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Random Fields II
Exercise Sheet 5

for the Exercises on December 11 from 12.00 - 2.00pm in Room 220

Exercise 1

Let $\{X(t) : t \in [0,1]\}$ be the Brownian motion with covariance function $C(s,t) = \min\{s,t\}$ for $s,t \in [0,1]$. Show that

$$H = \{g : [0,1] \to \mathbb{R} \text{ continuous} : g(0) = 0,\ \exists\ g'(t) \text{ for almost all } t \in [0,1], \int_0^1 (g'(t))^2 < \infty\}$$

is the corresponding reproducing kernel Hilbert space.

Exercise 2

A stochastic process $\{B(t), t \in [0,1]\}$ is called a *Brownian bridge* if

- the joint distribution of $B(t_1), B(t_2), ..., B(t_n)$, $t_1, ..., t_n \in [0,1]$, $n = 1, 2, ...$ is Gaussian with $\mathbb{E}(B(t)) = 0$ for all $t \in [0,1]$,
- the covariance function of $B(t)$ is given by
$$C(s,t) = \min\{s,t\} - st,$$
- the sample path function of $B(t,w)$ is continuous in t with probability one.

(a) Find the Karhunen-Loève expansion of the Brownian bridge.

(b) Show that $B(t) = W(t) - tW(1) = (1-t)W\left(\frac{t}{1-t}\right)$, $t \in [0,1]$, where $\{W(t) : t \in [0,1]\}$ is the Brownian motion.

(c) Find a representation of the Brownian bridge based on the Karhunen-Loève expansion of the Brownian motion. Compare this representation with the Karhunen-Loève expansion of the Brownian bridge.

Exercise 3

A stochastic process $\{X(t), t \geq 0\}$ is called *Ornstein-Uhlenbeck process* if it is a centered Gaussian process with covariance function $C(s,t) = \exp\{-\alpha|t-s|\}$, $s,t \geq 0$, $\alpha > 0$. Show that X is a transformed Brownian motion and determine the corresponding (transformed) Karhunen-Loève expansion.