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Winter Term 2009/2010

Random Fields II
Exercise Sheet 4

for the Exercises on December 4 from 12.00 - 2.00pm in Room 220

Exercise 1 For a covariance function $C(s,t)$, which is continuous on $[a,b] \times [a,b]$ we introduce the bounded linear *Fredholm operator* $A : L^2[a,b] \to L^2[a,b]$ by

$$Af(s) = \int_a^b C(s,t) f(t)\, dt, \quad f \in L^2[a,b].$$

The function $C = C(s,t)$ is called *integral kernel of the operator* A. Show that this operator is compact.

Exercise 2 Let W be a Gaussian white noise based on Lebesgue measure, and use it to define a random field on $\mathbb{R}_+^d = \{(t_1, ..., t_d) : t_i \geq 0\}$ by setting

$$W(t) = W([0,t]),$$

where $[0,t]$ is the rectangle $\prod_{i=1}^d [0, t_i]$. W_t is called the Brownian sheet on $\mathbb{R}_+^d$, or *multiparameter Brownian motion.* If $d = 1$, it is the standard Brownian motion.

(a) Show that W is a centered Gaussian field on $\mathbb{R}_+^N$ with covariance

$$\mathbb{E}(W_s W_t) = \min(s_1, t_1) \times \cdots \times \min(s_d, t_d).$$

(b) Suppose $d > 1$, and fix $d-k$ of the index variables t_i. Show that W is a scaled k-parameter Brownian sheet in the remaining variables.

(c) Find the Karhunen-Loève expansion for W on $[0,1]^d$.

Exercise 3 Let $T \in \mathbb{N}$ and $X = \{X(t), t \in [0,T]\}$ be a real-valued process on $[0,T]$ with $\mathbb{E}(X(t)) = 0$ and $\mathbb{E}(X(t)^2) < \infty$ for all $t \in [0,T]$.

(a) Suppose that $C(s,t) = \cos(2\pi(t-s))$. Show that the Karhunen-Loève expansion has only two summands and determine the terms of the expansion.

(b) Suppose that $C(s,t) = (1 - |t-s|)\mathbb{1}\{0 \leq t-s \leq 2T\}$. Determine the Karhunen-Loève expansion.

(c) Suppose that $C(s,t) = \sum_{n=0}^{\infty} \frac{1}{1+n^2} \cos\left(n \frac{2\pi}{T}(t-s)\right)$. Find the eigenvalues and eigenfunctions of the Fredholm operator A from Exercise 1.