Markov Chains and Monte Carlo Simulation
Exercise Sheet 1
for the Exercises on April 24, 2012 from 12 - 14 in Room 120

Exercise 1 (6 points)

(a) Set up a transition matrix $P$ corresponding to the state diagram

(b) Draw a state diagram corresponding to the following transition matrix:

$$
P = \begin{pmatrix}
0.25 & 0.15 & 0.2 & 0.4 \\
0 & 0.5 & 0.5 & 0 \\
0 & 0 & 1 & 0 \\
0.3 & 0.4 & 0.1 & 0.2 \\
\end{pmatrix}
$$

(c) Look closely at state 3 in your diagram of part b). What do you notice is strange about the way information flows in state 3? What effect do you think this might have on the long-range behaviour of this system?

Exercise 2 (4 points)

Prove the following statement: If $P$ is a stochastic $\ell \times \ell$ matrix, then $P^k$ is a stochastic matrix for every $k \in \mathbb{N}$.
Exercise 3 (8 points)

Two players $A$ and $B$ gamble. Before the game starts, player $A$ has $a$ Euro and player $B$ has $b$ Euro (where $a$ and $b$ are some positive integers). A fair coin is thrown. If the coin shows heads, then player $A$ pays 1 Euro to player $B$ and vice versa. The game ends (i.e. no money is exchanged any more) as soon as one of the players has no money left. Let $X_n$ represent the amount of money that player $A$ has after the $n$th throw, $n \geq 0$.

(a) Define an appropriate state space $E$ for the random variables $X_0, X_1, \ldots$ and find the initial distribution $\alpha$. Next, give a recursive representation $X_n = \phi(X_{n-1}, \xi_n)$ (with iid random variables $\xi_n$ and $X_0 \sim \alpha$, where $X_0$ is independent of $\{\xi_n\}$) of the sequence $\{X_n\}$, i.e. determine $\phi$ and $\{\xi_n\}$ appropriately.

(b) Show that any sequence $\{X_n\}$ defined recursively by $X_n = \phi(X_{n-1}, \xi_n)$ (as in part a)) fulfills the Markov property

$$P(X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_n = i_n | X_{n-1} = i_{n-1})$$

for any $i_0, \ldots, i_n \in E$, $n \geq 1$ such that $P(X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) > 0$.

(c) Find the transition matrix of the Markov chain defined in part a).