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Summer Term 2012

Markov Chains and Monte Carlo Simulation
Exercise Sheet 5

for the Exercises on June 19, 2012 from 12 - 14 in Room 120

Exercise 1 (4 points)

Let $P = (p_{ij})_{i,j=1,\ldots,N}$ be the transition matrix of a random walk on the N-cycle ($N \geq 3$). Find the smallest value of n such that $p_{ij}^{(n)} > 0$ for all states i and j. *Hint: You can find a detailed definition of the term N-cycle in the book 'Graph Theory' of R. Diestel.*

Exercise 2 (8 points)

Let $p, q \in [0, 1]$ be arbitrary fixed numbers. For each of the following matrices, check if it can be considered as the transition matrix of a reversible Markov chain with stationary initial distribution $\boldsymbol{\alpha}$, where $\alpha_i > 0$ for all $i \in E$.

(a)
$$P = \begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$$

(b)
$$P = \begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$$

(c) On $E = \{0, 1, 2, \ldots\}$ let $p_{01} = 1$, $p_{i,i+1} = p$, $p_{i,i-1} = q$ for $i \geq 1$ and $p_{ij} = 0$ otherwise

(d) $p_{ij} = p_{ji}$ for $i, j \in \{1, \ldots, l\}$

Hint: Use the fact that the uniform distribution on $E = \{1, \ldots, l\}$ is always a stationary initial distribution of a Markov chain with doubly stochastic transition matrix.

Exercise 3 (12 points)

Let $\{X_n\}$ be an ergodic Markov chain with transition matrix $\mathbf{P}$ and limit distribution $\pi^\top = (\pi_1, \ldots, \pi_l)$. Denote by $\tau_j^+ = \inf\{n \geq 1 : X_n = j\}$ the time of the first visit to j (called the *first-hitting time*), and define $\mu_{ij} = \mathbb{E}\{\tau_j^+ \mid X_0 = i\}$. Furthermore, write $\mathbf{E} = (1)_{i,j=1,\ldots,l}$. Verify the following claims:

(a) Let $\mathbf{M} := (\mu_{ij})_{i,j=1,\ldots,l}$. Then $\mathbf{M}$ can be written as $\mathbf{M} = \mathbf{P}(\mathbf{M} - \mathbf{M}_{\text{diag}}) + \mathbf{E}$, where $\mathbf{M}_{\text{diag}} = \text{Diag}(\mu_{11}, \ldots, \mu_{ll})$.

(b) For all states $i \in \{1, \ldots, l\}$ it holds that $\mu_{ii} = \pi_i^{-1}$.

(c) We assume that there exists exactly one matrix $\mathbf{M}$ that satisfies the equality of part (a). The matrix $\mathbf{M}$ of mean first-hitting times is then given by

$$\mathbf{M} = (\mathbf{I} - \mathbf{Z} + \mathbf{E}\mathbf{Z}_{\text{diag}})\mathbf{D}$$

with $\mathbf{Z} = (\mathbf{I} - (\mathbf{P} - \mathbf{\Pi}))^{-1} = \mathbf{I} + \sum_{n=1}^{\infty}(\mathbf{P}^n - \mathbf{\Pi})$, $\mathbf{D} = \text{Diag}(\frac{1}{\pi_1}, \ldots, \frac{1}{\pi_n})$ and $\mathbf{\Pi} = \mathbf{D}^{-1}\mathbf{E}$. *Hint: Show that* $(\mathbf{I} - \mathbf{P})\mathbf{Z} = \mathbf{I} - \mathbf{\Pi}$.

Exercise 4 (Monte Carlo integration) (6 points)

Consider the function $f : [0,1] \to \mathbb{R}$, $f(x) = (\cos(50x) + \sin(20x))^2$. Calculate the integral $F = \int_0^1 f(x)\,dx$ by Monte-Carlo simulation: For every number $n \in [1, 10.000]$ calculate the approximation $\hat{F}_n = \frac{1}{n}\sum_{j=1}^{n} f(u_j)$ and plot the values $\hat{F}_1, \ldots, \hat{F}_{10.000}$ together with a horizontal line showing the exact value F.