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Summer Term 2012

Markov Chains and Monte Carlo Simulation
Exercise Sheet 3

for the Exercises on May 22, 2012 from 12 - 14 in Room 120

Exercise 1 (5 points)

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_l)^\top$ be an initial distribution and $\mathbf{P} = (p_{ij})$, $i, j \in E = \{1, \ldots, l\}$ a transition matrix. Furthermore, let $\xi_0, \xi_1, \ldots$ be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$. Define the E-valued random variable X_0 by

$$X_0 = k \quad \Leftrightarrow \quad \xi_0 \in \left(\sum_{i=1}^{k-1} \alpha_i, \sum_{i=1}^{k} \alpha_i \right]$$

and the sequence $X_1, X_2, \ldots$ recursively by $X_n = \phi(X_{n-1}, \xi_n)$, where $\phi : E \times [0, 1] \to E$ is given by

$$\phi(i, z) = \sum_{k=1}^{l} k \mathbb{1} \left\{ \sum_{j=1}^{k-1} p_{ij} < z \leq \sum_{j=1}^{k} p_{ij} \right\}.$$

Verify that $\{X_n\}$ is a Markov chain with initial distribution $\boldsymbol{\alpha}$ and transition matrix $\mathbf{P}$.

Exercise 2 (4 points)

Compute the n-step transition matrix $\mathbf{P}^{(n)}$ for $\mathbf{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$ with $p, q \in (0, 1]$. Hint: Use the spectral representation of the transition matrix $\mathbf{P}$, i.e. $\mathbf{P} = \Phi(\mathrm{diag}(\theta))\Phi^{-1}$, where Φ is the matrix of eigenvectors of $\mathbf{P}$ and $\mathrm{diag}(\theta)$ is the diagonal matrix of the corresponding eigenvalues.

Exercise 3 (3 points)

Suppose your mood can be categorized in exactly three classes (good mood, even-tempered mood, bad mood) and is modelled by a Markov chain with the following transition probabilities:

	😁	😐	😫
😁	0.8	0.2	0.0
😐	0.4	0.4	0.2
😫	0.2	0.6	0.2

Here, in the transition matrix $\mathbf{P} = (p_{ij})$ the symbol i stands for your mood today and j for your mood tomorrow.

Suppose now that you are in a good mood on 1^{st} of July. Compute the probability distribution of your mood on Christmas eve (24^{th} of December). Hint: Use the spectral representation of the transition matrix $\mathbf{P}$, i.e. $\mathbf{P} = \Phi(\text{diag}(\theta))\Phi^{-1}$, where Φ is the matrix of eigenvectors of $\mathbf{P}$ and $\text{diag}(\theta)$ is the diagonal matrix of the corresponding eigenvalues. It is allowed to use maths software to calculate eigenvectors, eigenvalues etc..

Exercise 4 (8 points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a Markov chain with finite state space $E = \{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

(a) Is this Markov chain ergodic? Give a detailed proof of your statement.

(b) Determine the limits $\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}$, $i, j \in E$. Are these limits independent of i?