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Dipl.-Math. Ole Stenzel Summer Term 2012

Markov Chains and Monte Carlo Simulation
Exercise Sheet 2

for the Exercises on May 8, 2012 from 12 - 14 in Room 120

Exercise 1 (3 points)

Let $\{X_n\}_{n\geq 0}$ be a homogeneous Markov chain with state space $E = \{1, 2, 3\}$ and transition matrix $\mathbf{P} = (p_{ij})$, where $p_{12} = p_{23} = p_{31} = 1$. The initial distribution is $\boldsymbol{\alpha} = (1/3, 1/3, 1/3)^\top$. Show that the following sequence $\{X_n\}$ does not define a Markov chain:

$$Y_n = \begin{cases} 0 & \text{if } X_n = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Exercise 2 (8 points)

A tarif system used in automobile insurance is the so-called *bonus-malus system*. Here we consider the following example of a stochastic model:

- There exist four different premium classes $C_1, \ldots, C_4$ with insurance rates EUR 450, 350, 300 and 250, respectively.
- In the first year the policyholder is sorted in class C_1.
- The class of the policyholder in the following period is determined by the class of the preceding period and the number of claims reported. If no claim is reported, his policy is upgraded to the next higher class. If one claim is reported in the current period, then he is downgraded by one level. Two reported claims cause a downgrade by two levels. If three or more claims are reported in one period, then the policyholder is sorted in class C_1.
- The number of claims reported in one period is modelled by a sequence of independent and identically Poi(λ)-random variables $\{\xi_n\}$ for $\lambda = 2/3$.

Let X_n represent the class of the policyholder in the $(n+1)$-th period, $n \in\geq 0$.

(a) Give a recursion equation for X_n of the form $X_n = \phi(X_{n-1}, \xi_n)$, $n \geq 1$.

(b) Determine the corresponding transition matrix.

(c) Compute the expected premium value of a typical policyholder in the fourth period.

Remark. The n-step transition matrix $\mathbf{P}^{(n)}$ is given by $\mathbf{P}^{(n)} = \mathbf{P}^n$.

Exercise 3 (6 points)

Consider a *cyclic random walk* $\{X_n\}_{n\geq 0}$ with state space $E = \{0, 1, \ldots, 9\}$ and (deterministic) initial state $X_0 = 5$. If $X_{n-1} = i$, then we put

$$X_n = \begin{cases} (i+1) \mod 10 & p = 1/2, \\ (i+9) \mod 10 & 1-p = 1/2. \end{cases}$$

Calculate the following (conditional) probabilities:

(a) $\mathbb{P}(X_3 \in \{2, 3, 4\})$,

(b) $\mathbb{P}(X_3 = 6 \mid X_5 = 6)$,

(c) $\mathbb{P}(X_5 = 6 \mid X_7 = 7, X_8 = 6)$.