

Prof. Dr. V. Schmidt Dipl.-Math. Ole Stenzel

Summer Term 2012

Markov Chains and Monte Carlo Simulation Exercise Sheet 3

for the Exercises on May 22, 2012 from $12\ \text{--}\ 14$ in Room 120

Exercise 1 (5 points)

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_l)^{\top}$ be an initial distribution and $\mathbf{P} = (p_{ij}), i, j \in E = \{1, \ldots, l\}$ a transition matrix. Furthermore, let ξ_0, ξ_1, \ldots be a sequence of i.i.d. random variables uniformly distributed on [0, 1]. Define the *E*-valued random variable X_0 by

$$X_0 = k \quad \Leftrightarrow \quad \xi_0 \in \left(\sum_{i=1}^{k-1} \alpha_i, \sum_{i=1}^k \alpha_i\right)$$

and the sequence X_1, X_2, \ldots recursively by $X_n = \phi(X_{n-1}, \xi_n)$, where $\phi : E \times [0, 1] \to E$ is given by

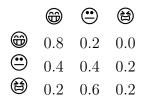
$$\phi(i,z) = \sum_{k=1}^{l} k \mathbb{1} \left\{ \sum_{j=1}^{k-1} p_{ij} < z \le \sum_{j=1}^{k} p_{ij} \right\}.$$

Verify that $\{X_n\}$ is a Markov chain with initial distribution $\boldsymbol{\alpha}$ and transition matrix **P**. **Exercise 2** (4 points)

Compute the *n*-step transition matrix $\mathbf{P}^{(n)}$ for $\mathbf{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$ with $p, q \in (0, 1]$. Hint: Use the spectral representation of the transition matrix \mathbf{P} , i.e. $\mathbf{P} = \Phi(\operatorname{diag}(\theta))\Phi^{-1}$, where Φ is the matrix of eigenvectors of \mathbf{P} and $\operatorname{diag}(\theta)$ is the diagonal matrix of the corresponding eigenvalues.

Exercise 3 (3 points)

Suppose your mood can be categorized in exactly three classes (good mood, even-tempered mood, bad mood) and is modelled by a Markov chain with the following transition probabilities:



Here, in the transition matrix $\mathbf{P} = (p_{ij})$ the symbol *i* stands for your mood today and *j* for your mood tomorrow.

Suppose now that you are in a good mood on 1^{st} of July. Compute the probability distribution of your mood on Christmas eve $(24^{th} \text{ of December})$. Hint: Use the spectral representation of the transition matrix \mathbf{P} , i.e. $\mathbf{P} = \Phi(\text{diag}(\theta))\Phi^{-1}$, where Φ is the matrix of eigenvectors of \mathbf{P} and $\text{diag}(\theta)$ is the diagonal matrix of the corresponding eigenvalues. It is allowed to use maths software to calculate eigenvectors, eigenvalues etc..

Exercise 4 (8 points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with finite state space $E = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

- (a) Is this Markov chain ergodic? Give a detailed proof of your statement.
- (b) Determine the limits $\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$, $i, j \in E$. Are these limits independent of *i*?