Dipl.-Math. oec. Florian Timmermann Winter Term 2009/2010

Random Fields II
Exercise Sheet 1

for the Exercises on November 13 from 12.00 - 2.00pm in Room 220

Exercise 1 There are several definitions for a stable random variable. Namely, a random variable X is said to have a stable distribution, if one of the following properties holds:

(a) For each $n \in \mathbb{N}$ there exist constants $c_n > 0$, $d_n \in \mathbb{R}$ such that $X_1 + \ldots + X_n \overset{d}{=} c_n X + d_n$, where $X_1, \ldots, X_n$ are independent copies of X.

(b) There are parameters $\alpha \in (0, 2]$, $\sigma \geq 0$, $\beta \in [-1, 1]$ and $\mu > 0$ such that the characteristic function $\varphi(t) = \mathbf{E} \exp(itX)$ of X has the form

$$\varphi(t) = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha (1 - i\beta \operatorname{sgn}(t) \tan \frac{\pi\alpha}{2}) + i\mu t\right) & \text{if } \alpha \neq 1 \\ \exp\left(-\sigma |t| (1 + i\beta \frac{2}{\pi} \operatorname{sgn}(t) \log |t|) + i\mu t\right) & \text{if } \alpha = 1. \end{cases}$$

Show that b) implies a). (In fact the two definitions are equivalent.)

Exercise 2 *(The parameters of a stable random variable)* According to the upper definition we can denote stable distributions by $S_\alpha(\sigma, \beta, \mu)$. The parameter α is often called the *index of stability* because it can be shown that the norming constants c_n in Exercise 1 are of the form $c_n = n^{1/\alpha}$ with $\alpha \in (0, 2]$ (see Feller, 1967).

(a) The parameter μ is a *shift parameter*. This is backed by the following property. Let $X \sim S_\alpha(\sigma, \beta, \mu)$ and let $c \in \mathbb{R}$ be a constant. Show that $X + c \sim S_\alpha(\sigma, \beta, \mu + c)$.

(b) The parameter σ is called the *scale parameter*. The reason lies in the following: Let $X \sim S_\alpha(\sigma, \beta, \mu)$ and let $c \in \mathbb{R} \setminus \{0\}$. Show that

$$cX \sim \begin{cases} S_\alpha(|c|\sigma, \beta \operatorname{sgn}(c), c\mu) & \text{if } \alpha \neq 1, \\ S_1(|c|\sigma, \beta \operatorname{sgn}(c), c\mu - \frac{2}{\pi} t \log |t| \sigma\beta) & \text{if } \alpha = 1. \end{cases}$$

(c) The parameter β is a *skewness parameter*. Show that, for any $0 < \alpha < 2$ it holds

$$X \sim S_\alpha(\sigma, \beta, 0) \text{ if and only if } -X \sim S_\alpha(\sigma, -\beta, 0).$$

Exercise 3

(a) Show that $X \sim S_\alpha(\sigma, \beta, \mu)$ is symmetric about zero if and only if $\beta = 0$ and $\mu = 0$.

(b) Show that the random variable $X \sim N(\mu, \sigma^2)$ is stable with $\alpha = 1/2$.

(c) Show that the Cauchy random variable X with probability density function $f_X(x) = \frac{\gamma}{\pi(x^2+\gamma^2)}$, $\gamma > 0$ is stable. What is the value of α?

Exercise 4

(a) Find a simple formula for the variogram $\gamma(s,t)$ of a random field $X = \{X(t),\, t \in \mathbb{R}^d\}$ with mean value function $\mu(t)$ and covariance function $C(s,t)$, $s,t \in \mathbb{R}^d$, $d \geq 1$.

(b) Now assume that X, defined on a compact $T \subset \mathbb{R}^d$ and $\operatorname{diam} T := \sup_{s,t\in\mathbb{R}^d} d(s,t) < \infty$ with the canonical pseudo-metric $d(s,t) := \sqrt{2\gamma(s,t)} < \infty$. The smoothness of the random field X is closely related to the behaviour of $\gamma(s,t)$ for points $s,t \in \mathbb{R}^d$ with infinitesimal small distance. We now try to find a sufficient condition for the a.s. continuity of the field X. It is known that Gaussian random fields are a.s. continuous and bounded with probability one if there exists a $\delta > 0$ such that

$$\int_0^\delta \sqrt{-\log u}\, dp(u) < \infty,$$

where $p^2(u) = \sup_{|s-t|\leq u} 2\gamma(s,t)$. Show that the existence of a constant $K \in (0,\infty)$ such that for all $\alpha, \eta > 0$ with $|s-t| < \eta$ it holds

$$\gamma(s,t) \leq \frac{K}{|\log|s-t||^{1+\alpha}}$$

is a sufficient condition for the a.s. continuity of X.