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Winter Term 2009/2010

Random Fields II
Exercise Sheet 3

for the Exercises on November 27 from 12.00 - 2.00pm in Room 220

Exercise 1 *An illustrative example of the Karhunen Theorem (1)*

(a) Show that the Wiener process $W = \{W(t),\ t \in [0, 2\pi]\}$ has the representation

$$W(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{1 - e^{-ikt}}{ik} z_k, \quad t \in [0, 2\pi],$$

where the z_k are uncorrelated centered random variables with unit variance and the series converges in the mean-square sense for every $t \in [0, 2\pi]$ (for $k = 0$ we set $(1 - e^{-ikt})/ik = -t$). Use the following steps:

Step 1: Determine the values $f(t, k)$, $t \in [0, 2\pi]$, $k \in \mathbb{Z}$ of the function f in the representation of the covariance function of the stochastic process, which are given as the coefficients of the Fourier series of $\mathbf{1}_{[0,t]}(u)$.

Step 2: Determine the value of the covariance function $C(s, t)$, $s, t \in [0, 2\pi]$, which arises by taking the space $E = \mathbb{Z}$, the counting measure ν on $\mathbb{Z}$, i.e. $\nu(\{k\}) = 1$, $k \in \mathbb{Z}$ and the function f from step 1. *Hint: Apply the Parseval equality.*

Step 3: Apply the Karhunen Theorem.

Exercise 2 *An illustrative example of the Karhunen Theorem (2)*
Show that the Wiener process $W = \{W(t),\ t \in [0, 1]\}$ has the representation

$$W(t) = \sum_{k=1}^{\infty} S_k(t) z_k, \quad t \in [0, 1],$$

where the $S_k(t)$, $t \in [0, 1]$, $k \geq 1$ are the *Schauder functions* and the z_k are uncorrelated centered random variables with unit variance and the series converges in the mean-square sense for every $t \in [0, 1]$. Use the same steps as in the preceding exercise.