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Random Fields II
Exercise Sheet 6

for the Exercises on December 18 from 12.00 - 2.00pm in Room 220

Exercise 1 Let $X = \{X_t,\, t \in \mathbb{R}\}$ be a random polynomial with $X_t = Y_0 + Y_1 t + \ldots + Y_n t^n$, $Y_i \sim \mathcal{N}(0,1)$ i.i.d., $i = 0, \ldots, n$. Determine the expected value, the variance, the covariance function and the characteristic function of X_t.

Exercise 2

(a) Calculate the spectral density of a random process $X = \{X_t,\, t \in \mathbb{R}\}$ with

- triangular covariance function $C(h) = (1 - |h|)\mathbf{1}\{|h| \le 1\}$,
- covariance function $C(h) = \frac{1}{8} e^{-|h|} (3\cos(h) + \sin|h|)$.

(b) Show that the spectral density of the Ornstein-Uhlenbeck process with covariance function $C(s,t) = \exp\{-\alpha|t-s|\}$, $s,t \in \mathbb{R}$, $\alpha > 0$, is given by

$$f(u) = \frac{1}{\pi} \frac{\alpha}{1 + \alpha^2 u^2}, \quad u \in \mathbb{R},$$

and verify that it can be interpreted as a Gamma(1/2)-mixture of Gaussian distributions, i. e.

$$f(u) = \int_0^\infty \left(\frac{t\alpha^2}{\pi} \right)^{\frac{1}{2}} e^{-t\alpha^2 u^2} \frac{1}{\sqrt{\pi}} e^{-t} t^{\frac{1}{2}-1} dt, \quad u \in \mathbb{R}.$$

(c) Calculate the spectral density of a random field $X = \{X_t,\, t \in \mathbb{R}^2\}$ with Gaussian covariance function $C(h) = e^{-\|h\|_2^2}$.

Exercise 3 Let T be a separable and compact Hausdorff space. Let $\mathcal{H}$ be a Hilbert space of functions $g : T \to \mathbb{R}$ which admits a reproducing kernel. Show that the reproducing kernel is uniquely determined by the Hilbert space $\mathcal{H}$.

Exercise 4 Show that under the same assumptions of Exercise 3, the reproducing kernel of a reproducing kernel Hilbert space is positive semi-definite.