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Markov Chains and Monte Carlo Simulation
Exercise Sheet 7

This exercise sheet will not be marked.

Bitte bis zum 11.07. im Hochschulportal für die Vorleistung anmelden (erste oder zweite Klausur).

Exercise 1

For each of the following densities, write down an algorithm based on the acceptance-rejection method to generate pseudorandom numbers according to each of the given distributions - call it G (with probability function $q = (q_1, \dots, q_{100})$ and density $g(y)$ respectively). Assume that the only available random number generator produces $U(0,1)$-pseudo random numbers.

(a) $q_j = a/j$, $j = 1, \dots, 100$ with $a = (\sum_{j=1}^{100} q_j)^{-1}$,

(b) $g(y) = \frac{1}{10} y^2 + \frac{7}{15} \mathbb{1}\{y \in (-1,1)\}$,

Hint: With regard to part (b): The auxiliary distribution F, from which candidates for the realizations of G are drawn, should be simple to generate, but not too far-off from the desired distribution.

Exercise 2 (Generalized hard core model)

Consider the hard core model introduced in Section 3.3.1 of the lecture notes, but now in 2D. Let the number of ones in a valid configuration $\mathbf{x}$ be weighted with a parameter $\lambda > 0$, i.e. one considers the probability function $\pi_\lambda = \{\pi_{\mathbf{x},\lambda},\ \mathbf{x} \in E\}$ with

$$\pi_{\mathbf{x},\lambda} = \frac{\lambda^{n(\mathbf{x})}}{l_\lambda} \quad \forall\, \mathbf{x} \in E,$$

where $n(\mathbf{x})$ denotes the number of ones in $\mathbf{x}$ and $l_\lambda = \sum_{\mathbf{x} \in E} \lambda^{n(\mathbf{x})}$.

(a) Determine the conditional probability $\pi_{1|\mathbf{x}(-v)}$ for $v \in V$, where

$$\pi_{x(v)|\,\mathbf{x}(-\mathbf{v})} = P\Big(X(v) = x(v) \mid \mathbf{X}(-\mathbf{v}) = \mathbf{x}(-\mathbf{v})\Big) \qquad (1)$$

- denote the conditional probability that the component $X(v)$ of $\mathbf{x}$ has the value $x(v)$
- given that the vector $\mathbf{X}(-\mathbf{v}) = (\mathbf{X}(\mathbf{w}),\ \mathbf{w} \in \mathbf{V} \setminus \{\mathbf{v}\})$ of the other components equals $\mathbf{x}(-\mathbf{v})$ where we assume $(x(v), \mathbf{x}(-\mathbf{v})) \in E$.

(b) Construct a Gibbs sampler for the generalized hard core model.

(c) Let V be an 8×8-grid in two dimensions (use the 8-neighbourhood for your experiments). Let Y_λ be the random number of vertices having value 1 as the probability function π_λ is used. Estimate the mean value $\mathbb{E}[Y_\lambda]$ in the following way: Generate $n\cdot k = 10^6$ steps of your Gibbs sampler ($n = 1000$) and put every n-th value into your sample. Provide estimates for $\mathbb{E}[Y_\lambda]$ putting $\lambda = 0.1, 1.0$ and 5.0, respectively.

Exercise 3 (will not be presented in tutorial)

Let $\{X_n\}_{n\geq 0}$ be a homogeneous Markov chain with state space $E = \{1,\ldots,l\}$ and transition matrix $\mathbf{P}$. Show that the Markov chain is ergodic if and only if $\mathbf{P}$ is quasi-positive.

Exercise 4 (Markov chain with countable state space. Example: queues)

Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space $E = \{0,1,2,\ldots\}$, where $X_0 = 0$ and

$$X_n = \max\{0, X_{n-1} + Z_n - 1\}\,,\ n \geq 1,$$

where the random variables $Z, Z_1, Z_2, \ldots$ are independent and identdically distributed and the transition matrix is given by

$$p_{ij} = \begin{cases} P(Z = j+1-i) & \text{if } j+1 \geq i > 0 \text{ or } j > i = 0, \\ P(Z=0) + P(Z=1) & \text{if } j = i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $\{X_n\}_{n\geq 0}$ is irreducible and aperiodic if

$$P(Z=0) > 0,\ P(Z=1) > 0, \quad \text{and } P(Z=2) > 0.$$

(b) Show that

$$X_n = \max\left(0, \max_{k\in\{1,\ldots,n\}} \sum_{r=k}^{n}(Z_r - 1)\right) \stackrel{d}{=} \max\left(0, \max_{k\in\{1,\ldots,n\}} \sum_{r=1}^{k}(Z_r - 1)\right).$$

(c) Show that the so-called negative drift condition, i.e., $\mathbf{E}(Z-1) < 0$, together with the conditions given in part (a), implies that $\mathbf{P} = (p_{ij})$ is ergodic, i.e., the equation $\alpha^T = \alpha^T\mathbf{P}$ has a uniquely determined probability solution $\alpha^T = (\alpha_0, \alpha_1, \ldots)$ which coincides with the limit distribution $\pi^T = (\pi_0, \pi_1, \ldots) = \lim_{n\to\infty} \alpha_n^T$, where $\pi_i > 0$ for each $i \geq 0$ and the limit distribution π does not depend on the choice of the initial distribution α_0.

(d) Show that the generating function $g_\pi : (-1,1) \to [0,1]$, where $g_\pi(s) = \sum_{i=0}^{\infty} s^i \pi_i$, is given by

$$g_\pi(s) = \frac{(1-\rho)(1-s)}{g_Z(s) - s} \quad \forall s \in (-1,1),$$

where $\rho = \mathbb{E}[Z]$ and $g_Z(s) = \mathbb{E}[s^Z]$.

Hint: the Markov chain can be interpreted as a queueing system, where in each time step ($n \to n+1$), one customer is served and Z_n new customers arrive.

Exercise 5 (will not be presented in tutorial)

- Let $m \geq 1$ and let $F, G : \mathbb{R}^m \to [0,1]$ be distribution functions (of m–dimensional random vectors) such that

$$g(\mathbf{x}) \leq c \text{ for some } c > 0 \qquad \text{and} \qquad G(\mathbf{y}) = \int_{(-\infty,\mathbf{y}]} g(\mathbf{x})\, dF(\mathbf{x})\,, \qquad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^m\,. \quad (2)$$

- Let $(U_1, \mathbf{X}_1), (U_2, \mathbf{X}_2), \ldots$ be a sequence of independent and identically distributed random vectors whose components are independent. Furthermore, let U_i be a $(0,1]$–uniformly distributed random variable and $\mathbf{X}_i$ be distributed according to F.
- Show that the random variable

$$I = \min\left\{k \geq 1 : U_k < \frac{g(\mathbf{X}_k)}{c}\right\} \qquad (3)$$

  is geometrically distributed with expectation c, i.e. $I \sim \mathrm{Geo}(c^{-1})$, and the random vector $\mathbf{Y} = \mathbf{X}_I$ is distributed according to G.

Exercise 6 (Uniform distribution on bounded Borel sets)

- Let the random vector $\mathbf{X} : \Omega \to \mathbb{R}^m$ (with distribution function F) be uniformly distributed on the cube $(-1,1]^m$ and let $B \in \mathcal{B}((-1,1]^m$ be an arbitrary Borel subset of $(-1,1]^m$ of positive Lebesgue measure $|B|$, where $m \geq 1$ is an arbitrary fixed integer.
- Show that distribution function $G : \mathbb{R}^m \to [0,1]$ of the uniform distribution on B is absolutely continuous with respect to F and determine the (Radon–Nikodym) density.
- Write down an algorithm (using the result of Exercise 5) to generate pseudo–random vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots$ according to the uniform distribution on B.

Exercise 7

Use the Metropolis algorithm to construct the transition matrix $\mathbf{P}$ of a Markov chain with state space $E = \{0, 1 \ldots, l\}$ and (ergodic) limit distribution $\pi^T = (\pi_0, \ldots, \pi_l)$ with

$$\pi_i = \frac{\mu^i \exp(-\mu)}{z \cdot i!}, \quad \forall\, i \in E,$$

where $\mu \in (0,1)$ and $z \in (0, \infty)$ is a normalizing constant. Hint: A potential successor of state i should be chosen uniformly on $\{i-1, i+1\}$ (set $-1 = 0$, $l + 1 = l$).