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Prof. Dr. Evgeny Spodarev
Dipl.-Math. oec. Florian Timmermann

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Random Fields II
Exercise Sheet 2

for the Exercises on November 20 from 12.00 - 2.00pm in Room 220

Exercise 1 Let ν be a finite measure, given on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $d \geq 1$. Let $Y, X_1, X_2, \ldots$ be independent random elements on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $Y \sim \text{Poi}(\nu(\mathbb{R}^d))$ and $X_1, X_2, \ldots$ i.i.d. d-dimensional $\mathcal{F}|\mathcal{B}(\mathbb{R}^d)$-measurable random vectors such that $\mathbf{P}_{X_1}(B) = \nu(B)/\nu(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$. We introduce the *Poisson random measure* $W : \mathcal{B}(\mathbb{R}^d) \times \Omega$ by

$$W(B, \omega) = \sum_{j=1}^{Y(\omega)} \mathbf{1}_B(X_j(\omega)), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Show that the Poisson random measure given on a semiring of bounded Borel sets in $\mathbb{R}^d$ is an orthogonal non-centered random measure.

Exercise 2 Let $\{W_t,\, t \in \mathbb{R}\}$ be a complex-valued L^2-process such that
i) $\mathbf{E}|W_s - W_t|^2 \to 0$ for any $s \in \mathbb{R}$ with $s \downarrow t$,
ii) it has independent increments, i.e. $\mathbf{E}(W_{t_2} - W_{t_1})\overline{(W_{t_3} - W_{t_2})} = 0$ for any $t_1 < t_2 < t_3$.
We introduce the family of random variables $W((a, b]) := W(b) - W(a)$ on the semiring $\mathcal{K} = \{(a, b],\, -\infty < a \leq b < \infty\}$, where $(a, a] = \emptyset$. Show that W is an orthogonal random measure on $\mathcal{K}$.

Exercise 3 Let E be a measurable space, and ν a σ-finite measure given on a semiring $\mathcal{K}(E)$ of subsets of E. Show that simple functions of the form $f : E \to \mathbb{C}$, $f = \sum_{i=1}^m c_i \mathbf{1}_{B_i}$, where $c_i \in \mathbb{C}$, $B_i \in \mathcal{K}(E)$, $i = 1, \ldots, m$, $\bigcup_{i=1}^m B_i = E$, $B_i \cap B_j = \emptyset$ for $i \neq j$, are dense in $L^2(E, \nu)$.

Exercise 4 Show Lemma 2.3.4 4): Let E be a measurable space, and ν a σ-finite measure given on a semiring $\mathcal{K}(E)$ of subsets of E. Let W be a centered orthogonal random measure, i.e. $\mathbf{E}W(A) = 0$, $A \in \mathcal{K}(E)$. Show that $\mathbf{E}J(f) = 0$ for $f \in L^2(E, \nu)$.