Prof. Dr. V. Schmidt
Dipl.-Math. Ole Stenzel

Summer Term 2012

Markov Chains and Monte Carlo Simulation
Exercise Sheet 4

for the Exercises on June 5, 2012 from 12 - 14 in Room 120

Exercise 1 (12 points)

A particle performs a random walk on the vertex* set $E = \{1, \dots, l\}$ of a connected graph G, which for simplicity we assume to have neither loops nor multiple edges. At each stage, the particle moves to a neighbour of its current position, each such neighbour chosen with equal probability. Let G have $\eta < \infty$ edges and X_n denote the position of the particle at step n.

(a) Is $\{X_n\}$ an ergodic Markov chain?

(b) Show that the stationary initial distribution $\pi^\top = (\pi_1, \dots, \pi_l)$ is given by $\pi_i = d_i/2\eta$, where d_i is the degree of vertex $i \in E$.

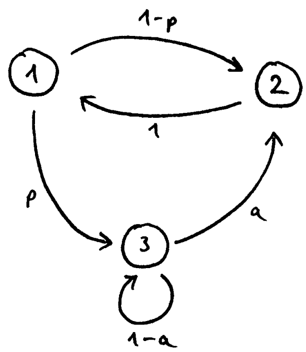
(c) A stationary Markov chain is called *reversible in equlibrium* if the transition matrix (p_{ij}) and the limit distribution π satisfy $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in E$. Show that the Markov chain $\{X_n\}$ given in this exercise is reversible in equilibrium.

(d) The *mean recurrence time of state i* (i.e. the expected time for the Markov chain $\{X_n\}$ to return to this state) is given by $\mu_i = 1/\pi_i$. A chess piece performs a random walk on a chessboard, where at each step it performs one of the available moves with the same probability. What is the mean recurrence time of a corner square if the piece is a i) king ii) queen iii) bishop iv) knight v) rook?

*: You can recall the introductory definitions of `vertex`, `edge`, `loop`, `neighbour`, `degree` and `connected graph` in the book 'Graph Theory' by R. Diestel.
See http://diestel-graph-theory.com/GrTh.html (English).

Exercise 2 (8 points)

Consider a Markov chain $\{X_n\}$ with three states, where the transition probabilities are given in the following state diagram:



(a) Under which conditions on a and p is the Markov chain irreducible and/or aperiodic?

(b) Determine the limit distribution $\pi^\top = (\pi_1, \pi_2, \pi_3)$ in the ergodic case.

(c) For which a and p is $\pi^\top$ the uniform distribution?

(d) Determine the mean recurrence time of state 2, i.e. $\mathbf{E}\{\tau_2 \mid X_0 = 2\}$ with $\tau_2 = \min\{n > 0 : X_n = 2\}$.

Exercise 3 (8 points)

Consider a birth and death process with two reflecting barriers, i.e. $E = \{0, 1, \dots, l\}$ and

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & & & & & \\ q_1 & r_1 & p_1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & q_i & r_i & p_i & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & q_{l-1} & r_{l-1} & p_{l-1} \\ & & & & & 1 & 0 \end{pmatrix},$$

where $q_i > 0$ and $p_i > 0$ for $i = 1, \dots, l-1$.

(a) Check that the vector $\alpha^T = (\alpha_0, \dots, \alpha_l)$ with

$$\alpha_i = \alpha_0 \frac{p_1 p_2 \cdot \ldots \cdot p_{i-1}}{q_1 q_2 \cdot \ldots \cdot q_i}, \qquad i = 1, \dots, l$$

and

$$\alpha_0 = \left(1 + \frac{1}{q_1} + \frac{p_1}{q_1 q_2} + \ldots + \frac{p_1 p_2 \cdot \ldots \cdot p_{l-1}}{q_1 q_2 \cdot \ldots \cdot q_l}\right)^{-1}$$

satisfies the system of linear equations $\alpha^\top = \alpha^\top \mathbf{P}$.

(b) Show that the given Markov chain is irreducible and aperiodic if $r_i > 0$ for some $i \in \{0, \dots, l\}$. What happens if $r_i = 0$ for all $i \in \{0, \dots, l\}$?