))^2] \longrightarrow \min
  \]

The minimization problem \( \implies \) solve the Lagrange equations:

\[
\begin{align*}
\sum_{j=1}^{n} \lambda_j \gamma(x_j - x_i) + \mu &= \gamma(x - x_i), \quad i = 1, \ldots, n, \\
\sum_{j=1}^{n} \lambda_j &= 1.
\end{align*}
\]
Ordinary kriging

To find the weights $\lambda_i$ from the system of linear equations, the variogram $\gamma(h)$ has to be known or estimated from the data $z(x_1), \ldots, z(x_n)$.

- **Matheron’s estimator:**
  \[
  \hat{\gamma}(h) = \frac{1}{2N(h)} \sum_{i,j: x_i - x_j \approx h} (Z(x_i) - Z(x_j))^2,
  \]
  $N(h)$ is the number of pairs $(x_i, x_j) : x_i - x_j \approx h$.
  Computations are made for $h$ on a grid in $\mathbb{R}^d$.

- $\hat{\gamma}(h)$ not conditionally negative definite $\Rightarrow$ a valid variogram model has to be fitted to $\hat{\gamma}(h)$.
Extrapolation of spatial data

What if $Z$ is not stationary? $Z(x) = m(x) + Y(x)$

$\implies$ Estimation of the drift $m$ and the residual $Y$

- **Estimation of $m$:** e.g. the moving average

  \[
  \hat{m}(x) = \frac{1}{N_x} \sum_{x_i \in R(x)} Z(x_i) \quad \text{where} \quad R(x) \text{ is the neighborhood of } x
  \]

  and $N_x = \# \{ i : x_i \in R(x) \}$.

- **Estimated residual** $Y^*(x_i) = Z(x_i) - \hat{m}(x_i)$

- **Extrapolation of $Y$** from the data $Y^*(x_1), \ldots, Y^*(x_n)$, e.g. by ordinary kriging provided that $Y$ is stationary of second order.
Applications

Synthetic data: disturbed Boolean models

Let \( \Xi \) be a stationary Boolean model with intensity \( \lambda \) and deterministic rectangular primary grain \( \Xi_0 = [a, b]^2 \). Let \( \xi = B_r(o) \) be a deterministic disturbance.

\[
m(x) = I(x \in \xi)
\]

\[
Y(x) = I(x \in \Xi) - p_\Xi \quad \text{where}
\]

\[
p_\Xi = E I\{o \in \Xi\} = P(o \in \Xi) = 1 - e^{-\lambda|\Xi_0|} = 1 - e^{-\lambda ab}
\]

is the area fraction of \( \Xi \).
$Y$ is a stationary random field of second order with the covariance function

$$C(h) = 2p_{\Xi} - 1 + (1 - p_{\Xi})^2 e^{\lambda|\Xi_0 \cap (\Xi_0 - h)|}$$

and the anisotropic variogram

$$\gamma(h) = C(0) - C(h) = 1 - p_{\Xi} - (1 - p_{\Xi})^2 e^{\lambda|\Xi_0 \cap (\Xi_0 - h)|}.$$ 

In the special case of $\Xi_0 = [a, b]^2$ it holds

$$\gamma(h) = e^{-\lambda ab} \left( 1 - e^{-\lambda(ab - |\Xi_0 \cap (\Xi_0 - h)|)} \right).$$
Disturbed Boolean models

Realisation of \( \Xi \) and \( \xi \)

\[
Z(x) = m(x) + Y(x) = I(x \in \xi) + I(x \in \Xi) - p_{\Xi}
\]

Measurement points

\( x_1 \ldots x_n \)
Applications

Disturbed Boolean models

Theoretical variogram $\gamma$  Estimated variogram $\hat{\gamma}^*$  Fitted variogram $\gamma^*$
Disturbed Boolean models

Estimated residual \( \hat{Y}^*(x) \)

Estimated drift \( \hat{m}(x) \)

Extrapolated field \( \hat{Z}(x) = \hat{m}(x) + \hat{Y}^*(x) \)
Applications

Disturbed Boolean models

Realisation of

\[ Z(x) = m(x) + Y(x) \]

Extrapolated field

\[ \hat{Z}(x) = \hat{m}(x) + \hat{Y}^*(x) \]
Applications

- Making road traffic maps

Taxi positions in Berlin

Traffic velocities: 13.02.2002, 18.00–18.30


Literature