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Transport networks with an infinite number of nodes

The following paper describes the ergodicity of a transport network with an infinite number of nodes, infinite queues of vehicles and zero queues of customers. The arrival process is assumed to be Poisson with intensity λ. The displacements of cars are instantaneous according to the routing matrix $P = (p_{ij})$.

1. Introduction

The present paper gives a description of the limit behavior of transport networks with an infinite number of nodes. Proofs are omitted due to the lack of space. The classification obtained is not complete and there are still some interesting open problems to solve.

Consider the following queueing system: mutually independent Poisson arrival processes with intensities λ_i, $\sum_{i=1}^{\infty} \lambda_i = \lambda \leq \infty$ enter a transport network with an infinite number of stations. There are x_i cars in an arbitrary node i $\left(\sum_{i=1}^{\infty} x_i = \infty\right)$ at the initial moment $t = 0$ which instantly transport clients to different nodes according to the infinite-dimensional routing matrix $\boldsymbol{P} = (p_{ij})_{i,j=1}^{\infty}$: if a client enters a node i with an awaiting vehicle the car instantly carries the customer to a station j with probability p_{ij}. If there are no cars present in the node i (we shall call such a node the *empty* one) the customer leaves the system (there are no accumulators for customers); accumulators for cars are of infinite capacity. Assume that $\forall i \in \mathbb{N}$ $\sum_{j=1}^{\infty} p_{ij} = 1,\ p_{ii} = 0$. If the contrary is not stated suppose that $\lambda < \infty$. Consider the space $\mathbb{Z}^{+\infty}$ where $\mathbb{Z}^{+} = \mathbb{N} \cup \{0\}$. Introduce the product $\sigma-$algebra $\mathcal{B}$ on $\mathbb{Z}^{+\infty}$. Let $\forall i$ $X_i(n)$ be a number of cars in the node i at the moment of n-th arrival to the system. Denote $\vec{X}(n) = \big(X_1(n), \dots X_i(n), \dots\big)$ $\forall n \in \mathbb{N}$. Let $\big(\Omega, \mathcal{L}, P\big)$ be a given probability space. Then $\vec{X}(n) : \big(\Omega, \mathcal{L}, P\big) \to \big(\mathbb{Z}^{+\infty}, \mathcal{B}\big)$ is a stochastic process. If we denote $P_{\vec{x}}(\cdot) = P\{\cdot \mid \vec{X}(0) = \vec{x}\}$ $\forall \vec{x} \in \mathbb{Z}^{+\infty}$ then $\boldsymbol{X} = \big(\vec{X}(n),\ P_{\vec{x}}\big)$ appears to be a non-countable Markov chain.

2. Definitions and results

Definition 2.1. *The network (the process $\boldsymbol{X}$) is called* ergodic *if the sequence of measures $P_{\vec{x}}^{n}$ which correspond to the distribution of $\boldsymbol{X}$ on every step n converges: $P_{\vec{x}}^{n}(\Gamma) = P_{\vec{x}}\{\vec{X}(n) \in \Gamma\} \longrightarrow \mu(\Gamma)$ as $n \to \infty$ $\forall \vec{x} \in \mathbb{Z}^{+\infty}$: $\sum_{i=1}^{\infty} x_i = \infty$, $\forall\, \Gamma \in \mathcal{B}$: Γ is a set of μ – continuity, where μ is a probability measure on $\big(\mathbb{Z}^{+\infty}, \mathcal{B}\big)$, $\mu\left(\vec{x} : \sum_{i=1}^{\infty} x_i = \infty\right) = 1$.*

Denote $\gamma_i = \lambda_i / \lambda$ $\forall i \in \mathbb{N}$, $\sum_{i=1}^{\infty} \gamma_i = 1$. Suppose the Markov chain with the state space $\mathbb{N}$ and the transition matrix $\boldsymbol{P}$ to be homogeneous, aperiodic, and irreducible. Introduce $G(i,j) = \sum_{n=0}^{\infty} p_{ij}^{(n)}$ $\forall i, j \in \mathbb{N}$ where $p_{ij}^{(n)}$ is the probability to reach the node i from the node j in n steps.

Definition 2.2. *Matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}}$ is called positive recurrent or ergodic (resp. null - recurrent, transient) if a countable Markov chain with transition matrix $\boldsymbol{P}$ is positive recurrent (resp. null - recurrent, transient).*

In the case of positive recurrence there exists such a probability distribution $\pi = \big(\pi_i\big)_{i \in \mathbb{N}}$ (later on called *stationary*) that $\forall i \in \mathbb{N}$ $\pi_i = \sum_{j=1}^{\infty} \pi_j p_{ji}$ (briefly, $\vec{\pi} = \vec{\pi}\boldsymbol{P}$).

Definition 2.3. *Measure $\mu(\cdot)$ on $\mathbb{N}$ (possibly σ-finite) is called (strictly) excessive for transition matrix $\boldsymbol{P} = (p_{ij})_{i,j \in \mathbb{N}}$ iff $\mu \geq \mu\boldsymbol{P}$: $\forall i$ $\mu_i \geq \sum_{j=1}^{\infty} \mu_j p_{ji}$ ($\mu > \mu\boldsymbol{P}$, respectively).*

P r o p o s i t i o n 2.1 [Necessary conditions of ergodicity]. *If the network is ergodic, then matrix* $\boldsymbol{P}$ *is positive recurrent.*

P r o p o s i t i o n 2.2 [D. Khmelev, V. Oseledets (1998)]. *If* $\boldsymbol{P}$ *is positive recurrent,* $\sum\limits_{i=1}^{\infty} \frac{\pi_i}{\lambda_i} < \infty$*, then the network is not ergodic.*

L e m m a 2.1. *If* $\boldsymbol{P}$ *is ergodic and* $\{\gamma_i\}_{i=1}^{\infty} \neq \{\pi_i\}_{i=1}^{\infty}$ *or if* $\boldsymbol{P}$ *is not ergodic then*

$$\exists\, i_0,\, j_0: \quad \gamma_{i_0} > \sum_{j \neq i_0} \gamma_j p_{j i_0}, \quad \gamma_{j_0} < \sum_{j \neq j_0} \gamma_j p_{j j_0}$$

T h e o r e m 2.1 [Stochastical boundedness]. *If* $\gamma_{i_0} > \sum\limits_{j=1}^{\infty} \gamma_j p_{j i_0}$ *for some* i_0*, then* $X_{i_0}(\cdot)$ *is stochastically bounded.*

For some $\{a_i\}_{i=1}^{\infty}$, $a_i \geq 0$, $\sum\limits_{i=1}^{\infty} a_i < \infty$ and for all $\vec{x} \in \mathbb{Z}^{+^{\infty}}$ define $f(\vec{x}) = \sum\limits_{i=1}^{\infty} a_i x_i \leq \infty$. Let us introduce the family of σ – algebras $\mathcal{F}_n = \sigma\left(\vec{X}(m),\ m \leq n\right)\ \ \forall n$.

T h e o r e m 2.2 [Non-ergodic case: existence of supermartingales]. *If the matrix* $\boldsymbol{P}$ *is transient and there exist numbers* $\{\varphi_i\}_{i=1}^{\infty}$ *such that* $\varphi_i \geq 0$, $\sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{\infty} G(i,j)\varphi_j < \infty$*, then for* $a_i = \left(G\varphi\right)_i = \sum\limits_{j=1}^{\infty} G(i,j)\varphi_j$ *and initial distributions of Markov chain* $\boldsymbol{X}$ *such that* $\sum\limits_{i=1}^{\infty} a_i X_i(0) < \infty$ *a. s. and* $E\sum\limits_{i=1}^{\infty} a_i X_i(0) < \infty$ *sequence* $\left(f\big(\vec{X}(n)\big), \mathcal{F}_n\right)_{n=1}^{\infty}$ *will be a supermartingale, and hence there exists some random variable* $X_\infty : EX_\infty < \infty$ *such that*

$$\sum_{i=1}^{\infty} a_i X_i(n) \stackrel{\text{w.p.1}}{\longrightarrow} X_\infty \ \text{as}\ n \to \infty.$$

C o r o l l a r y 2.1 [Stochastical boundedness]. *Under conditions of theorem 2.2 all nodes of the network are stochastically bounded.*

P r o p o s i t i o n 2.3. *If* $\exists\, j_0 : \sum\limits_{i=1}^{\infty} G(i, j_0) < \infty$*, then there exist such numbers* $\{\varphi_i\}_{i=1}^{\infty} : \varphi_i \geq 0$ *that the conditions of theorem 2.2 hold.*

R e m a r k 2.1. *Lemma 2.1 states that strictly excessive probability measures* γ *do not exist for* $\boldsymbol{P}$ *if* $\lambda < \infty$*. But, if* $\lambda = \infty$ *and* $\boldsymbol{P}$ *is transient, then there exist infinitely many* $\sigma-$*finite strictly excessive measures* μ : $\sum\limits_{i=1}^{\infty} \mu_i = \infty$, $\mu > \mu\mathbf{P}$*. Hence if we define our process* $\{X_i(t) : i \in \mathbb{N},\ \ t \geq 0\}$ *properly in case* $\lambda = \infty$ *then for* $\{\lambda_i\}_{i=1}^{\infty} : \lambda_i > \sum\limits_{j \neq i} \lambda_j p_{ji}\ \ \forall i \in \mathbb{N}$ *all coordinates* $X_i(t)$ *will be stochastically bounded.*

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3. References

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