Asset allocation, sustainable withdrawal, longevity risk and non-exponential discounting

The paper published in Insurance: Mathematics and Economics containing the Appendix with the proofs of the results

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Abstract

The present paper studies an optimal withdrawal and investment problem for a retiree who is interested in sustaining her retirement consumption above a pre-specified minimum consumption level. Apparently, the withdrawal and investment policy depends substantially on the retiree's health condition and her time preferences (subjective discount factor). We assume that the health of the retiree can worsen or improve in an unpredictable way over her lifetime and model the retiree's mortality intensity by a stochastic process. In order to make the decision about the consumption and investment policy more realistic, we assume that the retiree applies a non-exponential discount factor (an exponential discount factor with a small amount of hyperbolic discounting) to value her future income. In other words, we consider an optimization problem by combining four important aspects: asset allocation, sustainable withdrawal, longevity risk and non-exponential discounting. Due to the non-exponential discount factor, we have to solve a time-inconsistent optimization problem. We derive a non-local HJB equation which characterizes the equilibrium optimal investment and consumption strategy. We establish the first-order expansions of the equilibrium value function and the equilibrium strategies by applying expansion techniques. The expansion is performed on the parameter controlling the degree of discounting in the hyperbolic discounting that is added to the exponential discount factors. The first-order equilibrium investment and consumption strategies can be calculated in a feasible way by solving PDEs.

Keywords: Hyperbolic discounting, time-inconsistent optimization problem, non-local HJB equation, equilibrium strategies, PDE.

JEL: C6, G1, D9.

1 Introduction

The core objective of most retirees is to sustain her working life's living standard after retirement. In order to achieve this goal, a retiree needs to know how to invest the accumulated assets on financial markets and how to withdraw retirement income from the accumulated assets. Too little withdrawal might not ensure the desired living standard, while too much withdrawal might lead to an early exhaustion of the accumulated funds. Apparently, the investment and withdrawal policy depends on diverse factors, the two most important ones being individual longevity risk and the time preferences of the retiree (i.e. the retiree's subjective discount factor).

In this paper, we study an optimal consumption and investment problem for a retiree. The retiree buys a lifetime annuity providing a certain future income and invests the remaining available assets on the financial market with the aim of increasing the future consumption over a pre-specified minimum consumption level. The introduction of a minimum consumption level in the optimization problem takes account of the fact that the retiree cares about sustaining a minimum living standard during retirement. Sustainable consumption becomes especially relevant if the retiree lives longer than expected. We assume that the health of the retiree can worsen or improve in an unpredictable way over her lifetime and model the retiree's mortality intensity by a stochastic process. Consequently, the consumption and investment strategy must adapt to the health condition of the retiree. Furthermore, in order to make the decision about the consumption and investment policy more realistic, we assume that the retiree applies a non-exponential discount factor to value her future income. There is strong evidence that people discount the future income with non-constant rates of time preferences. Experimental studies by psychologists and economists suggest that rates of time preference tend to decline in time. In other words, people's valuation tends to decrease rapidly for short period delays and less rapidly for longer period delays, see for example Loewenstein and Prelec (1992), Luttmer and Mariotti (2003), Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and Ekeland et al. (2012). In the literature, a typical example points out that while people prefer two oranges in 21 days to one orange in 20 days, they also prefer one orange now to two oranges tomorrow. Such a feature is called the common difference effect which cannot be described by exponential discounting, yet by hyperbolic discounting. In the present paper, we consider discount factors which arise from exponential discount factors perturbed by adding a small amount of hyperbolic discounting. Such discount factors imply declining rates of time preferences.

Since the paper by Merton (1971) the problem of finding optimal consumption and investment strategy has become one of the most studied optimization problem in economy, finance and insurance. The present paper contributes to this stream of literature by combining four important aspects: asset allocation, sustainable withdrawal, individual longevity risk and non-exponential discounting. Our paper extends the result from Huang et al. (2012) and Huang and Milevsky (2011) by taking into account non-exponential preferences and minimum consumption. Huang and Milevsky (2011) concentrate on the effect of longevity risk on consumption and consider a deterministic force of mortality. Their study suggests that wealth managers should advocate dynamic spending in proportion to survival probabilities, adjusted upwards for exogenous pension income and adjusted downwards for high risk aversion. Huang et al. (2012) investigate a stochastic mortality intensity modelled by a diffusion process and the authors find an optimal consumption (without investment strategy) for an agent with no future income for power utility under exponential discounting. The current paper also extends the result from Guambe and Kufakunesu (2015) and Shen and Wei (2016) by considering non-exponential preferences, future income and minimum consumption level. We would like to remark that it is important to include future income in the retiree's decision making process since the retiree often receives a lifetime annuity during retirement.

The impact of non-exponential discounting on optimal asset allocation and consumption has already been considered in the literature. It is known that the optimization problem with non-exponential discounting leads to time-inconsistent strategies, prohibiting the application of the Bellman principle of optimality. Björk and Murgoci (2010) develop a theory for solving time-inconsistent optimization problems. They derive an extended version of the classical HJB equation which contains a non-local term, hindering the derivation of explicit solutions. Similar to our problem, Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and Ekeland et al. (2012) consider an optimal investment and consumption problem for an agent with general time-inconsistent preferences and deterministic future income. Closest related to our research is the paper by Ekeland et al. (2012) where the authors assume that the agent is exposed to mortality risk modelled by a deterministic mortality intensity function. The authors derive an HJB equation with a non-local term characterizing the value function of the optimization problem and the optimal strategies. Since the HJB equation cannot be solved explicitly, the authors present a numerical algorithm to solve the HJB equation. Similarly, Dong and Sircar (2014) suggest applying an expansion technique for solving non-local HJBs. Dong and Sircar (2014) solve the classical Merton problem for an agent with exponential discounting perturbed with a small amount of hyperbolic discounting and derive explicit formulas for the first-order approximations to the optimal consumption and investment strategies. Our paper extends the result from Dong and Sircar (2014) by taking account of stochastic mortality, future income and minimum consumption level, and the result from Ekeland et al. (2012) (and Ekeland and Lazrak (2006), Ekeland and Pirvu (2008)) by introducing a stochastic mortality intensity process, and a minimum consumption level.

The remainder of the paper is organized as follows. Section 2 describes the underlying financial market and the stochastic mortality process. Section 3 formulates the optimal asset allocation and withdrawal problem. In the subsequent Section 4, we solve our optimization problem. In Section 5, we provide an interpretation of the optimal equilibrium strategies and provide a numerical example. All proofs are included in the Appendix.

2 Model setup

We work on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ and a finite time horizon $T < \infty$. On the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ we define a non-negative random variable τ and a two-dimensional standard Brownian motion $W = (W_m, W_\lambda) = (W_m(t), W_\lambda(t), 0 \le t \le T)$. The Brownian motions W_m and W_λ are independent. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ consists of three subfiltrations. We set $\mathcal{F}_t = \mathcal{F}_t^m \vee \mathcal{F}_t^\tau \vee \mathcal{F}_t^\lambda$ for all $0 \le t \le T$, where $\mathcal{F}_t^m = \sigma(W_m(u), u \in [0,t])$ contains information about the financial market, $\mathcal{F}_t^\tau = \sigma(\mathbf{1}\{\tau \le u\}, u \in [0,t])$ contains information on the survival of the retiree and $\mathcal{F}_t^\lambda = \sigma(W_\lambda(u), u \in [0,t])$ contains information about the mortality intensity of the retiree. We assume that the subfiltrations $\mathcal{F}_t^m$ and $(\mathcal{F}_t^\tau, \mathcal{F}_t^\lambda)$ are independent, i.e. we assume that

(A1) the financial risk is independent of the demographic risk.

As always, the filtration $\mathbb{F}$ is completed with sets of measure zero.

The financial market consists of a risk-free bank account and a risky asset. The value of the risk-free bank account $B = (B(t), 0 \le t \le T)$ grows at an exponential rate, i.e. the value process B satisfies the differential equation

$$\frac{dB(t)}{B(t)} = rdt, \quad 0 \le t \le T, \quad B(0) = 1, \tag{2.1}$$

where $r \ge 0$ denotes the instantaneous rate of interest paid on cash deposited in the bank account. The price of the risky asset $S = (S(t), 0 \le t \le T)$ is modelled by a geometric Brownian motion, i.e. the price process S satisfies the stochastic differential equation (SDE)

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_m(t), \quad 0 \le t \le T, \quad S(0) = 1, \tag{2.2}$$

where $\mu > r$ denotes the instantaneous expected rate of return of the risky asset and $\sigma > 0$ its volatility.

The future lifetime of the retiree is modelled by the random variable τ which is assumed to take values on the set $[0, T] \cup \{+\infty\}$. We use the convention that $\tau = +\infty$ if $\tau > T$. The distribution of the future lifetime of the retiree is determined by the conditional survival probability

$$\mathbb{P}(\tau \geq t | \mathcal{F}_t^{\lambda}) = e^{-\int_0^t \lambda(u) du}, \quad 0 \leq t \leq T, \tag{2.3}$$

where λ denotes the mortality intensity of the retiree.[1] There are many possible stochastic models for the mortality intensity process, most of which are inspired by interest rate models, see e.g. Luciano and Vigna (2008) and Norberg (2010). In the latter reference, the author shows the fundamental difference between the interest rate and mortality rate modelling. Following Huang et al. (2012), Luciano and Vigna (2008) and Vigna et al. (2008), we use a Brownian motion to model randomness in the future mortality intensity λ. More specifically, we assume that the mortality intensity of the retiree $\lambda = (\lambda(t), 0 \leq t \leq T)$ evolves randomly in time in accordance with the dynamics

$$d\lambda(t) = p(t, \lambda(t))dt + \zeta(t, \lambda(t))dW_{\lambda}(t), \quad 0 \leq t \leq T, \quad \lambda(0) = \lambda_0. \tag{2.4}$$

By this way of modelling, we consider the fact that the health of the retiree can get worse or better in an unpredictable way during her lifetime. For example, at some future point in time the retiree might go to a hospital and have an operation. After that operation the retiree's health condition can get worse and the mortality intensity can increase at a faster rate than expected.[2] We assume that the retiree continuously

[1] In our paper, we only consider individual mortality risk, i.e. the remaining lifetime of the retiree is random and described through a conditional exponential distribution with a force of mortality which follows a stochastic process. This modelling is different from the mortality risk modelling in a collective model in which the force of mortality is random and follows a certain distribution function, see e.g. Olivieri and Pitacco (2009).

[2] It might be more reasonable to assume that the health of the retiree changes only at some random points in time due to some random events (such as a hospital visit and operation) and that the retiree can only be aware of significant changes in her health condition which are caused by important (and seldom) events. In such a case the dynamics of the mortality intensity λ can be modelled by the SDE

$$d\lambda(t) = p(t, \lambda(t))dt + \zeta(t, \lambda(t))dJ(t), \quad 0 \leq t \leq T, \quad \lambda(0) = \lambda_0, \tag{2.5}$$

"observes" her changing mortality intensity, i.e. the retiree has perfect knowledge about her current heath condition and the distribution of her future lifetime.

The dynamics (2.4) which defines the evolution of the mortality intensity in time is very general. We require that

(A2) the process λ is bounded from below and above, i.e. the process λ takes values in a bounded, open and connected set $\mathcal{K}$,

(A3) the functions $p, \zeta : [0,T] \times \mathcal{K} \mapsto \mathbb{R}$ are uniformly Lipschitz continuous in (t, λ).

An obvious lower bound for the mortality intensity λ is zero. It is known that under (A2)-(A3) there exists a unique solution to the SDE (2.4), see Theorem 2.9 in Karatzas and Shreve (1988).

Let us introduce an operator associated with the dynamics of the the process λ.

Definition 2.1. *Let $\mathcal{L}_\lambda$ denote a second-order differential operator given by*

$$\mathcal{L}_\lambda v(t,\lambda) = \frac{\partial}{\partial \lambda} v(t,\lambda) p(t,\lambda) + \frac{1}{2}\frac{\partial^2}{\partial \lambda^2} v(t,\lambda)\zeta^2(t,\lambda). \tag{2.6}$$

The operator $\mathcal{L}_\lambda$ is defined for $v \in \mathcal{C}^{1,2}([0,T] \times \mathcal{K})$. [3]

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where J denotes a compound Poisson process. In this paper we solve our optimization problem assuming the dynamics (2.4). However, we would like to point out that the solution to our optimization problem would be the same if we assumed the dynamics (2.5). We comment on this point later during our calculations.

[3] If we consider the dynamics (2.5), then the operator associated with the process λ would take the form

$$\mathcal{L}_\lambda v(t,\lambda) = \frac{\partial}{\partial \lambda} v(t,\lambda) p(t,\lambda) + \int_{\mathbb{R}} (v(t,\lambda + \zeta(t,\lambda)z) - v(t,\lambda))\chi\vartheta(dz), \tag{2.7}$$

where χ denotes the intensity of the compound Poisson process J and ϑ denotes the distribution of the jumps.

3 The optimal asset allocation and withdrawal problem

Think of a retiree who is endowed with a wealth level $\hat{x} > 0$ at her retirement date (in our framework at time $t = 0$). Since nowadays retirees are highly encouraged to invest in private retirement plans, we assume that at time $t = 0$ the retiree spends a part of her initial wealth $\alpha\hat{x}$, $0 \leq \alpha \leq 1$, on a lifetime annuity. From this investment the retiree will receive a lifetime income at the rate $a(\alpha\hat{x})$. More precisely, by spending $\alpha\hat{x}$ on the annuity the retiree buys an annuity which pays $a(\alpha\hat{x})$ for lifetime, but not longer than T years. The annuity income $a(\alpha\hat{x})$ is determined by the annuity provider. The remaining money $(1-\alpha)\hat{x}$ is used by the retiree to set up an investment plan and to increase the consumption which will be guaranteed by the lifetime annuity. In our optimization problem, the parameter α is assumed to be given. However, it is not difficult to incorporate α as a choice variable. We will come back to this point in the end of Section 4. The retiree is interested in maintaining a given living/consumption standard and has some bequest motives. The retiree needs to decide about how to invest and how much to withdraw from the available funds to consume during the retirement life.

In our model we consider a fixed finite time horizon T. The mathematical techniques which we use in this paper require that the time horizon T is fixed and finite. However, the time horizon T can be chosen so that $\mathbb{P}(\tau > T)$ is negligible. T could then be interpreted as the maximum future lifetime of the retiree, and we can deal with an uncertain time horizon determined by the retiree's death. Since we use the exponential distribution (2.3) for the future lifetime τ, the time horizon T cannot be formally defined as the maximum future lifetime of the retiree.

Let $\pi(t)$ denote the fraction of wealth that the retiree invests in the risky asset and $(1-\pi(t))$ the fraction in the risk-free bank account, and $c(t)$ the consumption rate.

The wealth process of the retiree $X = (X(t), 0 \le t \le T)$ satisfies the SDE

$$\begin{aligned} dX(t) &= \pi(t)X(t)\big(\mu dt + \sigma dW_m(t)\big) + (1-\pi(t))X(t)r dt \\ &\quad -c(t)dt + a(\alpha\hat{x})dt, \quad 0 \le t \le T, \\ X(0) &= (1-\alpha)\hat{x}. \end{aligned} \tag{3.1}$$

To sustain a certain living standard, the retiree's consumption c is assumed to be above a minimum consumption of c^*. Hence, we introduce the decomposition

$$c(t) = c^* + u(t), \quad 0 \le t \le T, \tag{3.2}$$

where u denotes the excess of the realized consumption over the minimum consumption c^*. In the literature on habit-formation, c^* is interpreted as a habit level, which is usually defined as a function of past consumption rates, see e.g. Munk (2008). Here, we assume that c^* is a positive constant.

A condition is needed which guarantees that the minimum consumption c^* can be sustained by the retiree who has capital $\hat{x}$ at her disposal and spends $\alpha\hat{x}$ on the lifetime annuity. We require that

(A4) $(1-\alpha)\hat{x} \ge \int_0^T (c^* - a(\alpha\hat{x}))e^{-rt}dt = (c^* - a(\alpha\hat{x}))\frac{1-e^{-rT}}{r}$.

Assumption (A4) means that the minimum consumption demanded by the retiree can be achieved with the lifetime annuity and the remaining wealth left for investment in the financial market. In other words, the remaining wealth $(1-\alpha)\hat{x}$ left for investment in the financial market must be sufficient to super-hedge the consumption difference $c^* - a(\alpha\hat{x})$ which arises after the annuity has been bought. Since the retiree demands to maintain the minimum consumption until her death and the combined financial and insurance market is incomplete, in condition (A4) we assume that the retiree lives until the terminal time T. Intuitively, it should be clear that we need a condition relating $\hat{x}$ and c^*. The retiree cannot demand very high (relative to her initial wealth $\hat{x}$) minimal consumption c^* since such a high minimal consumption may not be guaranteed by the lifetime annuity and the investment in the financial market.

In order to gain more insight into (A4), let us assume that the insurer uses the risk-free rate as a discount rate for valuing the annuity, i.e. $1 = a(1)\int_0^T e^{-rs}\hat{p}(s)ds$ where $(\hat{p}(s), 0 \le s \le T)$ denotes the survival probabilities specified by the insurer for the annuity valuation. The choice of the risk-free rate as a discount rate seems to be the most reasonable, especially if the market-consistent valuation is adopted by the insurer. Since $a(\alpha\hat{x}) = \alpha\hat{x}a(1)$ and $a(1)\frac{1-e^{-rT}}{r} > 1$, we can now rewrite (A4) as

$$\alpha \ge \frac{c^*\frac{1-e^{-rT}}{r} - \hat{x}}{(a(1)\frac{1-e^{-rT}}{r} - 1)\hat{x}}. \tag{3.3}$$

Assumption (A4) alternatively shows that in order to sustain the minimum consumption, a certain amount of lifetime annuity may have to be bought. This conclusion agrees with intuition. It is trivial to notice that if the retiree wants to consume 1 Euro in one year time in the market with zero interest rate, then she can buy a bond which costs her 1 Euro or an endowment contract which costs her less than 1 Euro (since there is a positive probability of death). Given a limited initial capital, some cash flows paid upon survival may not be replicated with financial instruments but with insurance contracts. In other words, if a higher minimum consumption is demanded, then a minimum amount of lifetime annuity may have to be bought in order to guarantee a part of this minimum consumption. Summing up, our assumption (A4) defines the upper bound for c^* or the lower bound for α. In the current low interest environment $(r = 0)$ conditions (A4) and (3.3) reduce to

$$(1-\alpha)\hat{x} \ge \big(c^* - \frac{\alpha\hat{x}}{\int_0^T \hat{p}(s)ds}\big)T, \quad \text{or} \quad \alpha \ge \frac{c^*T - \hat{x}}{(\frac{T}{\int_0^T \hat{p}(s)ds} - 1)\hat{x}}.$$

From a mathematical point of view, assumption (A4) defines the set of parameters for which we can solve our optimization problem. If (A4) is not satisfied, then the minimum consumption c^* cannot be sustained and we cannot solve our optimization problem (at least in the form stated below).

The controlled wealth process $X^{\pi,u}$ takes the form

$$\begin{aligned} dX^{\pi,u}(t) &= \pi(t)X^{\pi,u}(t)\big(\mu dt + \sigma dW_m(t)\big) + (1-\pi(t))X^{\pi,u}(t)rdt \\ &\quad -c^*dt - u(t)dt + a(\alpha\hat{x})dt, \quad 0 \le t \le T, \\ X^{\pi,u}(0) &= (1-\alpha)\hat{x}. \end{aligned} \tag{3.4}$$

We introduce an operator associated with the dynamics (3.4) and the set of admissible strategies.

Definition 3.1. *Let $\mathcal{L}_x$ denote a second-order differential operator given by*

$$\begin{aligned} \mathcal{L}_x^{\pi,u}v(t,x) &= \frac{\partial}{\partial x}v(t,x)\big(\pi x(\mu - r) + xr + a(\alpha\hat{x}) - c^* - u\big) \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,x)\pi^2x^2\sigma^2. \end{aligned} \tag{3.5}$$

The operator $\mathcal{L}_x^{\pi,u}$ is defined for $v \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$.

In the sequel the partial derivatives with respect to the state variables (the mortality intensity, the wealth process) and time are denoted by $v_\lambda, v_{\lambda\lambda}, v_x, v_{xx}, v_t$.

Definition 3.2. *A strategy $(\pi,u) = (\pi(t),u(t),0\le t\le T)$ is called admissible, $(\pi,u)\in\mathcal{A}$, if it satisfies the following conditions:*

1. *$\pi, u : [0,T]\times\Omega \to \mathbb{R}$ are progressively measurable mappings with respect to the filtration $\mathbb{F}$.*

2. *The process u is non-negative, $u(t)\ge 0,\ 0\le t\le T$, and the terminal wealth is non-negative, $X^{\pi,u}(T)\ge 0$.*

3. $\mathbb{E}\Big[\int_0^T|\pi(t)X^{\pi,u}(t)|^2dt + \int_0^T|u(t)|^2dt\Big] < \infty.$

4. *The stochastic differential equation (3.4) has a unique solution $X^{\pi,u}$ on $[0,T]$.*

5. *The strategy (π,u) is a Markov strategy.*

If (π, u) is a Markov strategy, then $\pi(t) = \pi(t, X^{\pi,u}(t), \lambda(t))$ and $u(t) = u(t, X^{\pi,u}(t), \lambda(t))$. If $(\pi, u) \in \mathcal{A}$, then the solution $X^{\pi,u}$ to the SDE (3.4) is a continuous, $\mathbb{F}$-adapted process, see Theorem 2.9 in Karatzas and Shreve (1988). Moreover, from (3.4) we get the representation

$$\begin{aligned} X^{\pi,u}(t) &= (1-\alpha)\hat{x}e^{rt} + \int_0^t \pi(s)X^{\pi,u}(s)e^{r(t-s)}(\mu - r)ds \\ &\quad + \int_0^t \pi(s)X^{\pi,u}(s)e^{r(t-s)}\sigma dW_m(s) \\ &\quad - \int_0^t e^{r(t-s)}(u(s) + c^* - a(\alpha\hat{x}))ds, \quad 0 \le t \le T, \end{aligned} \tag{3.6}$$

and using the Bulkholder inequality and point 3 from Definition 3.2 we can derive the moment estimate

$$\mathbb{E}[\sup_{t\in[0,T]} |X^{\pi,u}(t)|^2] \le K\Big(1 + \mathbb{E}\Big[\int_0^T |\pi(s)X^{\pi,u}(s)|^2 ds + \int_0^T |u(s)|^2 ds\Big]\Big) < \infty. \tag{3.7}$$

Hence, for $(\pi, u) \in \mathcal{A}$ the solution $X^{\pi,u}$ is square integrable.

We assume that the retiree's preference for consumption and bequest is modelled by a power utility function with a relative risk aversion coefficient $1 - \gamma$. We assume

(A5) the relative risk aversion coefficient of the power utility $1 - \gamma \in (0, 1)$.

At the retirement date $t = 0$, the retiree is interested in solving the following optimization problem

$$\sup_{(\pi,u)\in\mathcal{A}} \mathbb{E}\left[\int_0^{\tau\wedge T} \phi(t)(u(t))^\gamma dt + q\phi(T)(X^{\pi,u}(T))^\gamma 1_{\{\tau>T\}}\right], \tag{3.8}$$

where the parameter $q > 0$ describes how much the retiree weighs the bequest in the total expected utility and the function ϕ specifies subjective discount factors which are applied by the retiree to value future income. We choose the non-exponential discounting function

$$\phi(t) = e^{-\rho t}\frac{1}{(1+\delta t)^{\varepsilon}}, \quad 0 \le t \le T, \tag{3.9}$$

where $\rho \ge 0, \delta \ge 0, \varepsilon \ge 0$. The discount factors (3.9) are proposed for the first time by Luttmer and Mariotti (2003) and they are considered in utility optimization problems by Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), Ekeland et al. (2012). The discount factors (3.9) are exponential discount factors perturbed with hyperbolic discounting. The discount rates which are implied by the discount factors (3.9) and present the rates of time preference take the form

$$-\frac{\phi'(t)}{\phi(t)} = \rho + \frac{\delta}{1+\delta t}\varepsilon, \quad 0 \le t \le T. \tag{3.10}$$

Notice that the resulting discount rate is decreasing over time. In particular, short-term discount rates are higher than long-time discount rates, which is more appropriate to empirically model inter-temporal preferences compared to a constant discount rate (from an exponential discounting function) over time, see Figure 1. Our rates of time preference (3.10) are smoothly declining from $\rho + \delta\varepsilon$ at $t = 0$ to ρ at $t = \infty$. Hence, the desirable pattern of time preference rates, discussed in the introduction, can be modelled by the discounting function (3.9). The ability of (3.9) to capture the desirable pattern of rates of time preference is the main motivation for using our discounting function in the investment/withdrawal problem. Let us point out that increasing ρ increases our discount rates at all horizons and increasing δ raises our discount rates more at short horizons than at long horizons. The parameter ε determines how close our discount rates are to a constant rate. Hence, the discounting function (3.9) offers a flexible tool in modelling time preferences.

We define the value function for the optimization problem (3.8)

$$
\begin{aligned}
v^{\pi,u}(t,x,\lambda) = \mathbb{E}\Bigg[& \int_t^{\tau\wedge T} \phi(s-t)(u(s))^{\gamma} ds \\
& + q\phi(T-t)(X^{\pi,u}(T))^{\gamma} 1_{\{\tau>T\}} | X(t)=x, \lambda(t)=\lambda, \tau>t \Bigg], \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K},
\end{aligned}
\tag{3.11}
$$

together with the optimal value function

$$
v(t,x,\lambda) = v^{\pi^*,u^*}(t,x,\lambda) = \sup_{(\pi,u)\in\mathcal{A}} v^{\pi,u}(t,x,\lambda), \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K}. \tag{3.12}
$$

If the time horizon T is chosen so that $\mathbb{P}(\tau > T)$ is negligible, then the bequest is of little importance and the investment/withdrawal problem until the retiree's death is considered.

If an exponential discount factor is applied in (3.11), then the Dynamic Programming Principle can be applied to solve the optimization problem (3.12). The problem (3.12) is time consistent in the sense that if at time t_1 we solve (3.12) and find the optimal strategy for the period $[t_1, T]$, then this optimal strategy coincides with the optimal strategy for period $[t_2, T]$ found by solving (3.12) at time t_2. By classical techniques, we can conclude that the optimal value function (3.12) for an exponential discount factor, denoted by v^{exp}, solves the HJB equation:

$$
\begin{aligned}
& v_t^{exp}(t,x,\lambda) + \sup_{\pi,u}\left\{u^{\gamma} + \mathcal{L}_x^{\pi,u} v^{\exp}(t,x,\lambda)\right\} \\
& \qquad + \mathcal{L}_{\lambda} v^{exp}(t,x,\lambda) - \lambda v^{exp}(t,x,\lambda) = \rho v^{exp}(t,x,\lambda), \quad (t,x,\lambda)\in[0,T)\times\mathbb{R}\times\mathcal{K}, \\
& v^{exp}(T,x,\lambda) = qx^{\gamma}, \quad (x,\lambda)\in\mathbb{R}\times\mathcal{K},
\end{aligned}
\tag{3.13}
$$

and the corresponding optimal strategies π^{exp}, u^{exp} are given by

$$
\pi^{exp}(t,x,\lambda) = -\frac{v_x^{exp}(t,x,\lambda)}{x v_{xx}^{exp}(t,x,\lambda)} \frac{\mu - r}{\sigma^2}, \quad u^{exp}(t,x,\lambda) = \left(\frac{v_x^{exp}(t,x,\lambda)}{\gamma}\right)^{\frac{1}{\gamma-1}}.
$$

The solution to (3.13) represents a special case of our general solution which we derive in the next section. We would like to point out that although the form of the HJB (3.13) is known in this case, finding a solution to (3.13) is non-trivial.

For a general discount factor (3.9), it is known that the optimization problem (3.12) is time inconsistent and the Dynamic Programming Principle cannot be applied. More intuitively, if at time t_1, we solve (3.12) and find the optimal strategy for the period $[t_1, T]$, this optimal strategy is different from the optimal strategy for the period $[t_2, T]$ found by solving (3.12) at time t_2. Such an inconsistency arises since investor's preferences and discount rates change over time $[0, T]$.

For a general discount factor, the notion of an optimal strategy has to be properly defined for the time-inconsistent optimization problem (3.12). A game-theoretic approach to solving (3.12) is proposed in Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and Björk and Murgoci (2010). Following them, we can think of the problem (3.12) as a game played by a continuum of players. Each player, indexed with variable t, has her own utility function and controls the wealth only over an infinitesimal period of time $[t, t + dt]$. The player t has control over the wealth at time t and can freely choose a strategy at time t. Next, the player has to pass the wealth to the next player who has a different utility level and again can freely choose a strategy. We can define the equilibrium of this game as follows.

Definition 3.3. *Let us consider an admissible strategy $(\pi^*, u^*) \in \mathcal{A}$. Choose an arbitrary point $(t, x, \lambda) \in [0, T) \times \mathbb{R} \times \mathcal{K}$ and any admissible strategy $(\pi, u) \in \mathcal{A}$. We define a new admissible strategy*

$$(\pi_h(s), u_h(s)) = \begin{cases} (\pi(s), u(s)), & t \leq s \leq (t+h) \wedge \tau, \\ (\pi^*(s), u^*(s)), & (t+h) \wedge \tau < s \leq T \wedge \tau. \end{cases}$$

If

$$lim\ inf\ _{h \to 0} \frac{v^{\pi^*, u^*}(t, x, \lambda) - v^{\pi_h, u_h}(t, x, \lambda)}{h} \geq 0, \quad (t, x, \lambda) \in [0, T) \times \mathbb{R} \times \mathcal{K}, \quad (3.14)$$

then (π^*, u^*) *is called an equilibrium strategy and* $v^{\pi^*,u^*}(t,x,\lambda)$ *is called the equilibrium value function corresponding to the equilibrium strategy* (π^*, u^*).[4]

For a detailed characterization of a sub-game perfect equilibrium in a continuous-time optimization problem with a non-constant discounting, we refer to Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and Björk and Murgoci (2010).

When the player t decides on the strategy at time t, she should take into consideration that the next players in the future will have different objectives/preferences and are likely to choose different strategies. The idea of the equilibrium strategy defined in (3.14) is that the player t will not be better off by forcing the next players to choose her optimal strategy instead of letting the next players choose their best strategies. If the player at time t knows that all players after her will choose a strategy (π^*, u^*), then it is optimal for the player t to choose the strategy (π^*, u^*) at time t as well. The equilibrium strategy is time consistent and the players have no incentives to deviate from the equilibrium strategy. In fact, the time consistency defined in (3.14) is a minimal requirement for rationality, see Ekeland et al. (2012).

From Björk and Murgoci (2010) and Ekeland et al. (2012) we expect that our equilibrium value function for the problem (3.12) should satisfy the non-local Hamilton-Jacobi-Bellman equation

$$\begin{aligned}
&v_t(t,x,\lambda) + \sup_{\pi,u}\big\{u^\gamma + \mathcal{L}_x^{\pi,u} v(t,x,\lambda)\big\} + \mathcal{L}_\lambda v(t,x,\lambda) - \lambda v(t,x,\lambda) \\
&\quad = -\mathbb{E}\Big[\int_t^{\tau\wedge T} \phi'(s-t)(u^*(s))^\gamma ds \\
&\quad\quad + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau > T\} | X(t) = x, \lambda(t) = \lambda, \tau > t\Big], \quad (t,x,\lambda) \in [0,T) \times \mathbb{R} \times \mathcal{K}, \\
&v(T,x,\lambda) = qx^\gamma, \quad (x,\lambda) \in \mathbb{R} \times \mathcal{K},
\end{aligned} \tag{3.15}$$

where the equilibrium strategy (π^*, u^*) realizes the supremum in left hand side of equa-

[4]We are not considering a general equilibrium model. Here, we use the term "equilibrium" to be consistent with the literature on the similar topics, see e.g. Ekeland and Lazrak (2006), Ekeland and Pirvu (2008) and Björk and Murgoci (2010).

tion (3.15), i.e.

$$(\pi^*, u^*) = arg\ sup\ _{\pi,u}\ \Big\{u^\gamma + \mathcal{L}_x^{\pi,u} v(t,x,\lambda)\Big\}. \tag{3.16}$$

The operators $\mathcal{L}_x^{\pi,u},\ \mathcal{L}_\lambda$ are given by (2.6), (3.5).[5]

We can prove the following verification theorem.

Theorem 3.1. *Let (A1)-(A5) hold. Assume there exists a function* $v \in \mathcal{C}([0,T] \times \mathbb{R} \times \mathcal{K}) \cap \mathcal{C}^{1,2,2}([0,T) \times \mathbb{R} \times \mathcal{K})$ *and an admissible strategy* $(\pi^*, u^*) \in \mathcal{A}$ *which solve the HJB equation* (3.15)*. In addition, assume that the sequence*

$$\Big\{v(\mathcal{T}, X^{\pi^*,u^*}(\mathcal{T}), \lambda(\mathcal{T})),\ \mathcal{T}\ is\ an\ \mathbb{F}-stopping\ time, \mathcal{T} \in [0,T]\Big\}$$

is uniformly integrable. The strategy $(\pi^*, u^*) \in \mathcal{A}$ *is an equilibrium strategy and* $v(t,x,\lambda) = v^{\pi^*,u^*}(t,x,\lambda)$ *is the equilibrium value function corresponding to* (π^*, u^*)*.*

Details can be found in the Appendix.

4 The solution to the optimization problem

Solving the optimization problem (3.12) and the HJB equation (3.15) is challenging due to the non-local term in (3.15), see Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), Ekeland et al. (2012) and Dong and Sircar (2014). In order to solve the non-local HJB equation (3.15), we use the expansion method suggested in Dong and Sircar (2014). The expansion method allows us to find the first-order approximations of the equilibrium value function and the equilibrium strategies. This method provides a good approximation when the part of the hyperbolic discounting in the general discount factor (3.9) is small.

We assume that the discount factors (3.9) are derived from exponential discount

[5]If we choose the dynamics of the mortality intensity with jumps (2.5), we obtain the same HJB (3.15) but with a different operator $\mathcal{L}_\lambda$, see (2.7).

factors by adding a small amount of hyperbolic discounting, i.e. we consider the non-exponential discount factors of the form

$$\phi(t) = e^{-\rho t}\frac{1}{(1+\delta t)^{\varepsilon}} = e^{-\rho t-\varepsilon\ln(1+\delta t)}, \quad \varepsilon \to 0, \quad 0 \le t \le T. \tag{4.1}$$

We can expand the discount factors to the first-order at $\varepsilon = 0$ and get the approximation

$$\phi(t) = e^{-\rho t}(1+\vartheta(t)\varepsilon) + \mathcal{O}(\varepsilon^2), \quad \varepsilon \to 0, \quad 0 \le t \le T, \tag{4.2}$$

where

$$\vartheta(t) = -\ln(1+\delta t), \quad 0 \le t \le T.$$

Since $\varepsilon \to 0$, the discount factors (4.2) are positive. By $y^{\varepsilon}(w) \sim \mathcal{O}(\varepsilon^2)$, $\varepsilon \to 0$, we mean that $|y^{\varepsilon}(w)| \le K(w)\varepsilon^2$ for all w and $\varepsilon \in [0, \varepsilon_0]$, where $\varepsilon_0 > 0$ and K may depend on w but is independent of ε.

In Figure 1 we can see three discounting functions: exponential, non-exponential (4.1) and approximation (4.2). As already discussed, by using the discounting function (4.1) we can model discount factors that decrease rapidly for short period delays and less rapidly for longer period delays, which is clearly observed in Figure 1.

Unfortunately, we cannot solve our optimization problem for any non-exponential discounting function (4.1), i.e. we cannot provide a formal and feasible solution, see the proof of Theorem 7.1 in the Appendix. In this paper we do not fully analyze the impact of the non-exponential discounting factors on investment and consumption. Instead, we present an approximate solution and analyze the first-order impact. The related analysis is still interesting because our first-order strategies are able to capture the non-exponential form of the discounting function (4.1).

Let us now study the HJB equation (3.15). In the sequel, the conditional expectation $\mathbb{E}[\cdot|X(t) = x, \lambda(t) = \lambda, \tau > t]$ is denoted as $\mathbb{E}_{t,x,\lambda}[\cdot]$. The first-order conditions w.r.t u

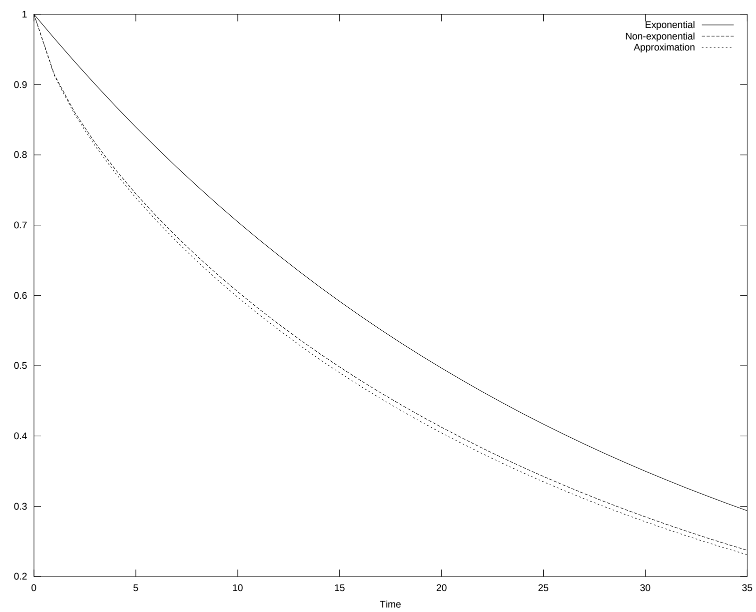

Figure 1: The discounting functions for $\rho = 0.035, \delta = 2, \varepsilon = 0.05$: exponential (solid line), non-exponential (4.1) and approximation (4.2) (dashed lines).

and π yield the following candidates for the optimal strategies

$$\pi^*(t) = -\frac{v_x(t,x,\lambda)}{xv_{xx}(t,x,\lambda)}\frac{\mu - r}{\sigma^2}, \quad u^*(t) = \left(\frac{v_x(t,x,\lambda)}{\gamma}\right)^{\frac{1}{\gamma-1}}. \tag{4.3}$$

Substituting the strategies (4.3) into the HJB equation (3.15), we obtain the equation

$$\begin{aligned}
&v_t(t,x,\lambda) + \big(xr + a(\alpha\hat{x}) - c^*\big)v_x(t,x,\lambda) - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{v_x^2(t,x,\lambda)}{v_{xx}(t,x,\lambda)} \\
&\quad + \Big(\frac{v_x(t,x,\lambda)}{\gamma}\Big)^{\frac{\gamma}{\gamma-1}}(1-\gamma) + \mathcal{L}_\lambda v(t,x,\lambda) - \lambda v(t,x,\lambda) \\
&= -\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \phi'(s-t)(u^*(s))^\gamma ds \\
&\quad + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau > T\}\Big], \quad (t,x,\lambda) \in [0,T) \times \mathbb{R} \times \mathcal{K}, \\
&v(T,x,\lambda) = qx^\gamma, \quad (x,\lambda) \in \mathbb{R} \times \mathcal{K},
\end{aligned} \tag{4.4}$$

where π^* and u^* are given by (4.3). Since we introduce the first-order expansion for the discounting function (4.1), it is reasonable to introduce the first-order expansions for the value function and the strategies

$$
\begin{aligned}
v(t,x,\lambda) &= v^0(t,x,\lambda)+v^1(t,x,\lambda)\varepsilon+\mathcal{O}(\varepsilon^2), \quad \varepsilon\to 0, \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K},\\
\pi^*(t) &= \pi^{0,*}(t)+\pi^{1,*}(t)\varepsilon+\mathcal{O}(\varepsilon^2), \quad \varepsilon\to 0, \quad 0\le t\le T,\\
u^*(t) &= u^{0,*}(t)+u^{1,*}(t)\varepsilon+\mathcal{O}(\varepsilon^2), \quad \varepsilon\to 0, \quad 0\le t\le T. \qquad (4.5)
\end{aligned}
$$

Our goal is to find v^0, v^1 together with $\pi^{0,*}, \pi^{1,*}$ and $u^{0,*}, u^{1,*}$.

If (4.5) holds, then we also have

$$
\begin{aligned}
&\Big(v_x^0(t,x,\lambda)+v_x^1(t,x,\lambda)\varepsilon+\mathcal{O}(\varepsilon^2)\Big)^{\frac{\gamma}{\gamma-1}}\\
&\quad= \big(v_x^0(t,x,\lambda)\big)^{\frac{\gamma}{\gamma-1}}+\frac{\gamma}{\gamma-1}\big(v_x^0(t,x,\lambda)\big)^{\frac{1}{\gamma-1}}v_x^1(t,x,\lambda)\varepsilon+\mathcal{O}(\varepsilon^2),\\
&\big(v_x^0(t,x,\lambda)+v_x^1(t,x,\lambda)\varepsilon+\mathcal{O}(\varepsilon^2)\big)^2\big(v_{xx}^0(t,x,\lambda)+v_{xx}^1(t,x,\lambda)\varepsilon+\mathcal{O}(\varepsilon^2)\big)^{-1}\\
&\quad= \frac{\big(v_x^0(t,x,\lambda)\big)^2}{v_{xx}^0(t,x,\lambda)}\\
&\qquad+\Big(2\frac{v_x^0(t,x,\lambda)}{v_{xx}^0(t,x,\lambda)}v_x^1(t,x,\lambda)-\Big(\frac{v_x^0(t,x,\lambda)}{v_{xx}^0(t,x,\lambda)}\Big)^2v_{xx}^1(t,x,\lambda)\Big)\varepsilon+\mathcal{O}(\varepsilon^2), \qquad (4.6)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[\int_t^{\tau\wedge T}\phi'(s-t)(u^*(s))^\gamma ds+q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma\mathbf{1}\{\tau>T\}]\\
&\quad= -\rho v^0(t,x,\lambda)-\rho v^1(t,x,\lambda)\varepsilon\\
&\qquad+\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\vartheta'(s-t)e^{-\rho(s-t)}(u^{0,*}(s))^\gamma ds\\
&\qquad+q\vartheta'(T-t)e^{-\rho(T-t)}(X^{\pi^{0,*},u^{0,*}}(T))^\gamma\mathbf{1}\{\tau>T\}\Big]\varepsilon+\mathcal{O}(\varepsilon^2). \qquad (4.7)
\end{aligned}
$$

Formula (4.7) is far from obvious and is derived in the Appendix in (7.24). After substituting our expansions (4.5)-(4.7) into the HJB equation (4.4), we collect the terms which are independent of ε, those which are proportional to ε and those of order $\mathcal{O}(\varepsilon^2)$. Subsequently, we obtain two equation systems which characterize the functions

v^0 and v^1:

$$
\begin{aligned}
&v_t^0(t,x,\lambda)+\big(xr+a(\alpha\hat{x})-c^*\big)v_x^0(t,x,\lambda)-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{(v_x^0(t,x,\lambda))^2}{v_{xx}^0(t,x,\lambda)}\\
&\quad+\Big(\frac{v_x^0(t,x,\lambda)}{\gamma}\Big)^{\frac{\gamma}{\gamma-1}}(1-\gamma)+\mathcal{L}_\lambda v^0(t,x,\lambda)-\lambda v^0(t,x,\lambda)\\
&=\rho v^0(t,x,\lambda),\quad (t,x,\lambda)\in[0,T)\times\mathbb{R}\times\mathcal{K},\\
&v^0(T,x,\lambda)=qx^\gamma,\quad (x,\lambda)\in\mathbb{R}\times\mathcal{K},
\end{aligned}\tag{4.8}
$$

and

$$
\begin{aligned}
&v_t^1(t,x,\lambda)+\Big(xr+a(\alpha\hat{x})-c^*-\frac{(\mu-r)^2}{\sigma^2}\frac{v_x^0(t,x,\lambda)}{v_{xx}^0(t,x,\lambda)}-\Big(\frac{v_x^0(t,x,\lambda)}{\gamma}\Big)^{\frac{1}{\gamma-1}}\Big)v_x^1(t,x,\lambda)\\
&\quad+\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\Big(\frac{v_x^0(t,x,\lambda)}{v_{xx}^0(t,x,\lambda)}\Big)^2v_{xx}^1(t,x,\lambda)+\mathcal{L}_\lambda v^1(t,x,\lambda)-\lambda v^1(t,x,\lambda)\\
&=\rho v^1(t,x,\lambda)-\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\vartheta'(s-t)e^{-\rho(s-t)}(u^{0,*}(s))^\gamma ds\\
&\quad+q\vartheta'(T-t)e^{-\rho(T-t)}(X^{\pi^{0,*},u^{0,*}}(T))^\gamma\mathbf{1}\{\tau>T\}\Big],\quad (t,x,\lambda)\in[0,T)\times\mathbb{R}\times\mathcal{K},\\
&v^1(T,x,\lambda)=0,\quad (x,\lambda)\in\mathbb{R}\times\mathcal{K}.
\end{aligned}\tag{4.9}
$$

If we compare our equation (4.8) with the HJB equation (3.13) for v^{exp} after substituting the optimal strategies π^{exp}, u^{exp}, we can notice that the function v^0 is the optimal value function for the optimization problem (3.12) with exponential discounting. This sounds reasonable, because for $\varepsilon=0$ our discounting function reduces to an exponential function. In the Appendix, see (7.22)-(7.23), we show that v^0 is the value function (3.11) with exponential discounting under the strategies $(\pi^{0,*},u^{0,*})$. Hence, our zeroth order strategies $(\pi^{0,*},u^{0,*})$ are the optimal strategies for the optimization problem (3.12) with exponential discounting. By (3.13), we get the zeroth order strategies

$$
\pi^{0,*}(t)=-\frac{v_x^0(t,x,\lambda)}{xv_{xx}^0(t,x,\lambda)}\frac{\mu-r}{\sigma^2},\quad u^{0,*}(t)=\left(\frac{v_x^0(t,x,\lambda)}{\gamma}\right)^{\frac{1}{\gamma-1}}.\tag{4.10}
$$

Let us assume

$$
\begin{aligned}
v^0(t,x,\lambda) &= (x+g(t))^\gamma f(t,\lambda), \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K},\\
v^1(t,x,\lambda) &= (x+G(t))^\gamma F(t,\lambda), \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K}.
\end{aligned}
$$

The terms $g(t)$ and $G(t)$ arise due to the income generated by the lifetime annuity.

Derivation of the functions f and g is rather standard. Substituting the above expression of $v^0(t,x,\lambda)$ in (4.8), we obtain the equations:

$$
\begin{aligned}
& f_t(t,\lambda)+\mathcal{L}_\lambda f(t,\lambda)-\Big(\lambda+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\Big)f(t,\lambda)\\
&\quad +(1-\gamma)(f(t,\lambda))^{\frac{\gamma}{\gamma-1}}=0, \quad (t,\lambda)\in[0,T)\times\mathcal{K},\\
& f(T,\lambda)=q, \quad \lambda\in\mathcal{K},
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
& g_t(t)-g(t)r+a(\alpha\hat{x})-c^*=0, \quad t\in[0,T),\\
& g(T)=0.
\end{aligned}
\tag{4.12}
$$

Deriving the functions F and G is much more complex, because we have to calculate the expectation in (4.9). Compared to equation (4.4), the non-local term in the equation for v^1 now involves strategies determined by v^0. i.e. the zeroth order strategies (4.10). This is the key point that simplifies the calculations and the derivation of the solution to our optimization problem. In the Appendix, see (7.17) and (7.21), we show that

$$
\begin{aligned}
& \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\vartheta'(s-t)e^{-\rho(s-t)}(u^{0,*}(s))^\gamma ds\\
&\quad +q\vartheta'(T-t)e^{-\rho(T-t)}(X^{\pi^{0,*},u^{0,*}}(T))^\gamma\mathbf{1}\{\tau>T\}\Big]\\
&= (x+g(t))^\gamma Q(t,\lambda), \quad (t,\lambda)\in[0,T]\times\mathcal{K},
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
&Q(t,\lambda)\\
&= \int_t^T \vartheta'(s-t)P^s(t,\lambda)ds + q^{\frac{1}{1-\gamma}}\vartheta'(T-t)P^T(t,\lambda), \quad (t,\lambda)\in[0,T]\times\mathcal{K}, \qquad (4.14)
\end{aligned}
$$

and the function P^s is obtained as a unique solution to the PDE

$$
\begin{aligned}
&P^s_t(t,\lambda) + \mathcal{L}_\lambda P^s(t,\lambda) - \big(\lambda + \rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\\
&\quad +\gamma(f(t,\lambda))^{\frac{1}{\gamma-1}}\big)P^s(t,\lambda) = 0, \quad t\in[0,s)\\
&P^s(s,\lambda) = (f(s,\lambda))^{\frac{\gamma}{\gamma-1}}, \quad \lambda\in\mathcal{K}. \qquad (4.15)
\end{aligned}
$$

By standard arguments, we can now derive equations for the functions F and G. We obtain the following equations:

$$
\begin{aligned}
&F_t(t,\lambda) + \mathcal{L}_\lambda F(t,\lambda) - \big(\lambda + \rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\\
&\quad +\gamma(f(t,\lambda))^{\frac{1}{\gamma-1}}\big)F(t,\lambda) + Q(t,\lambda) = 0, \quad (t,\lambda)\in[0,T)\times\mathcal{K},\\
&F(T,\lambda) = 0, \quad \lambda\in\mathcal{K}, \qquad (4.16)
\end{aligned}
$$

and

$$
\begin{aligned}
&G_t(t) - G(t)r + a(\alpha\hat{x}) - c^* = 0, \quad t\in[0,T),\\
&G(T) = 0. \qquad (4.17)
\end{aligned}
$$

It is clear that there exists a unique solution to equations (4.12) and (4.17) which is of the form

$$
\begin{aligned}
g(t) &= G(t) = \int_t^T (a(\alpha\hat{x}) - c^*)e^{-r(s-t)}ds\\
&= (a(\alpha\hat{x}) - c^*)\frac{1-e^{-r(T-t)}}{r}, \quad 0\le t\le T. \qquad (4.18)
\end{aligned}
$$

The existence of solutions to equations (4.11) and (4.16) is not trivial. We can prove the following result, see the Appendix.

Proposition 4.1. *Let (A2)-(A3), (A5) hold. There exist unique solutions* $f, F \in \mathcal{C}([0,T] \times \mathcal{K}) \cap \mathcal{C}^{1,2}([0,T) \times \mathcal{K})$ *to the PDEs* (4.11), (4.16)*. The functions* f *and* F *are bounded. Moreover, the function* f *is uniformly bounded away from zero, i.e.* $f(t,\lambda) \geq K > 0,\ (t,\lambda) \in [0,T] \times \mathcal{K}$*, and the function* F *is non-positive, i.e.* $F(t,\lambda) \leq 0, (t,\lambda) \in [0,T] \times \mathcal{K}$*.*

Recalling the formulas for the candidate optimal strategies (4.3) and the expansions (4.5)-(4.6), we can derive the first-order approximations to the equilibrium strategies and the equilibrium value function.

Theorem 4.1. [6] *Let (A1)-(A5) hold and* $\varepsilon \to 0$ *in the discount factors* (4.2)*. Consider the HJB equation* (3.15)*. The first-order approximation to the equilibrium value function* v *which solves the HJB equation takes the form*

$$v^{0,*}(t,x,\lambda) + v^{1,*}(t,x,\lambda)\varepsilon = (x+g(t))^{\gamma}\big(f(t,\lambda) + F(t,\lambda)\varepsilon\big), \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times \mathcal{K},$$

and the first-order approximations to the equilibrium investment and consumption strategies are of the form

$$\begin{aligned} \tilde{\pi}^*(t,x,\lambda) &= \frac{\mu - r}{\sigma^2(1-\gamma)} \frac{x+g(t)}{x}, \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times \mathcal{K}, \\ \tilde{u}^*(t,x,\lambda) &= (x+g(t))(f(t,\lambda))^{\frac{1}{\gamma-1}} \\ &\quad \cdot \Big(1 - \frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\varepsilon\Big), \quad (t,x,\lambda) \in [0,T] \times \mathbb{R} \times \mathcal{K}, \end{aligned} \tag{4.19}$$

where the functions $g,\ f,\ F$ *solve equations* (4.18), (4.11), (4.16)*. The first-order strategies* $(\tilde{\pi}^*, \tilde{u}^*)$ *are admissible, i.e.* $(\tilde{\pi}^*, \tilde{u}^*) \in \mathcal{A}$*.*

[6]If we choose the dynamics of the mortality intensity with jumps (2.5), we will derive the same results. Only a different generator $\mathcal{L}_\lambda$ for the mortality intensity process λ would arise in equations (4.11), (4.16). Proving existence and uniqueness of solutions to equations (4.11), (4.16) would be a bit more difficult if the dynamics (2.5) were used, see Delong and Klüppelberg (2008).

For the proof we refer to the Appendix where a formal version of the result on the first-order approximation of the equilibrium value function is presented in Theorem 7.1.

The key point of our result is that the PDEs (4.11), (4.16) can be easily solved by applying explicit/implicit finite difference methods. Consequently, the first-order equilibrium investment and consumption strategies can be calculated in a feasible way.

We now comment on the strategies. The optimal investment strategy $\tilde{\pi}^*(t, x, \lambda)$ is the famous Merton portfolio adjusted by the ratio $\frac{x+g(t)}{x}$. Depending on the magnitude of the annuity payment rate $a(\alpha\hat{x})$ and the minimum consumption rate c^*, the resulting optimal risky asset holding can be higher/equal to/lower than the Merton portfolio. If the annuity payment is higher (lower) than the minimum consumption, $g(t)$ is then positive (negative), and a higher (lower) risky asset holding results. Economically it makes sense. When the annuity payment is not sufficiently high to cover the minimum consumption level, the retiree consumes at a lower rate and invests more conservatively in order to ensure the minimum consumption level. In this case, the retiree's wealth is always positive and she must act in the financial market so that she will be able to consume at least the minimum consumption in the future. In the reversed case, when the annuity payment is above the minimum consumption level, the retiree consumes at a higher rate and invests more aggressively. In this case, a negative wealth is possible at some intermediate points in time and the retiree is not afraid that she will not be able to sustain the minimum consumption level. The key observation is that the investment strategy does not depend on the parameter ε of the non-exponential discounting. Neither does the investment strategy depend on the mortality intensity. This is reasonable since the retiree is simply interested in maximizing the expected return from the investment by choosing the best allocation strategy. Parameters like mortality and discount factors, which are independent of the financial market, do not affect this objective. The form for the optimal consumption is more complex. It dependents on the mortality intensity and the parameter ε of the non-exponential discounting. In the next section, we will explain the effect of these parameters on the optimal consumption in detail. At this point we can only conclude that the higher the parameter ε in the discounting function (4.1), i.e. the higher the amount of hyperbolic discounting added to exponential discounting, the higher the consumption rate, since f is positive and F is non-positive.

One may ask whether there is an optimal choice of α at the retirement date $t = 0$. In the following, we demonstrate briefly how an optimal α can be chosen, if we change the fixed α to a choice variable in our model. The first-order approximation to the equilibrium value function gives us

$$v(0,(1-\alpha)\hat{x},\lambda) = \big((1-\alpha)\hat{x} + g(0)\big)^{\gamma}\big(f(0,\lambda) + F(0,\lambda)\varepsilon\big) + \mathcal{O}(\varepsilon^2), \quad \varepsilon \to 0, \tag{4.20}$$

Although the function F is non-positive, the expansion (4.20) holds for sufficiently small ε and $f(0,\lambda) + F(0,\lambda)\varepsilon > 0$ as the function f is positive. We have

$$\begin{aligned} (1-\alpha)\hat{x} + g(0) &= (1-\alpha)\hat{x} + \int_0^T (a(\alpha\hat{x}) - c^*)e^{-rs}ds \\ &= \hat{x} - c^* \int_0^T e^{-rs}ds + \Big(a(1)\int_0^T e^{-rs}ds - 1\Big)\alpha\hat{x}. \end{aligned} \tag{4.21}$$

Let us assume that the insurer uses the risk-free rate as a discount rate for valuing the annuity. We can now conclude that the constant in front of $\alpha\hat{x}$ is positive. Hence, $\alpha = 1$ is the optimal choice which maximizes the expected discounted consumption. This conclusion agrees with the classical result by Yaari (1965). However, we would like to point out that in our model the retiree is free to choose any α at the retirement date $t = 0$, i.e. the retiree can buy an arbitrary amount of the lifetime annuity. We expect that in real life a retiree is very likely to choose $\alpha < 1$. Note that even for $\alpha = 1$ we might still be interested in finding the optimal investment and consumption strategies. If $\alpha = 1$ but $c^* < a(\hat{x})$, then the retiree is willing to invest the excess consumption $a(\hat{x}) - c^*$ in the risky asset in the financial market in order to consume the above $a(\hat{x})$ in the future.

5 Interpretation of the equilibrium strategy and numerical example

In this section we comment on the derived first-order equilibrium consumption (4.19) by considering a numerical example. We assume that the dynamics of the mortality intensity is modelled by a geometric Brownian motion (GBM), i.e. the mortality intensity process satisfies the SDE

$$d\lambda(t) = \mu_\lambda \lambda(t)dt + \sigma_\lambda \lambda(t)dW_\lambda(t). \tag{5.1}$$

Compared to the frequently used Ornstein-Uhlenbeck processes, the GBM has the advantage that the mortality intensity is always positive. Unfortunately, the GBM is not bounded which we require when solving our optimization problem. However, we can set a sufficiently high cap on the process and assume that the mortality intensity process is bounded. We would like to point out that Huang et al. (2012) and Shen and Wei (2016) also use a geometric Brownian motion as a stochastic mortality model in their papers in which they solve investment problems closely related to ours. We use the stochastic mortality model and the calibration method from Huang et al. (2012). To provide a better analysis, we estimate the mortality intensity parameters by calibrating the survival probabilities to the Polish life table. We focus on 10, 20 and 30 years survival probabilities for 65-year old men from the 2014 Polish life table and we first fit a Gompertz deterministic mortality law to those probabilities. The expected future lifetime for 65-year old men is estimated at the level of 15.86 years and the initial mortality intensity is $\lambda(0) = 0.0215$. Next, we choose the volatility for the geometric Brownian motion (we choose $\sigma_\lambda = 0.15$ as in Huang et al. (2012)) and we find the drift μ_λ so that the expected future lifetimes agree in the stochastic and the deterministic model at time $t = 0$, i.e. $\mu_\lambda = 0.096$.

In the numerical example, we consider the set of parameters presented in Table 1. We set the interest rate to $r = 0.02$. In many European countries interest rates are currently close to zero, but not in every country. E.g. in Poland the technical rate for 2016/2017 is set by the law at 1.94% and the long-term rate determined by the European Central Bank is 3.11%. Other parameters of our model are chosen arbitrarily just to illustrate our results. We would like to remark that the qualitative conclusions derived in this section will not change if different parameters or a different stochastic mortality model are used.

Parameter	Value	Parameter	Value
Retiree's age	65	$\lambda(0)$	0.0215
μ	0.05	σ	0.25
r	0.02	γ	0.1
μ_λ	0.096	σ_λ	0.15
ρ	0.035	δ	2
T	35	q	1

Table 1: Values of the parameters considered in the numerical example.

We use Monte Carlo methods and we estimate that the probability that the retiree survives 35 years is 0.037 (standard deviation=0.0009) in our model. The expected future lifetime is 15.86 (standard deviation=0.042), as already discussed.

First, we examine how the mortality intensity influences the equilibrium consumption strategy. Let us recall that the equilibrium investment strategy is independent of the mortality intensity. In order to focus on the effect of mortality, we assume that $x + g(t) = 1, 0 \leq t \leq T$. As a consequence of Theorem 4.1, the consumption at time t depends on the mortality intensity λ at time t through the functions $f(t, \lambda)$ and $F(t, \lambda)$. The functions f and F are obtained by numerically solving the PDEs by using explicit and implicit finite difference methods. The first-order equilibrium consumption strategy is presented in Figure 2. In accordance with our intuition, the higher the mortality intensity, the higher the consumption rate. Obviously, a retiree with a higher mortality intensity should consume more in order to exhaust the available funds before the death which is expected to come sooner. The impact of the mortality intensity on consumption is stronger in the first years, since at these times the bequest motive plays

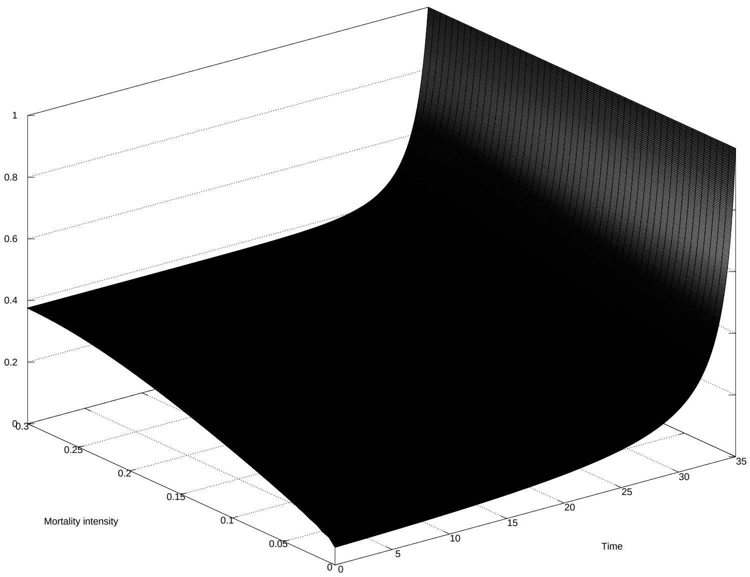

Figure 2: The first-order equilibrium consumption strategy (4.19) for $\varepsilon = 0.05$.

a negligible role (due to small probability of surviving the next 35 years).

One of the goals of this paper is to consider individual longevity risk and investigate its impact on consumption and investment. As emphasized before, the investment strategy does not depend on mortality or longevity risk, whereas longevity risk does affect the consumption strategy. In our model setup, longevity risk can be characterized by a varying mortality volatility σ_λ in the stochastic mortality intensity process (5.1). If we consider the stochastic mortality intensity process with $\sigma_\lambda = 0.3$, instead of the base value $\sigma_\lambda = 0.15$, then the probability that our 65-year old retiree survives 35 years increases from 0.037 to 0.1938 (standard deviation=0.0032) and the expected future lifetime increases from 15.86 to 18.67 (standard deviaton=0.082). Consequently, under a higher volatility of the stochastic mortality intensity process (5.1), the retiree has a longer life expectancy and she should consume less in order not to exhaust the available funds too soon. The numerical results confirm our intuition. In Figure 3 we fix the mortality intensity $\lambda(t)$ for all t (just for the ease of presentation of our result) and we plot the equilibrium consumption strategy over time for $\sigma_\lambda = 0.15$ and $\sigma_\lambda = 0.3$. We

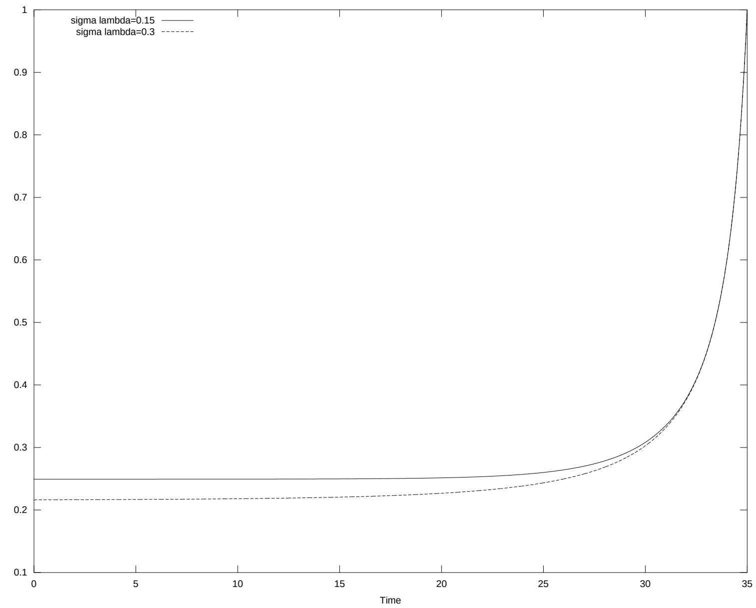

Figure 3: The first-order equilibrium consumption strategy (4.19) for $\varepsilon = 0.05$ and $\lambda(t) = 0.126, 0 \leq t \leq T$: $\sigma_\lambda = 0.15$ (upper solid line) and $\sigma_\lambda = 0.3$ (lower dashed line).

choose the mortality intensity $\lambda(t) = 0.126, 0 \leq t \leq T$, since this value corresponds to the average mortality intensity over the next 35 years for a 65-year old man under the deterministic Gompertz mortality law fitted to 2014 Polish mortality tables. We can indeed observe that the higher the longevity risk (the higher the volatility σ_λ of the stochastic mortality intensity process (5.1)), the lower the consumption rate.

Finally, we investigate the impact of perturbing exponential discounting with a small amount of hyperbolic discounting on investment and consumption strategies. The optimal investment strategy is the same for the time-inconsistent ($\varepsilon > 0$) and the time-consistent optimization problem ($\varepsilon = 0$). If exponential discounting is perturbed by adding hyperbolic discounting, then the discount rates applied to the future consumption streams are higher, see Figure 1, and the retiree prefers to consume at a higher rate since she is not willing to postpone her consumption. The higher parameter ε in the discounting function (4.1), i.e. the higher the amount of hyperbolic discounting added to exponential discounting, the higher the consumption rate. This is observed in

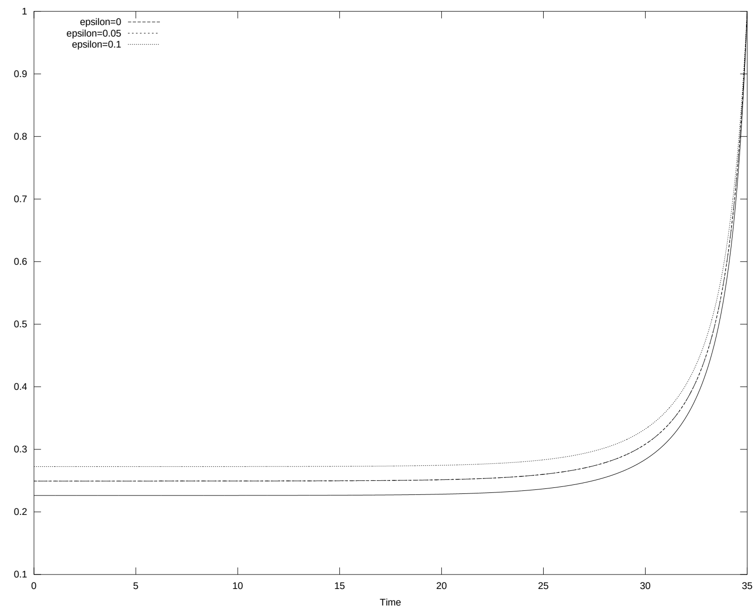

Figure 4: The first-order equilibrium consumption strategy (4.19) for $\lambda(t) = 0.126, 0 \leq t \leq T$: $\varepsilon = 0$ (lower solid line), $\varepsilon = 0.05$ (middle dashed line), $\varepsilon = 0.1$ (upper dashed line).

Figure 4, where we again fix the mortality intensity $\lambda(t) = 0.126, 0 \leq t \leq T$. We still assume that $x + g(t) = 1, 0 \leq t \leq T$ in order to focus on the impact of non-exponential discounting on the consumption strategy. In Table 2 we can find values of the consumption for $\lambda = 0.126$, as well as values of the consumption for a low and high mortality intensity ($\lambda = 0.03,\ \lambda = 0.27$).

Let us point out that there is a fundamental difference in the economic interpretations between an increase in the discount rate ρ in the exponential discounting and an increase in the parameter ε in the hyperbolic discounting, although both increases lead to a rise in the consumption rate. When ρ goes up, the retiree increases her discount rates for the entire time horizon, while when ε goes up, the retiree increases her discount rates more at the short horizon and less at the long horizon, see (3.10). In other words, a change in ρ or a change in ε leads to a significantly different change in the retiree's time preferences.

Time	0	10	20	30
$\lambda(t) = 0.03,\ \epsilon = 0$	0.1080	0.1085	0.1176	0.2063
$\lambda(t) = 0.03,\ \epsilon = 0.05$	0.1222	0.1228	0.1324	0.2255
$\lambda(t) = 0.03,\ \epsilon = 0.1$	0.1365	0.1371	0.1471	0.2448
$\lambda(t) = 0.126,\ \epsilon = 0$	0.2263	0.2263	0.2283	0.2838
$\lambda(t) = 0.126,\ \epsilon = 0.05$	0.2493	0.2494	0.2514	0.3082
$\lambda(t) = 0.126,\ \epsilon = 0.1$	0.2724	0.2725	0.2745	0.3325
$\lambda(t) = 0.27,\ \epsilon = 0$	0.3392	0.3392	0.3406	0.3866
$\lambda(t) = 0.27,\ \epsilon = 0.05$	0.3695	0.3695	0.3709	0.4171
$\lambda(t) = 0.27,\ \epsilon = 0.1$	0.3998	0.3998	0.4012	0.4475

Table 2: The first-order equilibrium consumption strategy (4.19).

In Table 3 we show the combined effect of the parameter ε and the mortality intensity λ on the consumption reduction caused by the increase of the mortality volatility for different points in time t. We first confirm the previous result: moving the mortality volatility σ_λ from 0.15 to 0.3 always leads to a reduction in the consumption rate, i.e., living longer in expectation reduces the optimal consumption level. This reduction is more substantial for a higher mortality intensity level λ. This sounds reasonable since for higher mortality intensity levels the consumption rate is higher, as we have already observed, and the consumption rate should be more adjusted if the life expectancy increases. Interestingly, this reduction becomes less substantial when the parameter ε moves from $\varepsilon = 0$ to $\varepsilon = 0.1$. Increasing ε means a decrease in the discount factors. Consequently, this leads to an increase in the consumption level, which offsets the opposite effect on consumption caused by the increase in the mortality volatility. This offsetting effect is more apparent for short-term than for long-term, because an increasing ε implies that the retiree discounts the short-term cash flows more than the long-term cash flows, see (3.10).

Compared to the time-consistent problem, the optimal consumption in the case of the time-inconsistent problem ($\varepsilon > 0$) is adjusted by the following form

$$\frac{\tilde{u}^*(t) - u^{0,*}(t)}{u^{0,*}(t)} = -\frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\varepsilon. \tag{5.2}$$

Time	0	10	20	30
$\lambda(t) = 0.03,\ \epsilon = 0$	11.48%	8.29%	2.89%	0.10%
$\lambda(t) = 0.03,\ \epsilon = 0.05$	10.72%	7.40%	2.72%	0.09%
$\lambda(t) = 0.03,\ \epsilon = 0.1$	10.19%	7.40%	2.65%	0.08%
$\lambda(t) = 0.126,\ \epsilon = 0$	14.10%	13.35%	10.34%	1.83%
$\lambda(t) = 0.126,\ \epsilon = 0.05$	13.28%	12.59%	9.79%	1.82%
$\lambda(t) = 0.126,\ \epsilon = 0.1$	12.59%	11.96%	9.36%	1.71%
$\lambda(t) = 0.27,\ \epsilon = 0$	17.42%	17.07%	15.47%	7.92%
$\lambda(t) = 0.27,\ \epsilon = 0.05$	16.56%	16.27%	14.83%	7.67%
$\lambda(t) = 0.27,\ \epsilon = 0.1$	15.86%	15.56%	14.23%	7.46%

Table 3: The reduction in the first-order equilibrium consumption strategy resulting from the increase in the mortality volatility σ_λ from 0.15 to 0.3 (The values should be interpreted with minus sign).

According to Proposition 4.1, the function f is positive and the function F is negative. Consequently, the first-order adjustment (5.2) of the optimal time-consistent consumption for the time-inconsistent problem is positive. The first-order adjustment for the equilibrium consumption (5.2) depends on the mortality intensity. We expect that the higher the mortality intensity, the lower the first-order adjustment since for high mortality intensity levels the consumption is already high, as we have already observed, and the retiree is expected to die soon so the effect of increasing the discount rates is smaller. Figure 5 confirms our intuition.

6 Conclusion

In this paper we have studied a version of the Merton problem for a retiree in which we combine four important aspects for the first time : asset allocation, sustainable withdrawal, individual longevity risk and non-exponential discounting. We have derived a non-local HJB equation which characterizes the equilibrium investment and consumption strategy for our time-inconsistent optimization problem. We have established the first-order expansions to the equilibrium value function and the equilibrium strategies by applying expansion techniques. The expansion is performed on the param-

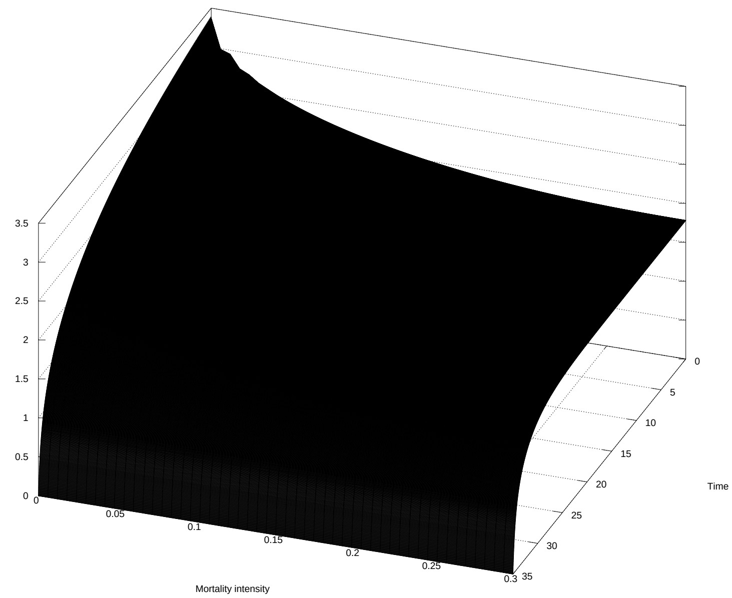

Figure 5: The first-order adjustment for the equilibrium consumption strategy (5.2) for $\varepsilon = 1$.

eter controlling the degree of discounting in the hyperbolic discounting that is added to the exponential discount factors. We have not fully analyzed the impact of the non-exponential discounting factors on retirement plans, but we have succeeded in analyzing the first-order impact of adding hyperbolic discounting to exponential discount factors. In our framework the first-order equilibrium investment and consumption strategies can be calculated in a feasible way by solving PDEs. We have investigated and discussed the impact of the key parameters of our model: the minimum consumption, the level of the bought annuity, the mortality intensity, the longevity risk (the volatility of the mortality intensity process), the parameter controlling the degree of hyperbolic discounting on the first-order equilibrium strategies.

There are still some unanswered and interesting questions related to the problem and the solution presented in this paper. Is it possible to find the true equilibrium strategies in a numerically feasible (and mathematically formal) way? How do the mortality intensity/the longevity risk and the hyperbolic discounting affect the consumption if we change the risk aversion parameter of the utility (in particular, if we consider a logarithmic utility)? How to define and solve our optimization problem with a weaker assumption for consumption sustainability (an investment/consumption optimization problem with ruin)? We hope to answer some of this questions in future research.

Acknowledgement: We would like to thank two referees for very useful and interesting remarks that allow us to improve the first version of this paper.

7 Appendix

Unless otherwise stated, we consider $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times \mathcal{K}$.

Proof of Theorem 3.1: We follow the ideas from Björk and Murgoci (2010) and Ekeland et al. (2012).

1. First we prove that a function v^* which solves the HJB equation (3.15) is the value function (3.11) corresponding to the strategy (π^*, u^*), i.e. we have the relation

$$\begin{aligned} v^*(t,x,\lambda) &= v^{\pi^*,u^*}(t,x,\lambda) \\ &= \mathbb{E}_{t,x,\lambda}\left[\int_t^{\tau\wedge T}\phi(s-t)(u^*(s))^\gamma ds + q\phi(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}_{\{\tau>T\}}\right], \end{aligned} \tag{7.1}$$

where (π^*, u^*) is given by (3.15)-(3.16). We introduce a localizing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \to T,\ n \to \infty$. By Itô's formula we get

$$\begin{aligned} &\mathbb{E}_{t,x,\lambda}\Big[v^*(T\wedge\tau_n, X^{\pi^*,u^*}(T\wedge\tau_n), \lambda(T\wedge\tau_n))\mathbf{1}\{\tau > T\wedge\tau_n\}\Big] = v^*(t,x,\lambda) \\ &\quad+\mathbb{E}_{t,x,\lambda}\Big[\int_t^{T\wedge\tau_n\wedge\tau} v_t^*(s, X^{\pi^*,u^*}(s), \lambda(s))ds + \int_t^{T\wedge\tau_n\wedge\tau}\mathcal{L}_x^{\pi^*,u^*}v^*(s, X^{\pi^*,u^*}(s),\lambda(s))ds \\ &\quad+\int_t^{T\wedge\tau_n\wedge\tau}\mathcal{L}_\lambda v^*(s, X^{\pi^*,u^*}(s),\lambda(s))ds - \int_t^{T\wedge\tau_n\wedge\tau} v^*(s, X^{\pi^*,u^*}(s),\lambda(s))\lambda(s)ds\Big]. \end{aligned} \tag{7.2}$$

The last term in (7.2) arises since there might be a jump in the value function $v^*(T \wedge \tau_n, X^{\pi^*,u^*}(T\wedge\tau_n), \lambda(T\wedge\tau_n))\mathbf{1}\{\tau > T\wedge\tau_n\}$ and the jump in the value function arrives with the intensity $\lambda(t)$ (which is the mortality intensity for τ). Recalling the HJB equation (3.15), using the property of conditional expectations and changing the order of integration, we derive

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^*(T\wedge\tau_n, X^{\pi^*,u^*}(T\wedge\tau_n), \lambda(T\wedge\tau_n))\mathbf{1}\{\tau > T\wedge\tau_n\}\Big] = v^*(t,x,\lambda)\\
&\quad+\mathbb{E}_{t,x,\lambda}\Big[-\int_t^{T\wedge\tau_n\wedge\tau}\Big(\int_s^{T\wedge\tau}\phi'(w-s)(u^*(w))^\gamma dw + q\phi'(T-s)(X^{\pi^*,u^*}(T))^\gamma\mathbf{1}_{\{\tau>T\}}\Big)ds\\
&\quad-\int_t^{T\wedge\tau_n\wedge\tau}(u^*(s))^\gamma ds\Big] = v^*(t,x,\lambda) + \mathbb{E}_{t,x,\lambda}\Big[-\int_t^{T\wedge\tau_n\wedge\tau}\phi(w-t)(u^*(w))^\gamma dw\\
&\quad+\int_{T\wedge\tau_n\wedge\tau}^{T\wedge\tau}\big(\phi(w-(T\wedge\tau_n\wedge\tau)) - \phi(w-t)\big)(u^*(w))^\gamma dw\\
&\quad+q\big(\phi(T-(T\wedge\tau_n)) - \phi(T-t)\big)(X^{\pi^*,u^*}(T))^\gamma\mathbf{1}\{\tau>T\}\Big].
\end{aligned}$$

We now take $n \to \infty$. Applying Lebesgue's dominated convergence theorem (recall that we assume that the family $\{v^*(\mathcal{T}, X^{\pi^*,u^*}(\mathcal{T}), \lambda(\mathcal{T})), \mathcal{T} \textit{ is an } \mathbb{F}-\textit{stopping time}\}$ is uniformly integrable) and using the terminal condition for $v^*(T,x,\lambda)$ from the HJB equation (3.15), we can prove (7.1).

2. We now prove that the strategy (π^*, u^*) defined by (3.15)-(3.16) is an equilibrium strategy. Fix $(t,x) \in [0,T) \times \mathbb{R}$ and choose $(\pi_h, u_h) \in \mathcal{A}$ as specified in Definition 3.3. We have

$$\begin{aligned}
&v^{\pi_h,u_h}(t,x,\lambda)\\
&\quad= v^{\pi_h,u_h}(t,x,\lambda) - \mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big]\\
&\qquad+\mathbb{E}_{t,x,\lambda}\Big[v^*(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big],
\end{aligned}\tag{7.3}$$

since (π^*, u^*) is applied after time $s \geq t+h$ and v^* is the value function corresponding to (π^*, u^*), as proved in the previous point. Equation (7.3) can be rewritten as

$$
\begin{aligned}
& v^{\pi_h,u_h}(t,x,\lambda) - v^*(t,x,\lambda) \\
&= \Big(v^{\pi_h,u_h}(t,x,\lambda) - \mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h)) \mathbf{1}\{\tau > t+h\}\Big]\Big) \\
&\quad + \Big(\mathbb{E}_{t,x,\lambda}\Big[v^*(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h)) \mathbf{1}\{\tau > t+h\}\Big] - v^*(t,x,\lambda)\Big). \qquad (7.4)
\end{aligned}
$$

We deal with the first term in (7.4). Recalling the definition of the value function $v^{\pi,u}$, see (3.11), we immediately get

$$
\begin{aligned}
& v^{\pi_h,u_h}(t,x,\lambda) \\
&= \mathbb{E}_{t,x,\lambda}\left[\int_t^{\tau\wedge T} \phi(s-t)(u^*(s))^\gamma ds + q\phi(T-t)(X^{\pi_h,u_h}(T))^\gamma \mathbf{1}_{\{\tau > T\}} \right],
\end{aligned}
$$

and using the property of conditional expectations we derive

$$
\begin{aligned}
& \mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h)) \mathbf{1}\{\tau > t+h\}\Big] \\
&= \mathbb{E}_{t,x,\lambda}\left[\int_{\tau\wedge(t+h)}^{\tau\wedge T} \phi(s-t-h)(u^*(s))^\gamma ds + q\phi(T-t-h)(X^{\pi_h,u_h}(T))^\gamma \mathbf{1}_{\{\tau > T\}} \right].
\end{aligned}
$$

Hence, the first term in (7.4) is equal to

$$
\begin{aligned}
& v^{\pi_h,u_h}(t,x,\lambda) - \mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h)) \mathbf{1}\{\tau > t+h\}\Big] \\
&= \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge(t+h)} \phi(s-t)(u(s))^\gamma ds \Big] \\
&\quad + \mathbb{E}_{t,x,\lambda}\Big[\int_{\tau\wedge(t+h)}^{\tau\wedge T} (\phi(s-t) - \phi(s-t-h))(u^*(s))^\gamma ds \Big] \\
&\quad + \mathbb{E}_{t,x,\lambda}\Big[q(\phi(T-t) - \phi(T-t-h))(X^{\pi_h,u_h}(T))^\gamma 1_{\{\tau > T\}} \Big]. \qquad (7.5)
\end{aligned}
$$

We now investigate three terms on the right hand side in (7.5). If we apply Fubini's theorem and differentiate the Lebesgue's integral, we obtain

$$\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{E}_{t,x,\lambda}\Big[\int_t^{t+h}\phi(s-t)(u(s))^{\gamma}\mathbf{1}\{s\le\tau\}ds\Big]}{h}\\
&\quad=\lim_{h\to 0}\frac{\int_t^{t+h}\mathbb{E}_{t,x,\lambda}\big[\phi(s-t)(u(s))^{\gamma}\mathbf{1}\{s\le\tau\}\big]ds}{h}=(u(t))^{\gamma}. \qquad (7.6)
\end{aligned}$$

Since

$$\begin{aligned}
|\phi(s-t)-\phi(s-t-h)| &\le Kh,\\
\frac{\int_t^{t+h}|\phi(s-t)-\phi(s-t-h)|(u^*(s))^{\gamma}ds}{h} &\le K\int_t^{t+h}(u^*(s))^{\gamma}ds,
\end{aligned}$$

we can apply Lebesgue's dominated convergence theorem and we can derive the limit

$$\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{E}_{t,x,\lambda}\Big[\int_{\tau\wedge(t+h)}^{\tau\wedge T}(\phi(s-t)-\phi(s-t-h))(u^*(s))^{\gamma}ds\Big]}{h}\\
&= \lim_{h\to 0}\mathbb{E}_{t,x,\lambda}\Big[\int_t^T\frac{\phi(s-t)-\phi(s-t-h)}{h}(u^*(s))^{\gamma}\mathbf{1}\{s\le\tau\}ds\mathbf{1}\{\tau>t+h\}\Big]\\
&\quad -\lim_{h\to 0}\mathbb{E}_{t,x,\lambda}\Big[\frac{\int_t^{t+h}(\phi(s-t)-\phi(s-t-h))(u^*(s))^{\gamma}ds}{h}\mathbf{1}\{\tau>t+h\}\Big]\\
&= \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\phi'(s-t)(u^*(s))^{\gamma}ds\Big]. \qquad (7.7)
\end{aligned}$$

Before we prove the limit for the third term in (7.5) we need some preliminary results. Let X^{t,x,π_h,u_h} denote the wealth process which starts at x at time t. We can notice that $X^{t,x,\pi_h,u_h}(T)=X^{t+h,X^{t,x,\pi_h,u_h}(t+h),\pi^*,u^*}(T)$. By continuity of integrals in (3.6) we can conclude that $X^{t,x,\pi_h,u_h}(t+h)\to x, h\to 0$. By classical results on SDEs, see e.g. Theorem II.5.2 in Kunita (1984) or Becherer and Schweizer (2005), we know that $(t,x)\mapsto X^{t,x,\pi_h,u_h}$ is continuous. Hence, we can conclude that $X^{t+h,X^{t,x,\pi_h,u_h}(t+h),\pi^*,u^*}(T)\to X^{t,x,\pi^*,u^*}(T),\ h\to 0$. By inequality (3.7) and admissability of (π_h,u_h) we obtain the estimate

$$\mathbb{E}\Big[\Big|\frac{\phi(T-t)-\phi(T-t-h)}{h}(X^{t,x,\pi_h,u_h}(T))^{\gamma}\Big|^2\Big] \leq K\Big(1+\mathbb{E}\Big[\int_t^{t+h}|\pi(s)X^{\pi,u}(s)|^2 ds$$
$$+\int_t^{t+h}|u(s)|^2 ds+\int_{t+h}^{T}|\pi^*(s)X^{t+h,X^{t,x,\pi,u}(t+h),\pi^*,u^*}(s)|^2 ds+\int_{t+h}^{T}|u^*(s)|^2 ds\Big]\Big) \leq K<\infty,$$

and we can deduce that the sequence $\left\{\frac{\phi(T-t)-\phi(T-t-h)}{h}(X^{\pi_h,u_h}(T))^{\gamma}, h \in [0,\epsilon]\right\}$ is uniformly integrable. Now we can establish the limit

$$\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{E}_{t,x,\lambda}\Big[q(\phi(T-t)-\phi(T-t-h))(X^{\pi_h,u_h}(T))^{\gamma}\mathbf{1}_{\{\tau>T\}}\Big]}{h}\\
&= \lim_{h\to 0}\mathbb{E}_{t,x,\lambda}\Big[q\frac{\phi(T-t)-\phi(T-t-h)}{h}(X^{\pi_h,u_h}(T))^{\gamma}\mathbf{1}_{\{\tau>T\}}\Big]\\
&= \mathbb{E}_{t,x,\lambda}\Big[q\phi'(T-t)(X^{\pi^*,u^*}(T))^{\gamma}1_{\{\tau>T\}}\Big]. \qquad (7.8)
\end{aligned}$$

Collecting (7.6)-(7.8), we get

$$\begin{aligned}
&\lim_{h\to 0}\frac{v^{\pi_h,u_h}(t,x,\lambda)-\mathbb{E}_{t,x,\lambda}\Big[v^{\pi_h,u_h}(t+h,X^{\pi_h,u_h}(t+h),\lambda(t+h))\mathbf{1}\{\tau>t+h\}\Big]}{h}\\
&= (u(t))^{\gamma}+\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T}\phi'(s-t)(u^*(s))^{\gamma}ds+q\phi'(T-t)(X^{\pi^*,u^*}(T))^{\gamma}\mathbf{1}_{\{\tau>T\}}\Big]. \qquad (7.9)
\end{aligned}$$

We deal with the second term in (7.4). Recall that an arbitrary (π,u) is applied on $[t,t+h]$. We introduce a localizing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that

$\tau_n \to T,\ n \to \infty$. Applying Itô's formula, as in point 1, we can derive

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^*((t+h)\wedge\tau_n, X^{\pi_h,u_h}((t+h)\wedge\tau_n), \lambda((t+h)\wedge\tau_n))\mathbf{1}\{\tau > (t+h)\wedge\tau_n\}\Big] \\
&\qquad -v^*(t,x,\lambda) \\
&\quad = \mathbb{E}_{t,x,\lambda}\Big[\int_t^{(t+h)\wedge\tau_n\wedge\tau} v_t^*(s, X^{\pi,u}(s), \lambda(s))ds + \int_t^{(t+h)\wedge\tau_n\wedge\tau} \mathcal{L}_x^{\pi,u} v^*(s, X^{\pi,u}(s), \lambda(s))ds \\
&\qquad + \int_t^{(t+h)\wedge\tau_n\wedge\tau} \mathcal{L}_\lambda v^*(s, X^{\pi,u}(s), \lambda(s))ds - \int_t^{(t+h)\wedge\tau_n\wedge\tau} v^*(s, X^{\pi,u}(s), \lambda(s))\lambda(s)ds\Big] \\
&\quad \leq \mathbb{E}_{t,x,\lambda}\Big[\int_t^{(t+h)\wedge\tau_n\wedge\tau} \sup_{\pi,u}\Big\{(u(s))^\gamma + v_t^*(s, X^{\pi,u}(s), \lambda(s)) + \mathcal{L}_x^{\pi,u} v^*(s, X^{\pi,u}(s), \lambda(s)) \\
&\qquad + \int_t^{(t+h)\wedge\tau_n\wedge\tau} \mathcal{L}_\lambda v^*(s, X^{\pi,u}(s), \lambda(s))ds - v^*(s, X^{\pi,u}(s), \lambda(s))\lambda(s)\Big\}ds \\
&\qquad - \int_t^{(t+h)\wedge\tau_n\wedge\tau} (u(s))^\gamma ds\Big],
\end{aligned}$$

where the supremum is with respect to u^γ and $\mathcal{L}_x^{\pi,u}$. Since the function v^* satisfies the HJB equation (3.15) we get the inequality

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^*((t+h)\wedge\tau_n, X^{\pi_h,u_h}((t+h)\wedge\tau_n), \lambda((t+h)\wedge\tau_n))\mathbf{1}\{\tau > (t+h)\wedge\tau_n\}\Big] \\
&\qquad -v^*(t,x,\lambda) \\
&\quad \leq \mathbb{E}_{t,x,\lambda}\Big[-\int_t^{(t+h)\wedge\tau_n\wedge\tau}\Big(\int_s^{\tau\wedge T} \phi'(w-t)(u^*(w))^\gamma dw + q\phi'(T-s)(X^{\pi^*,u^*}(T))^\gamma 1_{\{\tau>T\}}\Big)ds \\
&\qquad - \int_t^{(t+h)\wedge\tau_n\wedge\tau}(u(s))^\gamma ds\Big].
\end{aligned} \tag{7.10}$$

We now take $n \to \infty$. We apply Fatou's lemma on the left hand side of (7.10) together with Lebesgue's dominated convergence theorem on the right hand side (7.10) and we derive the inequality

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[v^*(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h))\mathbf{1}\{\tau > t+h\}\Big] - v^*(t,x,\lambda) \\
&\quad \leq \mathbb{E}_{t,x,\lambda}\Big[-\int_t^{(t+h)\wedge\tau}\Big(\int_s^{\tau\wedge T} \phi'(w-t)(u^*(w))^\gamma dw + q\phi'(T-s)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}_{\{\tau>T\}}\Big)ds \\
&\qquad - \int_t^{(t+h)\wedge\tau}(u(s))^\gamma ds\Big].
\end{aligned}$$

If we use Fubini's theorem and differentiate the Lebesque's integral, we obtain the inequality

$$
\begin{aligned}
&\lim_{h\to 0} \frac{\mathbb{E}_{t,x,\lambda}\Big[v^*(t+h, X^{\pi_h,u_h}(t+h), \lambda(t+h))\mathbf{1}\{\tau > t+h\}\Big] - v^*(t,x,\lambda)}{h} \\
&\leq \; -\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \phi'(w-t)(u^*(w))^\gamma dw + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma 1_{\{\tau>T\}}\Big] \\
&\quad -(u(t))^\gamma. \qquad (7.11)
\end{aligned}
$$

Finally, we substitute (7.9) and (7.11) into (7.4) and we prove that (π^*, u^*) is an equilibrium strategy, i.e. (π^*, u^*) satisfies the inequality (3.14). □

Proof of Proposition 4.1:

The proof is divided into several steps.

1. *We prove that there exists a unique solution* $f \in \mathcal{C}([0,T]\times\mathcal{K}) \cap \mathcal{C}^{1,2}([0,T)\times\mathcal{K})$ *to the PDE* (4.11) *and that* $0 < K_l \leq f(t,\lambda) \leq K_u$. We define the operator

$$
\begin{aligned}
(\mathcal{M}f)(t,\lambda) &= \mathbb{E}_{t,\lambda}\Big[q e^{-\left(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(T-t) - \int_t^T \lambda(u)du} \\
&+ \int_t^T (1-\gamma) e^{-\left(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(s-t) - \int_t^s \lambda(u)du} f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}} ds\Big]. \qquad (7.12)
\end{aligned}
$$

Recall that the process λ is assumed to be bounded from below. It is easy to see that if $f(t,\lambda) \geq K_l > 0$ then $f^*(t,\lambda) = (\mathcal{M}f)(t,\lambda) \geq K_l$ (the constant K_l is determined by the first term in (7.12)). Moreover, if $f(t,\lambda) \geq K_l > 0$ and $f(t,\lambda) \leq K_u < +\infty$ then $f^*(t,\lambda) = (\mathcal{M}f)(t,\lambda) \leq K_u$ (the constant K_u is determined by both terms in (7.12) and the lower bound K_l). Hence, the non-linear term in the PDE (4.11), i.e. the mapping $f \mapsto f^{\frac{\gamma}{\gamma-1}}$ is Lipschitz continuous on $[K_l, K_u]$. By Propositions 2.1 and 2.3 from Becherer and Schweizer (2005) the operator $\mathcal{M}$ is a contraction in the class of functions $\{f : 0 < K_l \leq f(t,\lambda) \leq K_u < +\infty\}$ and the fixed point of the operator $f^*(t,\lambda) = (\mathcal{M}f^*)(t,\lambda)$ is the unique solution to the PDE (4.11). Moreover, $f^* \in \mathcal{C}([0,T]\times\mathcal{K}) \cap \mathcal{C}^{1,2}([0,T)\times\mathcal{K})$.

2. *We prove that f is Lipschitz continuous in λ uniformly in t.* Let us consider a sequence $f_{n+1}(t,\lambda) = \mathcal{M} f_n(t,\lambda)$. We know that $f_n \to f,\ n \to \infty$, by the previous point. Let us initiate the iteration with a function f_0 which satisfies the Lipschitz condition

$$|f_0(t,\lambda) - f_0(t,\lambda')| \leq \theta(t)|\lambda - \lambda'|,$$

where θ will be specified in the sequel. Assume that

$$|f_n(t,\lambda) - f_n(t,\lambda')| \leq \theta(t)|\lambda - \lambda'|, \tag{7.13}$$

for some $n \in \mathbb{N}$. Clearly, inequality (7.13) holds for $n = 0$. We now show that inequality (7.13) holds for $n+1$. Let $(\lambda^{t,\lambda}(s), t \leq t \leq T)$ denote the process which solves the SDE (2.4) and starts at time t from $\lambda(t) = \lambda$. Recalling the definition of the operator from (7.12) we can derive

$$\begin{aligned}
&|f_{n+1}(t,\lambda) - f_{n+1}(t,\lambda')| \\
&\quad \leq \ K\Big(\mathbb{E}\Big[\int_t^T |\lambda^{t,\lambda}(s) - \lambda^{t,\lambda'}(s)|ds + \int_t^T |f_n(s,\lambda^{t,\lambda}(s)) - f_n(s,\lambda^{t,\lambda'}(s))|ds\Big]\Big) \\
&\quad \leq \ K\Big(1 + \int_t^T \theta(s)ds\Big)\mathbb{E}\Big[\sup_{s\in[t,T]} |\lambda^{t,\lambda}(s) - \lambda^{t,\lambda'}(s)|\Big] \\
&\quad \leq \ K\Big(1 + \int_t^T \theta(s)ds\Big)|\lambda - \lambda'|, \qquad (7.14)
\end{aligned}$$

where we use some standard estimates based on the mean-value theorem, Lipschitz continuity of the mapping $f \mapsto f^{\frac{\gamma}{\gamma-1}}$ on $[K_l, K_u]$, uniform boundedness of the process λ, uniform boundedness of the sequence f_n, inequality (7.13) and a classical estimate for $\mathbb{E}\big[\sup_{s\in[t,T]} |\lambda^{t,\lambda}(s) - \lambda^{t,\lambda'}(s)|\big]$ from the theory of SDEs, see e.g equation (4.6) in El Karoui et al. (1997). Let us choose θ such that

$$\theta(t) = K\Big(1 + \int_t^T \theta(s)ds\Big),$$

i.e. we choose $\theta(t) = Ke^{K(T-t)}$ with a sufficiently large K. With such a choice of θ, from (7.14) we immediately get the inequality

$$|f_{n+1}(t,\lambda) - f_{n+1}(t,\lambda')| \le \theta(t)|\lambda - \lambda'|. \tag{7.15}$$

Since the function θ in (7.15) does not depend on n, we conclude that the limit $f_n \to f,\ n \to \infty$, is also Lipschitz continuous in λ uniformly in t.

3. *We prove that f is Hölder continuous in t uniformly in λ.* Let us consider the BSDE

$$\begin{aligned} dY^{t,\lambda}(s) &= \Big(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2} + \lambda^{t,\lambda}(s)\Big)Y^{t,\lambda}(s)ds \\ &\quad -(1-\gamma)f(s,\lambda^{t,\lambda}(s))^{\frac{\gamma}{\gamma-1}}ds + Z^{t,\lambda}(s)dW_\lambda(s), \quad 0 \le s \le T, \\ Y^{t,\lambda}(T) &= q, \end{aligned} \tag{7.16}$$

and the forward dynamics of the process $(\lambda^{t,\lambda}(s), t \le s \le T)$ is given by the SDE (2.4). Recall that $\lambda^{t,\lambda}$ denotes the process λ which starts at time t from $\lambda(t) = \lambda$. Since λ and f are bounded, we deal with a linear BSDE. By Theorem 2.1 in El Karoui et al. (1997) there exists a unique solution to the BSDE (7.16) and the solution has the probabilistic representation $Y^{t,\lambda}(s) = (\mathcal{M}f)(s,\lambda^{t,\lambda}(s))$. Moreover, by uniqueness of solution to the BSDE (7.16) we must have $Y^{t,\lambda}(s) = f(s,\lambda^{t,\lambda}(s))$. We notice that the generator of the BSDE (7.16) is bounded and Lipschitz continuous in λ uniformly in (t,y) since the process Y is bounded and f together with $f^{\frac{\gamma}{\gamma-1}}$ are Lipschitz continuous in λ (as proved in the previous point). Hence, by Proposition 4.1 from El Karoui et al. (1997) we have the inequality

$$|f(t,\lambda) - f(t',\lambda)| \le K|t-t'|^{1/2},$$

and our assertion is proved.

4. *We prove that for any $s \in [0,T]$ there exists a unique solution $P^s \in \mathcal{C}([0,s] \times \mathcal{K}) \cap \mathcal{C}^{1,2}([0,s) \times \mathcal{K})$ to the PDE (4.15). The solution P^s is Lipschitz continuous in*

λ *uniformly in* t *and Hölder continuous in* t *uniformly in* λ. Since λ and f are bounded and $f(t,\lambda), f^{\frac{\gamma}{\gamma-1}}(t,\lambda), f^{\frac{1}{\gamma-1}}(t,\lambda)$ are uniformly Hölder continuous in (t,λ), by Proposition 2.3 from Becherer and Schweizer (2005) there exists a unique solution $P^s \in \mathcal{C}([0,s]\times\mathcal{K}) \cap \mathcal{C}^{1,2}([0,s)\times\mathcal{K})$ to the PDE (4.15) for any $s \in [0,T]$. Moreover, we have the representation

$$\begin{aligned} & P^s(t,\lambda) \\ = \;& \mathbb{E}_{t,\lambda}\Big[e^{-\left(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(s-t) - \int_t^s \left(\lambda(u)+\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}\right)du} f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\Big], \quad 0 \le t \le s. \end{aligned} \tag{7.17}$$

Let us now consider the BSDE

$$\begin{aligned} dY^{t,\lambda}(u) &= \Big(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2} + \lambda^{t,\lambda}(u) \\ &\quad + \gamma f(t,\lambda^{t,\lambda}(u))^{\frac{1}{\gamma-1}}\Big) Y^{t,\lambda}(u)du + Z^{t,\lambda}(u)dW_\lambda(u), \quad 0 \le u \le s, \\ Y^{t,\lambda}(s) &= f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}, \end{aligned} \tag{7.18}$$

and the forward dynamics of the process $(\lambda^{t,\lambda}(s), t \le s \le T)$ given by the SDE (2.4). As in the previous point, we can prove our second assertion by recalling Proposition 4.1 from El Karoui et al. (1997).

5. *We prove that there exists a unique solution* $F \in \mathcal{C}([0,T]\times\mathcal{K}) \cap \mathcal{C}^{1,2}([0,T)\times\mathcal{K})$ *to the PDE* (4.16) *and that* $K_l \le F(t,\lambda) \le 0$. Using our previous results and representation (4.14) for the function Q we can prove that the function Q is Lipschitz continuous in λ uniformly in t and Hölder continuous in t uniformly in λ. The assertion of this point and the representation

$$\begin{aligned} & F(t,\lambda) \\ = \;& \mathbb{E}_{t,x,\lambda}\Big[\int_t^T Q(s,\lambda(s)) e^{-\left(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(s-t) - \int_t^s \left(\lambda(u)+\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}\right)du} ds\Big], \end{aligned} \tag{7.19}$$

follow now from Proposition 2.3 in Becherer and Schweizer (2005). From (4.14) we conclude that the sign of Q, and consequently the sign of F, is determined by the sign of the derivative ϑ'. Since

$$\vartheta'(t) = -\frac{\delta}{1+\delta t} \leq 0,$$

the function F is non-positive. As in the previous points, we can also deduce that the function F is Lipschitz continuous in λ uniformly in t and Hölder continuous in t uniformly in λ (by using Proposition 4.1 from El Karoui et al. (1997)). $\square$

The proof of representation (4.13)**-**(4.14)**:** Since $v^0(t,x,\lambda) = (x+g(t))^\gamma f(t,\lambda)$, from (4.10) we get the formulas

$$\pi^{0,*}(t,x,\lambda) = \frac{\mu - r}{\sigma^2(1-\gamma)} \frac{x+g(t)}{x}, \quad u^{0,*}(t,x,\lambda) = (x+g(t)) f(t,\lambda)^{\frac{1}{\gamma-1}}.$$

Recall that $g(T) = 0$ and $f(T,\lambda) = q$. It is easy to see that in order to calculate the expectation in (4.13) we have to calculate the expectation

$$\mathbb{E}_{t,x,\lambda}\big[e^{-\rho(s-t)}(X^{\pi^{0,*},u^{0,*}}(s) + g(s))^\gamma f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}} \mathbf{1}\{s<\tau\}\big], \quad t \leq s \leq T. \tag{7.20}$$

Using similar techniques that lead to the SDE (7.26), see the proof below, we can deduce the dynamics

$$\begin{aligned} d(X^{\pi^{0,*},u^{0,*}}(s) + g(s)) &= (X^{\pi^{0,*},u^{0,*}}(s) + g(s)) dZ(s), \\ dZ(s) &= \big(\tilde{\mu} - f(s,\lambda(s))^{\frac{1}{\gamma-1}}\big) ds + \tilde{\sigma} dW_m(s), \end{aligned}$$

and, as in (7.27), we get the solution

$$\begin{aligned} & X^{\pi^{0,*},u^{0,*}}(s) + g(s) \\ &= (X^{\pi^{0,*},u^{0,*}}(t) + g(t)) e^{(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2)(s-t) - \int_t^s f(u,\lambda(u))^{\frac{1}{\gamma-1}} du + \tilde{\sigma}(W_m(s) - W_m(t))}, \quad t \leq s \leq T. \end{aligned}$$

By independence of (λ,τ) and W and the property of conditional expectation we derive

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[e^{-\rho(s-t)}\big(X^{\pi^{0,*},u^{0,*}}(s)+g(s)\big)^{\gamma}f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\mathbf{1}\{s<\tau\}\Big]\\
&= \big(x+g(t)\big)^{\gamma}\mathbb{E}_{t,x,\lambda}\Big[e^{-\rho(s-t)+\gamma(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)(s-t)+\gamma\tilde{\sigma}(W_m(s)-W_m(t))}\Big]\\
&\quad\cdot\mathbb{E}_{t,x,\lambda}\Big[e^{-\int_t^s\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}du}f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\mathbb{E}_{t,x,\lambda}\big[\mathbf{1}\{s<\tau\}|\sigma(\lambda(s),s\in[t,T])\big]\Big]\\
&= \big(x+g(t)\big)^{\gamma}e^{-\rho(s-t)+\gamma(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)(s-t)+\frac{1}{2}\gamma^2\tilde{\sigma}^2(s-t)}\\
&\quad\cdot\mathbb{E}_{t,\lambda}\Big[e^{-\int_t^s\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}du-\int_t^s\lambda^{t,\lambda}(u)du}f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\Big],\quad t\le s\le T. \qquad (7.21)
\end{aligned}$$

Since $g(T)=0$ and $f(T,\lambda)=q$, we get representation (4.14) with function P^s defined in (7.17). The fact that P^s satisfies the PDE (4.15) is proved while proving Proposition 4.1. □

The proof of formula (4.9)**:** We follow the idea from Dong and Sircar (2014). We assume that the following expansions hold:

$$\begin{aligned}
v(t,x,\lambda) &= v^0(t,x,\lambda)+v^1(t,x,\lambda)\varepsilon+\mathcal{O}(\varepsilon^2),\\
u^*(t) &= u^{0,*}(t)+u^{1,*}(t)\varepsilon+\mathcal{O}(\varepsilon^2),\\
\pi^*(t) &= \pi^{0,*}(t)+\pi^{1,*}(t)\varepsilon+\mathcal{O}(\varepsilon^2),\\
X^{\pi^*,u^*}(t) &= X^{\pi^{0,*},u^{0,*}}(t)+X^{\pi^{1,*},u^{1,*}}(t)\varepsilon+\mathcal{O}(\varepsilon^2),
\end{aligned}$$

where v denotes the equilibrium value function under an equilibrium strategy (π^*,u^*). Let us recall that we consider Markov strategies, i.e. $u(t)=u(t,X^{\pi,u}(t),\lambda(t))$. In the sequel we assume $u^*,u^{0,*},u^{1,*}$ are differentiable with respect to the state variable x. We have

$$
\begin{aligned}
(u^*(t))^\gamma = (u^*(X^{\pi^*,u^*}(t)))^\gamma &= \Big(u^*\big(X^{\pi^{0,*},u^{0,*}}(t) + X^{\pi^{1,*},u^{1,*}}(t)\varepsilon + \mathcal{O}(\varepsilon^2)\big)\Big)^\gamma \\
&= \Big(u^{0,*}\big(X^{\pi^{0,*},u^{0,*}}(t) + X^{\pi^{1,*},u^{1,*}}(t)\varepsilon\big) + u^{1,*}\big(X^{\pi^{0,*},u^{0,*}}(t) + X^{\pi^{1,*},u^{1,*}}(t)\varepsilon\big)\varepsilon + \mathcal{O}(\varepsilon^2)\Big)^\gamma \\
&= \Big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t)) + \Big((u^{0,*})'(X^{\pi^{0,*},u^{0,*}}(t))X^{\pi^{1,*},u^{1,*}}(t) + u^{1,*}(X^{\pi^{0,*},u^{0,*}}(t))\Big)\varepsilon + \mathcal{O}(\varepsilon^2)\Big)^\gamma \\
&= \big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t))\big)^\gamma \\
&\quad + \gamma\big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t))\big)^{\gamma-1}\Big((u^{0,*})'(X^{\pi^{0,*},u^{0,*}}(t))X^{\pi^{1,*},u^{1,*}}(t) + u^{1,*}(X^{\pi^{0,*},u^{0,*}}(t))\Big)\varepsilon + \mathcal{O}(\varepsilon^2) \\
&= \big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t))\big)^\gamma + R_1\Big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t)), u^{1,*}(X^{\pi^{0,*},u^{0,*}}(t)), X^{\pi^{1,*},u^{1,*}}(t)\Big)\varepsilon + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

with properly defined R_1. Similarly, we get

$$
\begin{aligned}
(X^{\pi^*,u^*}(T))^\gamma &= \Big(X^{\pi^{0,*},u^{0,*}}(T) + X^{\pi^{1,*},u^{1,*}}(T)\varepsilon + \mathcal{O}(\varepsilon^2)\Big)^\gamma \\
&= (X^{\pi^{0,*},u^{0,*}}(T))^\gamma + \gamma(X^{\pi^{0,*},u^{0,*}}(T))^{\gamma-1}X^{\pi^{1,*},u^{1,*}}(T)\varepsilon + \mathcal{O}(\varepsilon^2) \\
&= (X^{\pi^{0,*},u^{0,*}}(T))^\gamma + R_2\Big(X^{\pi^{0,*},u^{0,*}}(T), X^{\pi^{1,*},u^{1,*}}(T)\Big)\varepsilon + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

with properly defined R_2. Recall now the definition of the value function (3.11) and the discounting function (4.2). Using the expansions for u^* and X^{π^*,u^*} we derive

$$
\begin{aligned}
v(t,x,\lambda) = v^{\pi^*,u^*}(t,x,\lambda) &= \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} e^{-\rho(s-t)}\Big(1+\vartheta(s-t)\varepsilon\Big) \\
&\quad \cdot\Big(\big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(s))\big)^\gamma + R_1\Big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(s)), u^{1,*}(X^{\pi^{0,*},u^{0,*}}(s)), X^{\pi^{1,*},u^{1,*}}(s)\Big)\varepsilon\Big)ds \\
&\quad + qe^{-\rho(T-t)}\Big(1+\vartheta(T-t)\varepsilon\Big) \\
&\quad \cdot\Big(\big(X^{\pi^{0,*},u^{0,*}}(T)\big)^\gamma + R_2\Big(X^{\pi^{0,*},u^{0,*}}(T), X^{\pi^{1,*},u^{1,*}}(T)\Big)\varepsilon\Big)\mathbf{1}\{\tau > T\}\Big] + \mathcal{O}(\varepsilon^2). \qquad (7.22)
\end{aligned}
$$

We can now identify functions v^0 and v^1 so that

$$
v(t,x,\lambda) = v^0(t,x,\lambda) + v^1(t,x,\lambda)\varepsilon + \mathcal{O}(\varepsilon^2). \qquad (7.23)
$$

In particular, we see that v^0 is the value function under the strategy $(\pi^{0,*}, u^{0,*})$ for the optimization problem (3.11) with exponential discounting ρ.

We are now ready to handle the non-local term in the HJB (4.4). We use the expansion

$$\phi'(u) = \phi(u)(-\rho + \varepsilon\vartheta'(u)) = e^{-\rho u}(-\rho + (\vartheta'(u) - \rho\vartheta(u))\varepsilon) + \mathcal{O}(\varepsilon^2), \quad \varepsilon \to 0.$$

The non-local term in the HJB takes the form

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \phi'(s-t)(u^*(s))^\gamma ds + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau > T\}\Big] \\
&= \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} e^{-\rho(s-t)}\Big(-\rho + (\vartheta'(s-t) - \rho\vartheta(s-t))\varepsilon\Big) \\
&\quad \cdot\Big(\big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t))\big)^\gamma \\
&\quad\quad + R_1\Big(u^{0,*}(X^{\pi^{0,*},u^{0,*}}(t)), u^{1,*}(X^{\pi^{0,*},u^{0,*}}(t)), X^{\pi^{1,*},u^{1,*}}(t)\Big)\varepsilon\Big) ds \\
&\quad + qe^{-\rho(T-t)}\Big(-\rho + (\vartheta'(T-t) - \rho\vartheta(T-t))\varepsilon\Big) \\
&\quad \cdot\Big((X^{\pi^{0,*},u^{0,*}}(T))^\gamma + R_2\Big(X^{\pi^{0,*},u^{0,*}}(T), X^{\pi^{1,*},u^{1,*}}(T)\Big)\varepsilon\Big)\mathbf{1}\{\tau > T\}\Big] + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

Collecting the terms independent of ε, the terms proportional to ε and the terms of order $\mathcal{O}(\varepsilon^2)$, recalling v^0 and v^1 identified in (7.22), we get the key expansion

$$\begin{aligned}
&\mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \phi'(s-t)(u^*(s))^\gamma ds + q\phi'(T-t)(X^{\pi^*,u^*}(T))^\gamma \mathbf{1}\{\tau > T\}\Big] \\
&= -\rho v^0(t,x,\lambda) - \rho v^1(t,x,\lambda)\varepsilon \\
&\quad + \mathbb{E}_{t,x,\lambda}\Big[\int_t^{\tau\wedge T} \vartheta'(s-t)e^{-\rho(s-t)}(u^{0,*}(s))^\gamma ds \\
&\quad + q\vartheta'(T-t)e^{-\rho(T-t)}(X^{\pi^{0,*},u^{0,*}}(T))^\gamma \mathbf{1}\{\tau > T\}\Big]\varepsilon + \mathcal{O}(\varepsilon^2), \qquad (7.24)
\end{aligned}$$

which completes the proof. □

Proof of Theorem 4.1:

The form of the first-order equilibrium strategies (4.19) is obvious. We have to prove that the strategies (4.19) are admissible. The strategies are Markov and progressively measurable provided that the solution $X^{\tilde{\pi}^*,\tilde{u}^*}$ exists. Let us investigate the wealth process under the first-order equilibrium strategies. If we substitute the strategies from Theorem 4.1 into the SDE (3.4) which describes the retiree's wealth process, we get the dynamics

$$\begin{aligned} dX^{\tilde{\pi}^*,\tilde{u}^*}(t) &= \big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big)\frac{\mu-r}{\sigma^2(1-\gamma)}\big(\mu dt+\sigma dW_m(t)\big)\\ &+\Big(\big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big)-g(t)-(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t))\frac{\mu-r}{\sigma^2(1-\gamma)}\Big)rdt+a(\alpha\hat{x})dt-c^*dt\\ &-\big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big)f(t,\lambda(t))^{\frac{1}{\gamma-1}}\Big(1-\frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\varepsilon\Big)dt. \end{aligned} \tag{7.25}$$

Let

$$\tilde{\mu}=r+\frac{(\mu-r)^2}{\sigma^2(1-\gamma)},\quad \tilde{\sigma}=\frac{\mu-r}{\sigma(1-\gamma)},\quad \tilde{f}(t,\lambda)=f(t,\lambda)^{\frac{1}{\gamma-1}}\Big(1-\frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\varepsilon\Big).$$

If we recall the equation for g, see (4.12), then the dynamics (7.25) can be rewritten in the form

$$\begin{aligned} d\big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big) &= \big(X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)\big)dZ(t),\\ dZ(t) &= \big(\tilde{\mu}-\tilde{f}(t,\lambda(t))\big)dt+\tilde{\sigma}dW_m(t). \end{aligned} \tag{7.26}$$

We can conclude that the process $X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)$ is a stochastic exponential of Z and owns the solution

$$X^{\tilde{\pi}^*,\tilde{u}^*}(t)+g(t)=((1-\alpha)\hat{x}+g(0))e^{(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)t-\int_0^t\tilde{f}(s,\lambda(s))ds+\tilde{\sigma}W_m(t)}. \tag{7.27}$$

Hence, the SDE (3.4) under $(\tilde{\pi}^*,\tilde{u}^*)$ has a unique solution. The retiree's wealth process under the first-order equilibrium strategies (4.19) takes the form

$$X^{\tilde{\pi}^*,\tilde{u}^*}(t)=((1-\alpha)\hat{x}+g(0))e^{(\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2)t-\int_0^t\tilde{f}(s,\lambda(s))ds+\tilde{\sigma}W_m(t)}-g(t).$$

Since the function $\tilde{f}$ is bounded, we conclude that the process $X^{\tilde{\pi}^*,\tilde{u}^*}$ is square integrable and

$$\mathbb{E}\Big[\int_0^T |\tilde{\pi}^*(t)X^{\tilde{\pi}^*,\tilde{u}^*}(t)|^2 dt + \int_0^T |\tilde{u}^*(t)|^2 dt\Big] < \infty.$$

Consequently, the strategy $(\tilde{\pi}^*, \tilde{u}^*)$ is square integrable. Finally, we show that $\tilde{u}^*(t) \geq 0,\ 0 \leq t \leq T,\ X^{\tilde{\pi}^*,\tilde{u}^*}(T) \geq 0$. We can notice that

$$(1-\alpha)\hat{x} + g(0) = (1-\alpha)\hat{x} + \int_0^T (a(\alpha\hat{x}) - c^*)e^{-rs} ds. \tag{7.28}$$

Since (A4) holds, we get $(1-\alpha)\hat{x} + g(0) \geq 0$. Consequently, $X^{\tilde{\pi}^*,\tilde{u}^*}(t) + g(t) \geq 0,\ 0 \leq t \leq T$, by (7.27) and $\tilde{u}^*(t) \geq 0,\ 0 \leq t \leq T,\ X^{\tilde{\pi}^*,\tilde{u}^*}(T) \geq 0$, by (4.19) and Proposition 4.1. We would like to point out that this is the place where we use assumption (A4) which guarantees that our optimization problem has a solution, what we have already discussed when introducing (A4). □

We now give a formal version of the result on the first-order approximation of the equilibrium value function.

Theorem 7.1. *Let (A1)-(A5) hold and $\varepsilon \in [0, \varepsilon_0]$ where $\varepsilon_0 > 0$ is sufficiently small. Consider the HJB equation* (3.15) *and the discount factors* (4.1) *depending on ε. Assume there exists a solution $v(t,x,\lambda) = (x + g(t))^\gamma H^\varepsilon(t,\lambda)$ to the HJB equation such that*

a) g is given by (4.18)*,*

b) $H^\varepsilon \in \mathcal{C}([0,T] \times \mathcal{K}) \cap \mathcal{C}^{1,2}([0,T) \times \mathcal{K})$,

c) H^ε is Lipschitz continuous in λ uniformly in t and Hölder continuous in t uniformly in λ,

d) $0 < K_l \leq H^\varepsilon(t,\lambda) \leq K_u < \infty,\ (t,\lambda) \in [0,T] \times \mathcal{K}$. In addition, K_l, K_u are *constants independent of* ε.

Then

$$\left|H^{\varepsilon}(t,\lambda) - f(t,\lambda) - F(t,\lambda)\varepsilon\right| \le K\varepsilon^2, \quad (t,\lambda) \in [0,T]\times\mathcal{K},\ \varepsilon \in [0,\varepsilon_0],$$

where f, F *solve equations* (4.11), (4.16) *and* K *is a constant independent of* (t,λ,ε). *Consequently, the conclusions from Theorem 4.1 hold.*

The constants introduced in Theorem 7.1 are independent of ε but they can depend on ε_0.

Proof:
We omit the upper subscript ε in H^{ε}. If we substitute the candidate solution $v(t,x,\lambda) = (x+g(t))^{\gamma}H(t,\lambda)$ into the HJB equation (3.15) and do the calculations as in Section 4, then we can show that the function g must indeed solve (4.18) and the function H must solve the equation

$$\begin{aligned}
&H_t(t,\lambda) + \mathcal{L}_{\lambda}H(t,\lambda) - \Big(\lambda - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\Big)H(t,\lambda) + (1-\gamma)H(t,\lambda)^{\frac{\gamma}{\gamma-1}}\\
&= -\int_t^T \phi'(s-t)e^{\rho(s-t)}P^s(t,\lambda,H)ds\\
&\quad -q^{\frac{1}{1-\gamma}}\phi'(T-t)e^{\rho(T-t)}P^T(t,\lambda,H), \quad (t,\lambda)\in[0,T)\times\mathcal{K},\\
&H(T,\lambda) = q, \quad \lambda\in\mathcal{K},
\end{aligned} \tag{7.29}$$

where

$$\begin{aligned}
&P^s(t,\lambda,H)\\
&= \mathbb{E}_{t,\lambda}\Big[e^{-\left(\rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\right)(s-t) - \int_t^s\left(\lambda(u)+\gamma H(u,\lambda(u))^{\frac{1}{\gamma-1}}\right)du}H(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\Big], \quad 0\le t\le s.
\end{aligned} \tag{7.30}$$

Let us remark that the function $P^s(t,\lambda,H)$ defined in (7.30) is closely related to the function $P^s(t,\lambda)$ defined in (7.17). With our notation, the function $P^s(t,\lambda,f)$ coincides with the function $P^s(t,\lambda)$ defined in (7.17).

We would like to point out that equation (7.29) is a non-local, highly non-linear PDE and this equation is beyond our reach, both in terms of mathematics and numerics. In contrast to equations (4.11), (4.16), which are local, semi-linear and linear PDEs, and which solutions we have investigated in details and used to approximate H.

If there exists a solution H to (7.29), then the candidate optimal strategies take the form

$$\begin{aligned} \pi^*(t,x,\lambda) &= \frac{\mu - r}{\sigma^2(1-\gamma)}\frac{x+g(t)}{x}, \\ u^*(t,x,\lambda) &= (x+g(t))H(t,\lambda)^{\frac{1}{\gamma-1}}. \end{aligned} \tag{7.31}$$

As in Theorem 4.1, we can show that $(\pi^*, u^*) \in \mathcal{A}$ under our assumptions on H. We can also prove the uniform integrability demanded in Theorem 3.1. Notice that $X^{\pi^*,u^*}(t) + g(t)$ is a stochastic exponential of the form (7.27). By boundedness of H and Doob's inequality we get the estimate

$$\mathbb{E}\big[\sup_{t\in[0,T]} |X^{\pi^*,u^*}(t) + g(t)|^2\big] < \infty. \tag{7.32}$$

The uniform integrability follows from estimate (7.32) and boundedness of H. Hence, our candidates v and (π^*, u^*) defined in (7.29) and (7.31) are the equilibrium value function and the equilibrium strategies by Theorem 3.1.

The goal is to prove the accuracy of the approximation of $H^\varepsilon(t,\lambda)$ with $f(t,\lambda) + F(t,\lambda)\varepsilon$. Since $\phi'(u) = \phi(u)(-\rho + \vartheta'(u)\varepsilon)$ and formula (7.1) holds, the function H also satisfies the equation

$$
\begin{aligned}
&H_t(t,\lambda)+\mathcal{L}_\lambda H(t,\lambda)-\Big(\lambda+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\Big)H(t,\lambda)+(1-\gamma)H(t,\lambda)^{\frac{\gamma}{\gamma-1}}\\
&=-\Big(\int_t^T\phi(s-t)\vartheta'(s-t)e^{\rho(s-t)}P^s(t,\lambda,H)ds\\
&\qquad+q^{\frac{1}{1-\gamma}}\phi(T-t)\vartheta'(T-t)e^{\rho(T-t)}P^T(t,\lambda,H)\Big)\varepsilon,\quad (t,\lambda)\in[0,T)\times\mathcal{K},\\
&H(T,\lambda)=q,\quad \lambda\in\mathcal{K}.
\end{aligned}\tag{7.33}
$$

We define the residual $R(t,\lambda)=H(t,\lambda)-f(t,\lambda)-F(t,\lambda)\varepsilon$. From (7.33), (4.11), (4.16) we conclude that the function R satisfies the PDE

$$
\begin{aligned}
&R_t(t,\lambda)+\mathcal{L}_\lambda R(t,\lambda)-\Big(\lambda+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\Big)R(t,\lambda)\\
&\quad+C(t,\lambda)+D(t,\lambda)=0,\qquad (t,\lambda)\in[0,T)\times\mathcal{K},\\
&R(T,\lambda)=0,\quad \lambda\in\mathcal{K},
\end{aligned}\tag{7.34}
$$

where

$$
\begin{aligned}
C(t,\lambda) &= (1-\gamma)H(t,\lambda)^{\frac{\gamma}{\gamma-1}}-(1-\gamma)f(t,\lambda)^{\frac{\gamma}{\gamma-1}}+\gamma f(t,\lambda)^{\frac{\gamma}{\gamma-1}}F(t,\lambda)\varepsilon,\\
D(t,\lambda) &= \Big(\int_t^T\vartheta'(s-t)\big(\phi(s-t)e^{\rho(s-t)}P^s(t,\lambda,H)-P^s(t,\lambda,f)\big)ds\\
&\quad+q^{\frac{1}{1-\gamma}}\vartheta'(T-t)\big(\phi(T-t)e^{\rho(T-t)}P^T(t,\lambda,H)-P^T(t,\lambda,f)\big)\Big)\varepsilon.
\end{aligned}
$$

By properties of the functions H, f, F, which we prove above and assume in this theorem, the coefficients in the PDE (7.34) satisfy the assumptions of Proposition 2.3 from Becherer and Schweizer (2005). Consequently, we have the representation

$$
R(t,\lambda)=\mathbb{E}_{t,\lambda}\Big[\int_t^T e^{-\int_t^s(\lambda(u)+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2})du}\Big(C(s,\lambda(s))+D(s,\lambda(s))\Big)ds\Big].\tag{7.35}
$$

We now provide estimates for the functions C and D. First, we investigate the function

C. By Taylor's theorem and the mean value theorem we get

$$\begin{aligned} C(t,\lambda) &= (1-\gamma)H(t,\lambda)^{\frac{\gamma}{\gamma-1}} - (1-\gamma)\big(f(t,\lambda)+F(t,\lambda)\varepsilon\big)^{\frac{\gamma}{\gamma-1}} \\ &\quad + K_1\big(f(t,\lambda), f(t,\lambda)+F(t,\lambda)\varepsilon\big)\varepsilon^2 \\ &= K_2\big(H(t,\lambda), f(t,\lambda)+F(t,\lambda)\varepsilon\big)R(t,\lambda) + K_1\big(f(t,\lambda), f(t,\lambda)+F(t,\lambda)\varepsilon\big)\varepsilon^2. \end{aligned}$$

Let us recall that H, f, F are uniformly bounded and f, H are uniformly bounded away from zero. Moreover, for sufficiently small $\varepsilon \le \varepsilon_0$ the function $f(t,\lambda)+F(t,\lambda)\varepsilon$ is also uniformly bounded away from zero. Consequently, the functions K_1, K_2 arising due to Taylor's theorem and the mean value theorem are uniformly bounded and we have the first estimate

$$|C(t,\lambda)| \le K|R(t,\lambda)| + K\varepsilon^2, \quad (t,\lambda) \in [0,T]\times\mathcal{K},\ \varepsilon \in [0,\varepsilon_0], \tag{7.36}$$

where K is independent of (t,λ,ε).

We study the function D. Notice that $\phi(u) = e^{-\rho u} + K(u,\varepsilon)\varepsilon$, where $|K(u,\varepsilon)| \le K$ for all $u \in [0,T]$ and $\varepsilon \in [0,\varepsilon_0]$. We deal with

$$\begin{aligned} D(t,\lambda) &= \Big(\int_t^T \vartheta'(s-t)\big(P^s(t,\lambda,H) - P^s(t,\lambda,f)\big)ds \\ &\quad + q^{\frac{1}{1-\gamma}}\vartheta'(T-t)\big(P^T(t,\lambda,H) - P^T(t,\lambda,f)\big)\Big)\varepsilon \\ &\quad + \Big(\int_t^T \vartheta'(s-t)K(s-t,\varepsilon)e^{\rho(s-t)}P^s(t,\lambda,H)ds \\ &\quad + q^{\frac{1}{1-\gamma}}\vartheta'(T-t)K(T-t,\varepsilon)^{\rho(T-t)}P^T(t,\lambda,H)\Big)\varepsilon^2. \end{aligned} \tag{7.37}$$

The function in front of ε^2 is uniformly bounded for $(t,\lambda) \in [0,T]\times\mathcal{K}, \varepsilon \in [0,\varepsilon_0]$. Let us define $M^s(t,\lambda) = P^s(t,\lambda,H) - P^s(t,\lambda,f)$. From (4.15) we conclude that the function

M^s satisfies the PDE

$$
\begin{aligned}
&M_t^s(t,\lambda)+\mathcal{L}_\lambda M^s(t,\lambda)-\Big(\lambda+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}\Big)M^s(t,\lambda)\\
&\quad+\gamma H(t,\lambda)^{\frac{1}{\gamma-1}}P^s(t,\lambda,H)-\gamma f(t,\lambda)^{\frac{1}{\gamma-1}}P^s(t,\lambda,f)=0,\quad (t,\lambda)\in[0,s)\times\mathcal{K},\\
&M^s(T,\lambda)=H(s,\lambda)^{\frac{\gamma}{\gamma-1}}-f(s,\lambda)^{\frac{\gamma}{\gamma-1}},\quad \lambda\in\mathcal{K},
\end{aligned}
$$

and by Proposition 2.3 from Becherer and Schweizer (2005) we get the representation

$$
\begin{aligned}
M^s(t,\lambda(t)) &= \mathbb{E}\Big[e^{-\int_t^s(\lambda(u)+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2})du}\Big(H(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}-f(s,\lambda(s))^{\frac{\gamma}{\gamma-1}}\Big)\\
&\quad+\int_t^s e^{-\int_t^u(\lambda(z)+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2})dz}\Big(\gamma H(u,\lambda(u))^{\frac{1}{\gamma-1}}P^s(u,\lambda(u),H)\\
&\quad-\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}P^s(u,\lambda(u),f)\Big)du|\mathcal{F}_t\Big],\quad 0\le t\le s.
\end{aligned}
$$

By the mean value theorem we have

$$
\begin{aligned}
&M^s(t,\lambda(t))\\
&= \mathbb{E}\Big[e^{-\int_t^s(\lambda(u)+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2})du}K_1\big(H(s,\lambda(s)),f(s,\lambda(s))\big)\big(H(s,\lambda(s))-f(s,\lambda(s))\big)\\
&\quad+\int_t^s e^{-\int_t^u(\lambda(z)+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2})dz}\\
&\quad\cdot\Big(P^s(u,\lambda(u),H)K_2\big(H(u,\lambda(u)),f(u,\lambda(u))\big)\big(H(u,\lambda(u))-f(u,\lambda(u))\big)\\
&\quad+\gamma f(u,\lambda(u))^{\frac{1}{\gamma-1}}M^s(u,\lambda(u))\Big)du|\mathcal{F}_t\Big], \qquad (7.38)
\end{aligned}
$$

where the functions K_1,K_2 are uniformly bounded for $(t,\lambda)\in[0,T]\times\mathcal{K},\varepsilon\in[0,\varepsilon_0]$ since H,f are uniformly bounded away from zero and uniformly bounded from above. We can notice that the process M^s, which satisfies (7.38), satisfies a linear BSDE, e.g.

Proposition 2.2 in El Karoui et al. (1997). Consequently, we obtain the representation

$$
\begin{aligned}
&M^s(t,\lambda(t)) \\
&= \mathbb{E}\Big[e^{-\int_t^s(\lambda(u)+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}+\gamma(f(u,\lambda(u))^{1/(\gamma-1)})du} \\
&\qquad \cdot K_1\big(H(s,\lambda(s)),f(s,\lambda(s))\big)\big(H(s,\lambda(s))-f(s,\lambda(s))\big) \\
&\qquad + \int_t^s e^{-\int_t^u(\lambda(z)+\rho-r\gamma-\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}+\gamma(f(z,\lambda(z))^{1/(\gamma-1)})dz} \\
&\qquad \cdot\Big(P^s(u,\lambda(u),H)K_2\big(H(u,\lambda(u)),f(u,\lambda(u))\big)\big(H(u,\lambda(u))-f(u,\lambda(u))\big)\Big)du|\mathcal{F}_t\Big].
\end{aligned}
$$

We can immediately derive the inequality

$$
\begin{aligned}
|M^s(t,\lambda(t))| &\leq K\mathbb{E}\Big[|H(s,\lambda(s))-f(t,\lambda(s))|+\int_t^s|H(u,\lambda(u))-f(u,\lambda(u))|du|\mathcal{F}_t\Big] \\
&\leq K\mathbb{E}\Big[|R(s,\lambda(s))|+\int_t^s|R(u,\lambda(u))|du|\mathcal{F}_t\Big] \\
&\quad +K\mathbb{E}\Big[|F(s,\lambda(s))|+\int_t^s|F(u,\lambda(u))|du|\mathcal{F}_t\Big]\varepsilon \\
&\leq K\mathbb{E}\Big[|R(s,\lambda(s))|+\int_t^s|R(u,\lambda(u))|du|\mathcal{F}_t\Big]+K\varepsilon, \quad 0\leq t\leq s,
\end{aligned}
$$

which, substituted into (7.37), allows us to establish the second estimate

$$
|D(t,\lambda)|\leq K\mathbb{E}_{t,\lambda}\Big[\int_t^T|R(s,\lambda(s))|ds\Big]\varepsilon+K\varepsilon^2, \quad (t,\lambda)\in[0,T]\times\mathcal{K},\ \varepsilon\in[0,\varepsilon_0], \tag{7.39}
$$

where K is independent of (t,λ,ε). Recalling (7.35) and using our two estimates (7.36), (7.39) for C and D, we arrive at the key estimate

$$
|R(t,\lambda)|\leq K\Big(\int_t^T\mathbb{E}_{t,\lambda}\big[|R(s,\lambda(s))|\big]ds\Big)(1+\varepsilon)+K\varepsilon^2, \quad (t,\lambda)\in[0,T]\times\mathcal{K},\ \varepsilon\in[0,\varepsilon_0].
$$

If we choose $t_0\in[0,T]$, then

$$
\mathbb{E}\big[|R(t,\lambda(t))||\mathcal{F}_{t_0}\big]\leq K\Big(\int_t^T\mathbb{E}\big[|R(s,\lambda(s))||\mathcal{F}_{t_0}\big]ds\Big)(1+\varepsilon)+K\varepsilon^2, \quad t\in[t_0,T],\ \varepsilon\in[0,\varepsilon_0].
$$

By Grönwall's inequality we finally get

$$\mathbb{E}\big[|R(t,\lambda(t))||\mathcal{F}_{t_0}\big] \leq K\varepsilon^2 e^{K(1+\varepsilon)(T-t_0)} \leq K\varepsilon^2, \quad t\in[t_0,T],\ \varepsilon\in[0,\varepsilon_0],$$

and, in particular,

$$|R(t_0,\lambda)| \leq K\varepsilon^2, \quad t_0\in[0,T], \quad (t_0,\lambda)\in[0,T]\times\mathcal{K},\ \varepsilon\in[0,\varepsilon_0],$$

where K is independent of $(t_0,\lambda,\varepsilon)$. This proves that $f(t,\lambda)+F(t,\lambda)\varepsilon$ uniformly approximates $H^\varepsilon(t,\lambda)$ with accuracy of order $\mathcal{O}(\varepsilon^2)$, and consequently that $(x+g(t))^\gamma\big(f(t,\lambda)+F(t,\lambda)\varepsilon\big)$ provides the first-order approximation to the equilibrium value function $v(t,x,\lambda)=(x+g(t))^\gamma H^\varepsilon(t,\lambda)$ for our optimization problem as $\varepsilon\to 0$, i.e.

$$v(t,x,\lambda)=(x+g(t))^\gamma\big(f(t,\lambda)+F(t,\lambda)\varepsilon\big)+\mathcal{O}(\varepsilon^2), \quad (t,x,\lambda)\in[0,T]\times\mathbb{R}\times\mathcal{K},\ \varepsilon\to 0.$$

The accuracy of the approximation for v depends on x. $\square$

Notice that the strategy (7.31) shows that the first-order equilibrium investment strategy (4.19) is the true equilibrium investment strategy.

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