Exercise 1 Let \( X = \{X_t, t \in \mathbb{R}\} \) be a random polynomial with \( X_t = Y_0 + Y_1 t + \ldots + Y_n t^n, \) \( Y_i \sim \mathcal{N}(0, 1) \) i.i.d., \( i = 0, \ldots, n. \) Determine the expected value, the variance, the covariance function and the characteristic function of \( X_t. \)

Exercise 2

(a) Calculate the spectral density of a random process \( X = \{X_t, t \in \mathbb{R}\} \) with
- triangular covariance function \( C(h) = (1 - |h|) 1\{|h| \leq 1\}, \)
- covariance function \( C(h) = \frac{1}{8} e^{-|h|} (3 \cos(h) + \sin|h|). \)

(b) Show that the spectral density of the Ornstein-Uhlenbeck process with covariance function \( C(s, t) = \exp\{-\alpha|t-s|\}, s, t \in \mathbb{R}, \alpha > 0, \) is given by
\[
f(u) = \frac{1}{\pi} \frac{\alpha}{1 + \alpha^2 u^2}, \quad u \in \mathbb{R},
\]
and verify that it can be interpreted as a Gamma(1/2)-mixture of Gaussian distributions, i. e.
\[
f(u) = \int_0^\infty \left(\frac{t \alpha^2}{\pi}\right)^{\frac{1}{2}} e^{-t \alpha^2 u^2} \frac{1}{\sqrt{\pi}} e^{-t \frac{1}{2}} dt, \quad u \in \mathbb{R}.
\]

(c) Calculate the spectral density of a random field \( X = \{X_t, t \in \mathbb{R}^2\} \) with Gaussian covariance function \( C(h) = e^{-||h||^2_2}. \)

Exercise 3 Let \( T \) be a separable and compact Hausdorff space. Let \( \mathcal{H} \) be a Hilbert space of functions \( g : T \to \mathbb{R} \) which admits a reproducing kernel. Show that the reproducing kernel is uniquely determined by the Hilbert space \( \mathcal{H}. \)

Exercise 4 Show that under the same assumptions of Exercise 3, the reproducing kernel of a reproducing kernel Hilbert space is positive semi-definite.