Efficiently pricing double barrier derivatives in stochastic volatility models[1]

Marcos Escobar
Department of Mathematics,
Ryerson University, Toronto,
254 Church Street, Ontario, Canada,
email: escobar@ryerson.ca,

Peter Hieber
Lehrstuhl für Finanzmathematik (M13),
Technische Universität München,
Parkring 11, 85748 Garching-Hochbrück, Germany,
email: hieber@tum.de,

Matthias Scherer
Lehrstuhl für Finanzmathematik (M13),
Technische Universität München,
Parkring 11, 85748 Garching-Hochbrück, Germany,
email: scherer@tum.de.

Abstract

Imposing a symmetry condition on returns, Carr and Lee [2009] show that (double) barrier derivatives can be replicated by a portfolio of European options and can thus be priced using fast Fourier techniques (FFT). We show that prices of barrier derivatives in stochastic volatility models can alternatively be represented by rapidly converging series, putting forward an idea by Hieber and Scherer [2012]. This representation turns out to be faster and more accurate than FFT. Numerical examples and a toolbox of a large variety of stochastic volatility models illustrate the practical relevance of the results.

`Keywords: first-passage time, barrier options, stochastic volatility, stochastic clock.`

Barrier derivatives are among the most liquidly traded over-the-counter (OTC) products. Their payout depends on whether the underlying crosses some prespecified level(s) until the maturity of the contract. If the final payoff depends on an upper and a lower threshold (contracts termed "double barrier derivatives"), barrier products constitute a simple possibility to obtain a long/short position in volatility.

Closed-form prices for barrier derivatives were first obtained in the Black–Scholes model; the single barrier pricing formulas can be referred to, e.g., Black and Cox [1976] or Reiner and Rubinstein

[1]Peter Hieber acknowledges funding by the German Academic Exchange Service (DAAD). This version may differ from the final published version *Efficiently pricing double barrier derivatives, Review of Derivatives Research, Vol. 17, No. 2 (2014), pp. 191–216* in typographical detail.

[1991]. Later, those results were extended to (stochastic and local) volatility models that fulfill certain symmetry conditions in the return distribution (see, e.g., Derman et al. [1994], Carr et al. [1998], Dupont [2002], Carr and Lee [2009], Carr et al. [2011]). Related to this work, several authors price barrier derivatives analytically (e.g. by fast Fourier techniques (FFT)) for special cases of the stochastic volatility models of Heston (see, e.g., [Lipton, 2001, p. 235], Carr et al. [2003], Sepp [2006], Kammer [2007], Escobar et al. [2011]) and Stein–Stein (see, e.g., Götz [2011]). Those extensions allow to include important stylized facts that are criticized in the seminal Black–Scholes model: (1) volatility varies over time (stochastic volatility) and (2) implied volatility depends on the strike price (smile feature).

The contribution of this paper is as follows:

- We review existing results on the pricing and risk management of double barrier derivatives under stochastic volatility. We aim at providing a reader friendly recipe on pricing and hedging double barrier derivatives under stochastic volatility. We provide a toolbox of (single and multi-factor) stochastic volatility models that allow to price barrier derivatives in closed-form, an aspect that has not been the prime focus of earlier works. Examples include single and multi-factor CIR-type stochastic volatility (which is the type of volatility used in the Heston [1993], Schöbel and Zhu [1999], or Christoffersen et al. [2009] model), or the Stein and Stein [1991] model. Jump processes for the volatility are also discussed, an idea that was, for example, applied in the Barndorff-Nielsen and Shephard [2001] model.
- We show that the existing results based on FFT techniques can be significantly improved in terms of computational efficiency. Double barrier derivatives can (in the same stochastic volatility setting) be priced by rapidly converging infinite series, extending an idea by Hieber and Scherer [2012]. In contrast to Fourier techniques, this result avoids the integration over contours of the complex plane and is (in all examples we considered) faster than FFT. We provide error bounds that allow for a straightforward implementation of the results.

Increasingly popular and numerically demanding tasks, like the pricing and risk management of large portfolios of barrier derivatives or their calibration to over-the-counter prices (see, e.g., Carr and Crosby [2010] and Kilin [2011]), have flagged the need for fast and reliable numerical techniques. Closed-form prices for barrier derivatives in (special cases of) several well-known models can be used as a benchmark to assess the performance of other numerical techniques or as a control variate for variance reduction in Monte-Carlo simulations.

The paper is organized as follows: After introducing the basic notation in Section 1, Section 2 presents the toolbox of specific stochastic volatility models. Sections 3.1 and 3.2 review the pricing results of single and double barrier derivatives and show that they can fast and accurately be priced by Fourier inversion, using results by Carr and Madan [1999], Bakshi and Madan [2000], and Raible [2000]. Section 4 shows that (double) barrier derivatives can alternatively be priced by rapidly converging infinite series. Numerical results are presented in Section 6, validating the improvement in terms of computation time and accuracy of the proposed methodology compared to FFT methods.

1 Model description

We consider on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ the process

$$\frac{dS_t}{S_t} = r_t\, dt + \sigma_t\, dW_t, \quad S_0 > 0, \tag{1}$$

where $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion, $\{\sigma_t\}_{t\geq 0}$ the (stochastic) volatility, independent of W, and $\{r_t\}_{t\geq 0}$ the (deterministic) risk-less interest rate[2]. The processes $\{\sigma_t\}_{t\geq 0}$ and $\{r_t\}_{t\geq 0}$ are adapted to the filtration $\mathbb{F}$ and satisfy the regularity conditions $\int_0^t |r_s|\, ds < \infty$ and $\mathbb{E}_{\mathbb{Q},S_0}\big[\int_0^t \sigma_s^2\, ds\big] < \infty$ $\mathbb{Q}$-a.s. for all $t \geq 0$. We define a (lower) barrier $D_t := D \exp(\int_0^t r_s ds)$ and an (upper) barrier $P_t := P \exp(\int_0^t r_s ds)$, where $D < S_0 < P$. We define a bank account $B_t = \exp(\int_0^t r_s\, ds)$ and denote the first-exit time by

$$\tau := \inf\big\{t \geq 0 \,|\, S_t \notin (D_t, P_t)\big\}, \tag{2}$$

where $\inf \emptyset := \infty$. Besides, $\tau_+ := \tau$ if $S_\tau = P_\tau$ and $\tau_- := \tau$ if $S_\tau = D_\tau$, i.e. if the upper barrier is hit first, we set $\tau_+ := \tau$; if the lower barrier is hit first, we set $\tau_- := \tau$.

For a given maturity T, our objective is to price derivatives that depend on whether or not $\{S_t\}_{t\geq 0}$ crosses the thresholds $\{D_t\}_{t\geq 0}$ or $\{P_t\}_{t\geq 0}$. We consider contracts $X_{D,P}^{g(S_T)}(S_0)$ that consist of a positive payoff $g(S_T)$ (where $\mathbb{E}[g(S_T)] < \infty$) if none of the barriers $\{D_t\}_{t\geq 0}$, $\{P_t\}_{t\geq 0}$ is hit until maturity T (and 0 otherwise). The price of those contracts is given by

$$X_{D,P}^{g(S_T)}(S_0) = \frac{1}{B_T}\, \mathbb{E}_{\mathbb{Q},S_0}\big[\mathbb{1}_{\{\tau > T\}}\, g(S_T)\big], \tag{3}$$

where we denote $\mathbb{E}_{\mathbb{Q},x}[\,\cdot\,] := \mathbb{E}_{\mathbb{Q}}[\,\cdot \mid S_0 = x]$. A special case are single barrier contracts, i.e.

$$X_{D,\infty}^{g(S_T)}(S_0) = \frac{1}{B_T}\, \mathbb{E}_{\mathbb{Q},S_0}\big[\mathbb{1}_{\{\tau_- > T\}}\, g(S_T)\big]. \tag{4}$$

2 Stochastic volatility models

Various parameterizations of the stochastic volatility model (1) have been proposed in the literature. This section discusses some of the most famous examples that are used by both practitioners and academics. These include the CIR-type stochastic volatility model (Section 2.1), which is the type of volatility used in the Heston [1993] or Christoffersen et al. [2009] model. In Section 2.2, the volatility follows an Ornstein–Uhlenbeck (OU) process, which results in the Stein and Stein [1991] model. Finally, jump processes for the volatility are discussed (Section 2.3), an idea that was, for example, applied in the Barndorff-Nielsen and Shephard [2001] model.

We explicitly discuss richer volatility structures including multiple risk factors. Multi-factor models have become very popular for modeling short rates, where it is widely accepted that one factor is not sufficient to capture the time and cross-sectional variation in the term structure; however, their application has only recently (see, e.g., Christoffersen et al. [2009]) reached the area of option pricing.

[2]A comment on generalizations to stochastic interest rates is given in Remark 17.

2.1 CIR-type stochastic volatility

This section discusses two models that rely on a CIR-type stochastic volatility: The 2-factor stochastic volatility model considered in Christoffersen et al. [2009] (allowing for a rich volatility structure, see Example 2), and the (1-factor) Heston model (Example 1). Extensions to more than two factors are straightforward, due to their high number of parameters, however, they are not frequently used in practice.

Example 1 (Heston-type stochastic volatility)
The Heston [1993] model was introduced as

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t\,dt + \sqrt{v_t}\,dW_t, \quad S_0 > 0, \\ dv_t &= \theta(\nu - v_t)dt + \gamma\sqrt{v_t}d\tilde{W}_t, \quad v_0 > 0, \end{aligned} \tag{5}$$

where θ, ν, and γ are non-negative constants; $\{\tilde{W}_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ one-dimensional Brownian motions with correlation ρ. The Feller [1951] condition $2\theta\nu > \gamma^2$ guarantees that the process is almost surely positive. The characteristic function of the log-asset process in the Heston model is given by, see Heston [1993], Rollin et al. [2011]

$$\begin{aligned} \varphi_T(u, S_0) = \mathbb{E}\left[e^{iu\ln(S_T)}\right] &= \exp\left(iu\ln(S_0) + iu\int_0^T r_t\,dt\right) \\ &\cdot \left(\frac{\exp(\theta T/2)}{\cosh(\varrho T/2) + \frac{\xi}{\varrho}\sinh(\varrho T/2)}\right)^{\frac{2\theta\nu}{\gamma^2}} \exp\left\{-\frac{v_0}{\varrho}\,\frac{(iu+u^2)\sinh(\varrho T/2)}{\cosh(\varrho T/2) + \frac{\xi}{\varrho}\sinh(\varrho T/2)}\right\}, \end{aligned} \tag{6}$$

where $\varrho = \sqrt{(\theta - \gamma\rho iu)^2 + \gamma^2(iu + u^2)}$, $\xi = \theta - \gamma\rho ui$. The special case $\rho = 0$ is considered in, e.g., Ball and Roma [1994], [Lipton, 2001, p. 235], Carr et al. [2003], Sepp [2006], Kammer [2007], and Escobar et al. [2011].

Example 2 (2-factor stochastic volatility)
Following Christoffersen et al. [2009], a 2-factor stochastic volatility model can be introduced as

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t\,dt + \sqrt{v_t^{(1)}}dW_t^{(1)} + \sqrt{v_t^{(2)}}dW_t^{(2)}, \quad S_0 > 0, \\ dv_t^{(1)} &= \theta_1(\nu_1 - v_t^{(1)})dt + \gamma_1\sqrt{v_t^{(1)}}d\tilde{W}_t^{(1)}, \quad v_0^{(1)} > 0, \\ dv_t^{(2)} &= \theta_2(\nu_2 - v_t^{(2)})dt + \gamma_2\sqrt{v_t^{(2)}}d\tilde{W}_t^{(2)}, \quad v_0^{(2)} > 0, \end{aligned} \tag{7}$$

where $\{W_t^{(1)}\}_{t\geq 0}$, $\{W_t^{(2)}\}_{t\geq 0}$, $\{\tilde{W}_t^{(1)}\}_{t\geq 0}$, and $\{\tilde{W}_t^{(2)}\}_{t\geq 0}$ are Brownian motions. $\{W_t^{(1)}\}_{t\geq 0}$ has correlation ρ_1 with $\{\tilde{W}_t^{(1)}\}_{t\geq 0}$, $\{W_t^{(2)}\}_{t\geq 0}$ has correlation ρ_2 with $\{\tilde{W}_t^{(2)}\}_{t\geq 0}$. The remaining correlations are assumed to be zero. The parameters θ_1, θ_2, ν_1, ν_2, γ_1, and γ_2 are positive constants.

Using independence between the volatility processes, the characteristic function of the log-asset process is a straightforward deduction from Equation (6), i.e.

$$\varphi_T(u,S_0)=\mathbb{E}\big[e^{iu\ln(S_T)}\big]=\exp\left(iu\ln(S_0)+iu\int_0^T r_t\,dt\right)$$
$$\cdot\exp\left\{-\frac{v_0^{(1)}}{\varrho_1}\frac{(iu+u^2)\sinh(\varrho_1T/2)}{\cosh(\varrho_1T/2)+\frac{\xi_1}{\varrho_1}\sinh(\varrho_1T/2)}\right\}\exp\left\{-\frac{v_0^{(2)}}{\varrho_2}\frac{(iu+u^2)\sinh(\varrho_2T/2)}{\cosh(\varrho_2T/2)+\frac{\xi_2}{\varrho_2}\sinh(\varrho_2T/2)}\right\}$$
$$\cdot\left(\frac{\exp(\theta_1T/2)}{\cosh(\varrho_1T/2)+\frac{\xi_1}{\varrho_1}\sinh(\varrho_1T/2)}\right)^{\frac{2\theta_1\nu_1}{\gamma_1^2}}\left(\frac{\exp(\theta_2T/2)}{\cosh(\varrho_2T/2)+\frac{\xi_2}{\varrho_2}\sinh(\varrho_2T/2)}\right)^{\frac{2\theta_2\nu_2}{\gamma_2^2}},\tag{8}$$

where $\varrho_j=\sqrt{(\theta_j-\gamma_j\rho_j iu)^2+\gamma_j^2(iu+u^2)}$, $\xi_j=\theta_j-\gamma_j\rho_j ui$, *for* $j=1,2$. *To stay within our model framework (1), volatility and asset process have to be independent. Thus, we have to restrict ourselves to the case where* $\rho=\rho_1=\rho_2=0$. *This case is used in, e.g., Götz [2011], Kiesel and Lutz [2011].*

2.2 OU-type stochastic volatility

If the volatility is of OU-type, we obtain the stochastic volatility model of Stein and Stein [1991], an approach that was later extended to include dependence between S and σ by Schöbel and Zhu [1999].

Example 3 (Stein–Stein model)
Stein and Stein [1991] introduce the stochastic volatility model

$$\frac{dS_t}{S_t}=r_t\,dt+\sigma_t\,dW_t,\quad S_0>0,\tag{9}$$
$$d\sigma_t=\xi(\sigma_t-\varkappa)dt+k\,d\tilde{W}_t,\quad \sigma_0>0,$$

where ξ, $\varkappa$, *and* k *are positive constants;* $\{\tilde{W}_t\}_{t\geq0}$ *and* $\{W_t\}_{t\geq0}$ *independent one-dimensional Brownian motions. In this model, the volatility is governed by an arithmetic Ornstein–Uhlenbeck process, with a tendency to revert back to a long-run average level of* $\varkappa$. *The characteristic function of the log-asset process is given by, see, e.g., Stein and Stein [1991]*

$$\varphi_T(u)=\mathbb{E}\big[e^{iu\ln(S_T)}\big]=\exp\left(iu\ln(S_0)+iu\int_0^T r_t\,dt\right)$$
$$\cdot\exp\Big(L\big((iu+u^2)/2\big)\sigma_0^2/2+M\big((iu+u^2)/2\big)\sigma_0+N\big((iu+u^2)/2\big)\Big),\tag{10}$$

where the functions $L(u)$, $M(u)$, *and* $N(u)$ *are defined in Appendix A.*

Similar to the Heston-type extension (Example 2), this model can also be extended to several factors, allowing for richer volatility structures.

2.3 Volatility with jumps

Jumps in the volatility process have also become recognized stylized facts, see – among many others – Naik [1993], Barndorff-Nielsen and Shephard [2001], and Eraker et al. [2003]. While a jump in returns has no impact on the distribution of future returns, jumps in volatility are highly persistent. Bates

[1996] and Barndorff-Nielsen and Shephard [2001] assume that (external) shocks lead to a sudden increase in volatility. Then, volatility gradually returns to its original level (see Example 4). Those kind of processes are also popular in insurance applications, see, e.g., Dassios and Jang [2003].

Example 4 (Jumps in the volatility)
Barndorff-Nielsen and Shephard [2001] propose to include jumps in the volatility process. Different applications in insurance or finance can be found in, e.g., Cox and Isham [1980], Dassios and Jang [2003].

$$\frac{dS_t}{S_t} = r_t\,dt + \sqrt{v_t}\,dW_t, \quad S_0 > 0, \tag{11}$$
$$v_t = v_0 \exp(-\delta t) + \sum_{s_i \le t} M_i \exp\big(-\delta(t - s_i)\big), \quad v_0 > 0,$$

where $v_0 > 0$ is the initial variance, $\delta > 0$ the exponential decay rate, $\{s_i\}_{i=1}^{\infty}$ are the jump times of a time-homogeneous Poisson process with intensity $\psi > 0$, and M_i are the jump sizes with distribution $G(y)$, $y > 0$. The characteristic function is given by, see, e.g., Dassios and Jang [2003]

$$\varphi_T(u, S_0) = \mathbb{E}\big[e^{iu\ln(S_T)}\big] = \exp\left(iu\ln(S_0) + iu\int_0^T r_t\,dt\right)$$
$$\cdot \exp\left(-\frac{(iu+u^2)v_0}{2\delta}\big(1-\exp(-\delta T)\big) - \psi\int_0^T \left[1 - \hat{g}\Big(\frac{iu+u^2}{2\delta}\big(1-\exp(\delta(T-t))\big)\Big)\right]dt\right), \tag{12}$$

where $v_0 > 0$ and $\hat{g}(u) := \int_0^\infty \exp(-uy)\,dG(y)$ is the Laplace transform of the jump size distribution $G(y)$, $y > 0$. Special cases include, for example, exponential jump diffusions $\hat{g}(u) = 1/(1 + u/\zeta)$, in which case the integral in (12) can be computed explicitly.

There are many other parameterizations of a stochastic volatility not covered in this paper. Frequently used is the Hull–White model (see, e.g., Hull and White [1987]). This type of model also occurs as a continuous diffusion limit of GARCH models (see, e.g., Klüppelberg et al. [2004], Brockwell et al. [2006]). However, the characteristic function of the log-asset price is not known explicitly in these models and has to be evaluated numerically. Furthermore, it is also possible to consider 1-dimensional marginals of multivariate stochastic volatility models (e.g. da Fonseca et al. [2007], Pigorsch and Stelzer [2009]).

3 Fourier pricing

The following sections are devoted to the pricing of *down-and-out* contracts on one barrier (Section 3.1). *Up-and-out* contracts can be treated similarly, as well as products on two barriers (Section 3.2).

3.1 One barrier

First, the pricing results for single barrier derivatives are reviewed. The price of a *down-and-out* contract under the stochastic volatility model (1) is given in Theorem 5.

Theorem 5 (One barrier: Down-and-out contract)
In model (1), consider a lower barrier $\{D_t\}_{t\geq 0}$ *with* $D := D_0 < S_0$, $\mathbb{E}[g(S_T)] < \infty$, *and a derivative with payoff* $\mathbb{1}_{\{\tau_- > T\}}\, g(S_T)$. *Then*

$$X_{D,\infty}^{g(S_T)}(S_0) = \frac{1}{B_T}\left(\mathbb{E}_{\mathbb{Q},S_0}\Big[\mathbb{1}_{\{S_T > D_T\}}\, g(S_T)\Big] - \frac{S_0}{D}\,\mathbb{E}_{\mathbb{Q},D^2/S_0}\Big[\mathbb{1}_{\{S_T > D_T\}}\, g\left(S_T\right)\Big]\right).$$

Note that the included expectations do not depend on the whole path $\{S_t\}_{t\geq 0}$, *but on the integrated quantities* $S_T = S_0 \exp\big(\int_0^T (r_t - \sigma_t^2/2)\,dt + \int_0^T \sigma_t\, dW_t\big)$ *and* $D_T = D\exp(\int_0^T r_t\, dt)$.

Proof
See, e.g., Carr and Lee [2009]. □

In the following, several well-known examples for the payoff $g(S_T)$, digital and barrier options as well as bonus certificates, are presented. The results allow for an interpretation as a static replication of the exotic barrier derivatives by path-independent standard vanilla calls, puts, and digitals. If the characteristic function of the log-asset price $\ln(S_T)$ is known, Fourier inversion techniques by Carr and Madan [1999], Bakshi and Madan [2000], and Raible [2000] allow for an efficient evaluation of the given expectations.

3.1.1 Digital options

First, we consider digital options, i.e. options that pay \$1 at maturity T if the barrier is not hit during the lifetime of the contract and $S_T > K_T$, where $K_T := K\exp(\int_0^T r_t\, dt) > D_T$, i.e. their conditional payoff function is $g(S_T) = \mathbb{1}_{\{S_T > K_T\}}$. Lemma 6 presents risk-neutral prices of this payoff.

Lemma 6 (Digital options)
Consider the stochastic volatility model (1). The price of a digital option with maturity T *(i.e. conditional payoff function* $g(S_T) = \mathbb{1}_{\{S_T > K_T\}}$, $K_T := K\exp(\int_0^T r_t\, dt) > D_T$, *in Theorem 5) is given by*

$$I_K(S_0; D, T) = \frac{1}{B_T}\Big[\mathbb{Q}_{S_0}\big(S_T > K_T\big) - \frac{S_0}{D}\,\mathbb{Q}_{D^2/S_0}\big(S_T > K_T\big)\Big], \tag{13}$$

where $\mathbb{Q}_x(\,\cdot\,) := \mathbb{Q}(\,\cdot \mid S_0 = x)$,

$$\mathbb{Q}_{S_0}(S_T > K_T) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\ln(K_T/S_0)}\,\varphi_T\big(u-i, S_0\big)}{iu\,\varphi_T(-i, S_0)}\right] du \tag{14}$$

and $\varphi_T(u, S_0) = \mathbb{E}\big[e^{iu\ln(S_T)}\big]$ *is the characteristic function of the log-asset price* $\ln(S_T)$.

The integrals in Lemma 6 can be evaluated using FFT, see, e.g., Carr and Madan [1999]. This result is a straightforward application of Theorem 5. For Equation (14), we refer to, e.g., Bakshi and Madan [2000]. In the Black–Scholes model (i.e. $\sigma_t = \sigma$, $r_t = r$), we obtain

$$I_K\big(S_0; D, T\big) = \frac{1}{B_T}\left[\Phi\left(\frac{\ln(S_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}}\right) - \frac{S_0}{D}\,\Phi\left(\frac{\ln(D^2/(S_0 K)) - \sigma^2 T/2}{\sigma\sqrt{T}}\right)\right]. \tag{15}$$

3.1.2 Barrier options

Down-and-out call, respectively put, options are also a special case of Theorem 5 with payoff $g(S_T) = \max(S_T - K_T, 0)$, respectively $g(S_T) = \max(K_T - S_T, 0)$, at time T for a given strike $K_T := K \exp(\int_0^T r_t\, dt) > D_T$. Before pricing those contracts, we want to consider the simpler case of call options. If the characteristic function $\varphi_T(u, S_0) := \mathbb{E}\big[\exp\big(iu \ln(S_T)\big)\big]$ of the log-asset price $\ln(S_T)$ is known, Carr and Madan [1999] and Raible [2000] propose to price a call option with strike K_T by

$$C_K(S_0, T) = \frac{1}{B_T} \frac{e^{-\alpha \ln(K_T)}}{\pi} \int_0^\infty e^{-iu \ln(K_T)} \frac{\varphi_T\big(u - (1+\alpha)i, S_0\big)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}\, du. \tag{16}$$

The latter integral can – for many strikes simultaneously – be evaluated using FFT. The damping factor $\alpha > 0$ is usually chosen from the interval $[1, 2]$, for a more detailed discussion, we refer to Carr and Madan [1999].

Lemma 7 (Barrier options)
Consider the stochastic volatility model (1). The price of a down-and-out call, respectively put, option with strike $K_T := K \exp(\int_0^T r_t\, dt) > D_T$ and maturity T is

$$DOC_K(S_0; D, T) = C_K\big(S_0, T\big) - \frac{S_0}{D} C_K\big(D^2/S_0, T\big), \tag{17}$$

$$\begin{aligned} DOP_K(S_0; D, T) &= P_K\big(S_0, T\big) - \frac{S_0}{D} P_K\big(D^2/S_0, T\big) \\ &= DOC_K(S_0; D, T) + (K - S_0) - \frac{S_0}{D}(K - S_0), \end{aligned} \tag{18}$$

where $P_K(S_0, T) := C_K(S_0, T) - S_0 + K$. If the characteristic function $\varphi_T(u, S_0)$ of the log-asset price $\ln(S_T)$ is known, the price of a call option $C_K(S_0, T)$ on $\{S_t\}_{t \geq 0}$ with strike K_T and maturity T is given by Equation (16).

Lemma 7 is again a straightforward corollary to Theorem 5 using the conditional payoff function $g(S_T) = \max(S_T - K_T, 0)$, respectively $g(S_T) = \max(K_T - S_T, 0)$. The case $0 \leq K \leq D$ can be treated similarly.

3.1.3 Bonus certificates

Many exotic derivatives can be replicated by using the results in the previous sections. In this section, we present one example called "bonus certificates" and show how the results in Theorem 5 can be applied. For a given bonus level $L_T := L \exp(\int_0^T r_t\, dt) > D_T$ and a barrier $\{D_t\}_{t \geq 0}$, the payoff at maturity T is given by

$$\text{payoff}(T) = \begin{cases} \max(S_T, L_T), & \tau_- > T, \\ S_T, & \text{else}. \end{cases} \tag{19}$$

Under the risk-neutral measure $\mathbb{Q}$, its price is given by

$$BO_L(S_0; D, T) = \frac{1}{B_T} \mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\{\tau_- > T\}} \max(S_T, L_T) + \mathbb{1}_{\{\tau_- \leq T\}} S_T\big]$$

$$= \frac{1}{B_T}\,\mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\{\tau_- > T\}}\max(0, L_T - S_T) + S_T\big]$$
$$= \frac{1}{B_T}\,\mathbb{E}_{\mathbb{Q}}\big[S_T\big] + X_{D,\infty}^{\max(0,L_T-S_T)}(S_0) = S_0 + DOP_L(S_0; D, T). \tag{20}$$

This leads to the result in Lemma 8.

Lemma 8 (Bonus certificates)
Consider the stochastic volatility model (1). The price of bonus certificates with bonus level $L_T := L\,\exp(\int_0^T r_t\,dt) > D_T$ and payoff (19) at maturity T is given by

$$BO_L(S_0; D, T) = S_0 + DOP_L(S_0; D, T). \tag{21}$$

If the characteristic function $\varphi_T(u, S_0)$ of the log-asset price $\ln(S_T)$ is known, the price of a bonus certificate $BO_L(S_0; D, T)$ can thus be found via Equation (16).

3.2 Two barriers

As a second step, we investigate *exit-and-out* contracts on two barriers, i.e. we derive $X_{D,P}^{g(S_T)}(S_0)$ (as defined in Section 1) for certain payoff functions. Examples include double barrier options, double digital options, or corridor bonus certificates. The price of those contracts is represented as an infinite series of path-independent derivatives, a result that has been obtained by, e.g., Carr and Lee [2009].

Theorem 9 (Two barriers: Down-and-out contract I)
In model (1), consider a derivative with payoff $\mathbb{1}_{\{\tau > T\}}\,g(S_T)$, where $\mathbb{E}[g(S_T)] < \infty$. Its price is given by

$$X_{D,P}^{g(S_T)}(S_0) = \frac{1}{B_T}\sum_{n=-\infty}^{\infty}\frac{D^n}{P^n}\left(\mathbb{E}_{\mathbb{Q},S_0^{(2n)}}\Big[\mathbb{1}_{\{S_T\in(D_T,P_T)\}}\,g(S_T)\Big] - \frac{S_0}{D}\,\mathbb{E}_{\mathbb{Q},S_0^{(2n-1)}}\Big[\mathbb{1}_{\{S_T\in(D_T,P_T)\}}\,g(S_T)\Big]\right), \tag{22}$$

where $S_0^{(2n)} = S_0 P^{2n}/D^{2n}$ and $S_0^{(2n-1)} = P^{2n}/(D^{2n-2}S_0)$, $n \in \mathbb{Z}$.

Note that the included expectations do not depend on the whole path $\{S_t\}_{t\geq 0}$, but on the integrated quantities $S_T = S_0\,\exp\big(\int_0^T (r_t - \sigma_t^2/2)\,dt + \int_0^T \sigma_t\,dW_t\big)$, $D_T = D\,\exp(\int_0^T r_t\,dt)$, and $P_T = P\,\exp(\int_0^T r_t\,dt)$.

Proof
See, e.g., Carr and Lee [2009]. □

3.2.1 Double digital options

As a first application of Theorem 9, we consider double digital options, i.e. options that pay \$1 at maturity T if the barriers $\{D_t\}_{t\geq 0}$ and $\{P_t\}_{t\geq 0}$ are not hit during the lifetime of the contract and $S_T > K_T$, where $K_T := K\,\exp(\int_0^T r_t\,dt) \in [D_T, P_T]$, i.e. their conditional payoff function is $g(S_T) = \mathbb{1}_{\{S_T > K_T\}}$. In the Black–Scholes model, risk-neutral prices for double digital options are presented in many different representations (see, e.g., Darling and Siegert [1953], Geman and Yor [1996], Lin [1999]). Lemma 10 provides the price in the more general model framework (1).

Lemma 10 (Double digital options I)
Consider the stochastic volatility model (1). The price of double digital options with strike $K_T = K\,\exp(\int_0^T r_t\,dt) \in [D_T, P_T]$ *and maturity* T *(i.e. payoff function* $g(S_T) = \mathbb{1}_{\{S_T > K_T\}}$ *in Theorem 9) is*

$$I_K(S_0; D, P, T) = \frac{1}{B_T} \sum_{n=-\infty}^{\infty} \frac{D^n}{P^n} \Big(\mathbb{Q}_{S_0^{(2n)}}\big(S_T \in (K_T, P_T)\big) - \frac{S_0}{D}\, \mathbb{Q}_{S_0^{(2n-1)}}\big(S_T \in (K_T, P_T)\big) \Big), \tag{23}$$

where $S_0^{(2n)} = S_0 P^{2n}/D^{2n}$ *and* $S_0^{(2n-1)} = P^{2n}/(D^{2n-2} S_0)$, $n \in \mathbb{Z}$.

This lemma is a straightforward application of Theorem 9.

3.2.2 Double barrier options

The second derivative we consider are double barrier options. Conditional on survival, i.e. staying within the boundaries, they have the same payoff $g(S_T) = \max(S_T - K_T, 0)$ (where $K_T = K\,\exp(\int_0^T r_t\,dt) \in [D_T, P_T]$) as in the single barrier case. Applying Theorem 9, Lemma 11 presents the corresponding prices. In the Black–Scholes model, prices for double barrier options have – for different parameterizations – been presented in the literature (see, e.g., Geman and Yor [1996], Lin [1999], Pelsser [2000]). The presented equations, however, often tend to be rather complicated and usually lack the intuitive (and for replication very convenient) interpretation as a portfolio of infinitely many standard vanilla options.

Lemma 11 (Double barrier options I)
Consider the stochastic volatility model (1). The price of double barrier options with strike $K_T = K\,\exp(\int_0^T r_t\,dt) \in [D_T, P_T]$ *and maturity* T *(i.e. conditional payoff function* $g(S_T) = \max(S_T - K_T, 0)$ *in Theorem 9) is*

$$\begin{aligned} EOC_K(S_0; D, P, T) = &\sum_{n=-\infty}^{\infty} \frac{D^n}{P^n} \Big(C_K\big(S_0^{(2n)}, T\big) - C_P\big(S_0^{(2n)}, T\big) + (P-K)\, I_P\big(S_0^{(2n)}; P, T\big) \Big) \\ &- \frac{D^n}{P^n} \frac{S_0}{D} \Big(C_K\big(S_0^{(2n-1)}, T\big) - C_P\big(S_0^{(2n-1)}, T\big) + (P-K)\, I_P\big(S_0^{(2n-1)}; P, T\big) \Big), \end{aligned} \tag{24}$$

where $S_0^{(2n)} = S_0 P^{2n}/D^{2n}$ *and* $S_0^{(2n-1)} = P^{2n}/(D^{2n-2} S_0)$, $n \in \mathbb{Z}$.

Proof
Apply Theorem 9 with $g(S_T) = \max(S_T - K_T, 0)$. *We find that*

$$\begin{aligned} EOC_K(S_0; D, P, T) &= \frac{1}{B_T} \sum_{n=-\infty}^{\infty} \frac{D^n}{P^n} \Big(\mathbb{E}_{\mathbb{Q}, S_0^{(2n)}}\Big[\mathbb{1}_{\{\, S_T \in (D_T, P_T)\}}\, g(S_T)\Big] - \frac{S_0}{D}\, \mathbb{E}_{\mathbb{Q}, S_0^{(2n-1)}}\Big[\mathbb{1}_{\{\, S_T \in (D_T, P_T)\}}\, g(S_T)\,\Big] \Big) \\ &= \frac{1}{B_T} \sum_{n=-\infty}^{\infty} \frac{D^n}{P^n} \Big(C_K\big(S_0^{(2n)}, T\big) - C_P\big(S_0^{(2n)}, T\big) + (P-K)\, I_P\big(S_0^{(2n)}; P, T\big) \Big) \\ &\quad - \frac{D^n}{P^n} \frac{S_0}{D} \Big(C_K\big(S_0^{(2n-1)}, T\big) - C_P\big(S_0^{(2n-1)}, T\big) + (P-K)\, I_P\big(S_0^{(2n-1)}; P, T\big) \Big) \square \end{aligned}$$

Concluding this section, we have established how single and double barrier derivatives can be priced using the FFT results by Carr and Madan [1999], Bakshi and Madan [2000], and Raible [2000].

Alternatively, double barrier derivatives be priced exploiting that model (1) can be represented as a time-changed geometric Brownian motion, see Section 4. In financial applications, this time change can be interpreted as a measure of activity or “business clock”.

4 Time-change representations

For the discounted process $\tilde{S} = \{\tilde{S}_t\}_{t\geq 0} = \{S_t/B_t\}_{t\geq 0}$, representations as a time-changed Brownian motion are available. The time-change representations are interesting from a numerical point of view: They allow for fast converging infinite series instead of Laplace or Fourier inversions. Digital options in this setting can be priced using first-passage time results by Hieber and Scherer [2012]. We describe $\tilde{S}$ as a time-changed geometric Brownian motion G_{Λ_t}, i.e.

$$\frac{dG_t}{G_t} = dW_t, \quad G_0 := S_0 > 0, \tag{25}$$

and $\Lambda = \{\Lambda_t\}_{t\geq 0}$ is a (pathwise) continuous and increasing stochastic process with $\Lambda_0 = 0$ and $\lim_{t\to\infty} \Lambda_t = \infty$ $\mathbb{Q}$-a.s.. If the Laplace transform of Λ_T is known, it is denoted by $\vartheta^c_T(u) := \mathbb{E}\big[\exp\big(-u\Lambda_T\big)\big]$. Then, the characteristic function of $\ln(\tilde{S}_t) = \ln(G_{\Lambda_t})$ is given by $\tilde{\varphi_T}(u, S_0) = \exp(iu\ln(S_0)) \cdot \vartheta^c_T\big((iu + u^2)/2\big)$ (see, e.g., Equation (2.3) in Hurd [2009]). Theorem 12 presents the time-change representations for all models from Section 2.

Theorem 12 (Time-change representations)
$\tilde{S} = \{S_t/B_t\}_{t\geq 0}$ can be represented as a time-changed geometric Brownian motion G_{Λ_T} in the following cases of interest[3]:

- *In Examples 1 and 2, this is achieved by $\Lambda_T := \int_0^T \lambda_s ds$ and $\lambda_t = v_t$ for all $t \geq 0$. The Laplace transform of the integrated process Λ_T is given by*

$$\begin{aligned}\vartheta^c_T(u) &:= \mathbb{E}\Big[\exp\Big(-u\int_0^T \lambda_s ds\Big)\Big] \\ &= \left(\frac{\exp(\theta_1 T/2)}{\cosh(\varrho_1 T/2) + \frac{\theta_1}{\varrho_1}\sinh(\varrho_1 T/2)}\right)^{\frac{2\theta_1\nu_1}{\gamma_1^2}} \left(\frac{\exp(\theta_2 T/2)}{\cosh(\varrho_2 T/2) + \frac{\theta_2}{\varrho_2}\sinh(\varrho_2 T/2)}\right)^{\frac{2\theta_2\nu_2}{\gamma_2^2}} \\ &\quad \cdot \exp\left\{-\frac{\lambda_0^{(1)}}{\varrho_1}\frac{u\sinh(\varrho_1 T/2)}{\cosh(\varrho_1 T/2) + \frac{\theta_1}{\varrho_1}\sinh(\varrho_1 T/2)} - \frac{\lambda_0^{(2)}}{\varrho_2}\frac{u\sinh(\varrho_2 T/2)}{\cosh(\varrho_2 T/2) + \frac{\theta_2}{\varrho_2}\sinh(\varrho_2 T/2)}\right\},\end{aligned} \tag{26}$$

  where $\varrho_j = \sqrt{\theta_j^2 + \gamma_j^2 u}$, for $j = 1, 2$. Model (5) is obtained if one sets $\nu = \nu_1$, $\theta = \theta_1$, $\gamma = \gamma_1$, $\lambda_0 = \lambda_0^{(1)}$, and $\theta_2 = \lambda_0^{(2)} = 0$.

- *In Example 3, one sets $\Lambda_T := \int_0^T \lambda_s ds$ and $\lambda_t = \sigma_t^2$ for all $t \geq 0$. The Laplace transform of the integrated process Λ_T is given by*

$$\vartheta^c_T(u) := \mathbb{E}\Big[\exp\Big(-u\int_0^T \lambda_s ds\Big)\Big] = \exp\big(L(u)\lambda_0/2 + M(u)\sqrt{\lambda_0} + N(u)\big), \tag{27}$$

[3]For Examples 1 and 2, we refer to, e.g., Cox et al. [1985], Dufresne [2001]; for Example 3 to Stein and Stein [1991]; for Example 4 to, e.g., Dassios and Jang [2003].

where the functions $L(u)$, $M(u)$, and $N(u)$ are defined in Appendix A.

- *In Example 4, this is achieved by $\Lambda_T := \int_0^T \lambda_s ds$ and $\lambda_t = v_t$ for all $t \geq 0$. Then,*

$$\begin{aligned}\vartheta_T^c(u) &:= \mathbb{E}\Big[\exp\Big(-u\int_0^T \lambda_s ds\Big)\Big] \\ &= \exp\left(-\frac{u\lambda_0}{\delta}(1-\exp(-\delta T)) - \psi\int_0^T \Big[1-\hat{g}\Big(\frac{u}{\delta}(1-\exp(\delta(T-t)))\Big)\Big]\,dt\right),\end{aligned} \tag{28}$$

where the parameters are defined as in Example 4.

Then, exit-and-out contracts can be priced by rapidly converging infinite series. In contrast to Section 3.2, where one had to compute two Fourier integrals per term, Theorem 13 presents infinite series that need a single evaluation of the Laplace transform of the time change per term. This allows for a faster computation and an easier control of the truncation error (we present error bounds in Section 5). Theorem 13 presents the general pricing result for exit-and-out contracts that pay $g(S_T)$ at maturity T if the path survives until T.

Theorem 13 (Two barriers: Exit-and-out contract II)
Consider a time-changed geometric Brownian motion G_{Λ_t} with a (pathwise) continuous time-change Λ, independent of G. Denote the Laplace transform of Λ_T by $\vartheta_T^c(u) := \mathbb{E}[\exp(-u\Lambda_T)]$, $u \geq 0$. Then, the price of a derivative with payoff $\mathbb{1}_{\{\tau>T\}}\, g(S_T)$ (where $\mathbb{E}[g(S_T)] < \infty$) at maturity T is given by

$$X_{D,P}^{g(S_T)}(S_0) = \frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b}\sum_{n=1}^{\infty}\vartheta_T^c\left(\frac{1}{8}+\frac{n^2\pi^2}{2(a-b)^2}\right)\sin\left(\frac{n\pi(x-b)}{a-b}\right)Z_n^{g(S_T)}, \tag{29}$$

where $Z_n^{g(S_T)} := \int_b^a e^{-\frac{y}{2}}\sin\left(\frac{n\pi(y-b)}{a-b}\right)g(e^y)dy$, $x := \ln(S_0)$, $a := \ln(P)$, and $b := \ln(D)$.

Proof
The transition density function describes the probability density that the process $\{\ln(S_t)\}_{t\geq 0}$ starts at $x := \ln(S_0)$, survives until time T, and ends up in $y := \ln(S_T)$. In the Black–Scholes model, this density is given by (see, e.g., Cox and Miller [1965], Pelsser [2000])

$$f_{ab}(T,y) = \frac{2e^{\frac{x}{2}}}{a-b}\sum_{n=1}^{\infty} e^{-\left(\frac{1}{8}+\frac{n^2\pi^2}{2(a-b)^2}\right)T}\, e^{-\frac{y}{2}}\sin\left(\frac{n\pi(x-b)}{a-b}\right)\sin\left(\frac{n\pi(y-b)}{a-b}\right).$$

In the Black–Scholes model, we then find that the price of an exit-and-out contract with payoff $\mathbb{1}_{\{\tau>T\}}\, g(S_T)$ at maturity T is given by

$$\begin{aligned}BS_{D,P}^{g(S_T)}(S_0) &:= \frac{1}{B_T}\int_b^a f_{ab}(T,y)\, g(e^y)\, dy \\ &= \frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b}\sum_{n=1}^{\infty} e^{-\left(\frac{1}{8}+\frac{n^2\pi^2}{2(a-b)^2}\right)T}\sin\left(\frac{n\pi(x-b)}{a-b}\right)\left(\int_b^a e^{-\frac{y}{2}}\sin\left(\frac{n\pi(y-b)}{a-b}\right)g(e^y)dy\right).\end{aligned}$$

If the interval $[0,T]$ is continuously transformed to $[0,\Lambda_T]$, the latter expression is the price of an exit-and-out contract conditional on the time change $T=\Lambda_T$. If this time-change has Laplace transform $\vartheta^c_T(u) := \mathbb{E}[\exp(-u\Lambda_T)]$, $u \geq 0$, we conclude that

$$\mathbb{E}\left[\mathbb{E}\left[e^{-\left(\frac{1}{8}+\frac{n^2\pi^2}{2(a-b)^2}\right)T}\,\middle|\, T=\Lambda_T\right]\right] = \vartheta^c_T\left(\frac{1}{8}+\frac{n^2\pi^2}{2(a-b)^2}\right)$$

and thus obtain the price of exit-and-out contracts on time-changed geometric Brownian motion

$$X^{g(S_T)}_{D,P}(S_0) = \frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b}\sum_{n=1}^{\infty}\vartheta^c_T\left(\frac{1}{8}+\frac{n^2\pi^2}{2(a-b)^2}\right)\sin\left(\frac{n\pi(x-b)}{a-b}\right)\left(\int_b^a e^{-\frac{y}{2}}\sin\left(\frac{n\pi(y-b)}{a-b}\right)g(e^y)dy\right).$$

□

The result in Theorem 13 can also be used to price options on one barrier. Therefore, e.g., the upper barrier is set to a very high value (i.e. $a = 10\sigma\sqrt{T}$) that guarantees that the probability of hitting the upper barrier is negligible (i.e. smaller than 1e-16). In our numerical examples (see Section 6), this approach is still significantly faster than FFT techniques.

Theorem 14 is a first application of Theorem 13 to price (double) digital options.

Theorem 14 (Double digital options II)
Consider a time-changed geometric Brownian motion G_{Λ_t} with a (pathwise) continuous time-change Λ, independent of G. Denote the Laplace transform of Λ_T by $\vartheta^c_T(u) := \mathbb{E}[\exp(-u\Lambda_T)]$, $u \geq 0$. If the strike price is denoted $K_T := K\exp(\int_0^T r_t\,dt) > D_T$, the price of a double digital option with payoff $\mathbb{1}_{\{\tau>T,\,S_T>K_T\}}$ at maturity T is given by

$$I_K\big(S_0;D,P,T\big) = \frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b}\sum_{n=1}^{\infty}\vartheta^c_T\left(\frac{1}{8}+\frac{n^2\pi^2}{2(a-b)^2}\right)\sin\left(\frac{n\pi(x-b)}{a-b}\right)Z^{g(S_T)}_n,$$

where

$$Z^{g(S_T)}_n = \frac{e^{-\frac{a}{2}}\frac{n\pi(-1)^{n+1}}{a-b} + e^{-\frac{k}{2}}\left(\frac{1}{2}\sin\left(\frac{n\pi(k-b)}{a-b}\right)+\frac{n\pi}{a-b}\cos\left(\frac{n\pi(k-b)}{a-b}\right)\right)}{\frac{1}{4}+\frac{n^2\pi^2}{(a-b)^2}},$$

$x := \ln(S_0)$, $k := \ln(K)$, $a := \ln(P)$, *and* $b := \ln(D)$.

Proof
From Theorem 13, we can conclude that

$$\begin{aligned}Z^{g(S_T)}_n &= \int_b^a e^{-\frac{y}{2}}\sin\left(\frac{n\pi(y-b)}{a-b}\right)g(e^y)dy = \int_k^a e^{-\frac{y}{2}}\sin\left(\frac{n\pi(y-b)}{a-b}\right)dy\\ &= \frac{e^{-\frac{y}{2}}}{\frac{1}{4}+\frac{n^2\pi^2}{(a-b)^2}}\left(-\frac{1}{2}\sin\left(\frac{n\pi(y-b)}{a-b}\right)-\frac{n\pi}{a-b}\cos\left(\frac{n\pi(y-b)}{a-b}\right)\right)\Bigg|_k^a\end{aligned}$$

$$= \frac{e^{-\frac{a}{2}} \frac{n\pi(-1)^{n+1}}{a-b} + e^{-\frac{k}{2}} \left(\frac{1}{2} \sin\left(\frac{n\pi(k-b)}{a-b} \right) + \frac{n\pi}{a-b} \cos\left(\frac{n\pi(k-b)}{a-b} \right) \right)}{\frac{1}{4} + \frac{n^2\pi^2}{(a-b)^2}}.$$

In the special case $k = b$, this is a result derived in Hieber and Scherer [2012][4], i.e.

$$Z_n^{g(S_T)} = \frac{e^{-\frac{a}{2}} \frac{n\pi(-1)^{n+1}}{a-b} + e^{-\frac{k}{2}} \frac{n\pi}{a-b}}{\frac{1}{4} + \frac{n^2\pi^2}{(a-b)^2}}. \qquad \square$$

The same idea can now be used to price (double) barrier options. The result in Theorem 15 builds on a representation of the Black–Scholes price of a double barrier option that is rarely used in the literature, see, e.g., Pelsser [2000].

Theorem 15 (Double barrier options II)
Consider a time-changed Brownian motion G_{Λ_t} with a (pathwise) continuous time-change Λ, independent of G. Denote the Laplace transform of Λ_T by $\vartheta_T^c(u) := \mathbb{E}[\exp(-u\Lambda_T)]$, $u \geq 0$. If the strike price is denoted $K_T := K \exp(\int_0^T r_t \, dt) > D_T$, $t \geq 0$, the price of a double barrier option with payoff $\mathbb{1}_{\{\tau > T\}}(S_T - K_T)^+$ is given by

$$EOC_K(S_0; D, P, T) = \frac{2e^{\frac{x}{2}}}{a-b} \sum_{n=1}^{\infty} \vartheta_T^c \left(\frac{1}{8} + \frac{n^2\pi^2}{2(a-b)^2} \right) \sin\left(\frac{n\pi(x-b)}{a-b} \right) Z_n^{g(S_T)}, \tag{30}$$

where

$$Z_n^{g(S_T)} = \frac{\frac{2n\pi}{a-b}(-1)^{n+1} \sinh\left(\frac{a-k}{2}\right) - \sin\left(\frac{n\pi(k-b)}{a-b}\right)}{e^{-\frac{k}{2}} \left(\frac{1}{4} + \frac{n^2\pi^2}{(a-b)^2} \right)}, \; x := \ln(S_0),\; k := \ln(K),\; a := \ln(P), \text{ and } b := \ln(D).$$

Proof
Applying the results from Theorem 13, we get

$$Z_n^{g(S_T)} = \int_b^a e^{-\frac{y}{2}} \sin\left(\frac{n\pi(y-b)}{a-b} \right) g(e^y) dy = \int_k^a e^{-\frac{y}{2}} \sin\left(\frac{n\pi(y-b)}{a-b} \right) (e^y - e^k) dy$$

$$= \left. \frac{\frac{e^{\frac{y}{2}} + e^{k-\frac{y}{2}}}{2} \sin\left(\frac{n\pi(y-b)}{a-b} \right) - \frac{n\pi\left(e^{\frac{y}{2}} - e^{k-\frac{y}{2}} \right)}{a-b} \cos\left(\frac{n\pi(y-b)}{a-b} \right)}{\frac{1}{4} + \frac{n^2\pi^2}{(a-b)^2}} \right|_k^a$$

$$= \frac{\frac{n\pi(-1)^{n+1}}{a-b} \left(e^{\frac{a}{2}} + e^{k-\frac{a}{2}} \right) - e^{\frac{k}{2}} \sin\left(\frac{n\pi(k-b)}{a-b} \right)}{\frac{1}{4} + \frac{n^2\pi^2}{(a-b)^2}} = \frac{\frac{2n\pi(-1)^{n+1}}{a-b} \sinh\left(\frac{a-k}{2}\right) - \sin\left(\frac{n\pi(k-b)}{a-b} \right)}{e^{-\frac{k}{2}} \left(\frac{1}{4} + \frac{n^2\pi^2}{(a-b)^2} \right)}.$$

In the case of Brownian motion ($\Lambda_T = T$) this pricing result is given in, e.g., Pelsser [2000]. $\square$

[4]Using that $\sin\left(\frac{n\pi(x-b)}{a-b}\right) = (-1)^n \sin\left(\frac{n\pi(x-a)}{a-b}\right)$, one obtains the results in Hieber and Scherer [2012], Theorem 2 ($\mu = -1/2$, $\sigma = 1$, a generalization to $\mu \in \mathbb{R}$ and $\sigma > 0$ is straightforward).

Remark 16 (Single barrier limit as special case)
One can show that in the limit $P \to \infty$, the prices of the exit-and-out contracts in Theorems 13, 14, and 15 converge to the already presented single barrier expression

$$X_{D,\infty}^{g(S_T)}(S_0) = \frac{1}{B_T}\left(\mathbb{E}_{\mathbb{Q},S_0}\left[\mathbb{1}_{\{S_T>D_T\}}\, g(S_T)\right] - \frac{S_0}{D}\,\mathbb{E}_{\mathbb{Q},D^2/S_0}\left[\mathbb{1}_{\{S_T>D_T\}}\, g\left(S_T\right)\right]\right)$$

in Theorem 5 (see Appendix B for the computations).

*However, it turns out that if one wants to evaluate the latter expectations numerically, it is very convenient to still use the series representation for the double barrier contract (Theorem 13). Therefore, the upper barrier $a = \ln(P)$ is, for example, set to $10\sigma\sqrt{T}$, a value that guarantees that the probability of hitting the upper barrier is negligible, i.e. it is smaller than 1*e-*16.*

Remark 17 (Stochastic interest rates)
If necessary, it is possible to include stochastic interest rates in the model framework (1) while still keeping the analytical tractability. If $\{r_t\}_{t\geq 0}$ is independent of $W = \{W_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$, this is straightforward: $1/B_T$ must simply be replaced by $\mathbb{E}_{\mathbb{Q}}[1/B_T]$. Dependence between $\{r_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ can be introduced as follows: One defines $r_t := \gamma_t - \rho_*\,\sigma_t^2$, where $\{\gamma_t\}_{t\geq 0}$ is independent of $W = \{W_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$. $\rho_* \in \mathbb{R}$ can be used to include either a positive or a negative dependence between volatility and interest rates. The results in Theorems 14 and 15 can rather easily be modified; $\vartheta_T^c\left(\frac{1}{8} + \frac{n^2\pi^2}{2(a-b)^2}\right)$ then changes into $\vartheta_T^c\left(\frac{1}{8} + \frac{n^2\pi^2}{2(a-b)^2} + \rho_*\right)$. The Fourier pricing results can also be adapted easily.

5 Error bounds

To implement the pricing formulas in Theorems 13, 14, and 15, the infinite series have to be approximated by finite series. Lemma 18 presents error bounds if the Laplace transform of the time change is exponentially bounded, i.e. if $\vartheta_T(u) \leq J\exp(-Mu)$, where J, M are positive constants, and if the Laplace transform is bounded by $J^*\exp(-M^*\sqrt{u})$, where J^*, M^* are again positive constants. This applies to all examples presented in Section 2.

Lemma 18 (Error bounds)
Consider a time-changed geometric Brownian motion $\{G_{\Lambda_t}\}_{t\geq 0}$ with a (pathwise) continuous time-change Λ, independent of G. Set again $a := \ln(P)$, $b := \ln(D)$, and $x := \ln(S_0)$ and denote the Laplace transform of Λ_T by $\vartheta_T^c(u) := \mathbb{E}[\exp(-u\Lambda_T)]$, $u \geq 0$. Assume that the conditional payoff function $g(e^y)$ is bounded for $y \in [a, b]$. Set

$$K^* := \int_b^a e^{-\frac{y}{2}}\, |g(e^y)|\, dy\,.$$

If the infinite series in Theorem 13 is truncated after N summands, the (absolute) computation error of the option price is defined as

$$\epsilon := \left|\frac{2e^{\frac{x}{2}}}{a-b}\sum_{n=N+1}^{\infty}\vartheta_T^c\left(\frac{1}{8} + \frac{n^2\pi^2}{2(a-b)^2}\right)\sin\left(\frac{n\pi(x-b)}{a-b}\right) Z_n^{g(S_T)}\right|. \tag{31}$$

For a given precision $\epsilon > 0$, a lower bound for the summation index $N \in \mathbb{N}$ is required.

(a) If the Laplace transform of the time change is exponentially bounded, i.e. if $\vartheta_T(u) \leq J\exp(-Mu)$, where J, M are positive constants, we find that

$$N > \sqrt{\left|\frac{2(a-b)^2}{\pi^2 M} \ln\left(\frac{B_T M \pi^2 \epsilon}{2K^* e^{\frac{x}{2}}(a-b)J}\right)\right|}. \tag{32}$$

(b) If $\vartheta_T(u) \leq J^* \exp(-M^*\sqrt{u})$, where $J^*, M^* > 0$ are positive constants, then

$$N > -\frac{\sqrt{2}(a-b)}{\pi M^*} \ln\left(\frac{B_T M^* \pi \epsilon}{2\sqrt{2}K^* e^{\frac{x}{2}} J^*}\right). \tag{33}$$

Proof
Note that

$$\left|\int_b^a e^{-\frac{y}{2}} \sin\left(\frac{n\pi(y-b)}{a-b}\right) g(e^y)\, dy\right| \leq \int_b^a e^{-\frac{y}{2}} \; |g(e^y)| \; dy = K^* < \infty\,.$$

Similarly to Hieber and Scherer [2012], if $\vartheta_T(u) \leq J\exp(-Mu)$, where J, M are positive constants, we get from Equation (31)

$$\begin{aligned}
\epsilon &= \left|\frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b} \sum_{n=N+1}^{\infty} \vartheta_T^c\left(\frac{1}{8} + \frac{n^2\pi^2}{2(a-b)^2}\right) \sin\left(\frac{n\pi(x-b)}{a-b}\right) Z_n^{g(S_T)}\right| \\
&\leq \frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b} K^* \sum_{n=N+1}^{\infty} n\, \vartheta_T^c\left(\frac{n^2\pi^2}{2(a-b)^2}\right) \\
&\leq \frac{1}{B_T}\frac{2K^* e^{\frac{x}{2}}}{a-b} \int_N^{\infty} n\, J \exp\left(-M\frac{n^2\pi^2}{2(a-b)^2}\right) dn \\
&= \frac{2K^* e^{\frac{x}{2}}(a-b)}{B_T M \pi^2} J \exp\left(-M\frac{N^2\pi^2}{2(a-b)^2}\right).
\end{aligned}$$

From this, a lower bound for the summation index N is obtained as

$$N > \sqrt{\left|\frac{2(a-b)^2}{\pi^2 M} \ln\left(\frac{B_T M \pi^2 \epsilon}{2K^* e^{\frac{x}{2}}(a-b)J}\right)\right|}. \tag{34}$$

If $\vartheta_T(u) \leq J^* \exp(-M^*\sqrt{u})$, where J^*, M^* are positive constants, we get analoguously

$$\begin{aligned}
\epsilon &\leq \frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b} K^* \int_N^{\infty} J^* \exp\left(-M^*\frac{n\pi}{\sqrt{2}(a-b)}\right) dn \\
&= \frac{1}{B_T}\frac{2K^* e^{\frac{x}{2}}}{a-b}\frac{\sqrt{2}(a-b)}{M^*\pi} J^* \exp\left(-M^*\frac{N\pi}{\sqrt{2}(a-b)}\right).
\end{aligned}$$

Then,

$$N > -\frac{\sqrt{2}(a-b)}{\pi M^*} \ln\left(\frac{B_T M^* \pi \epsilon}{2\sqrt{2} K^* e^{\frac{x}{2}} J^*}\right). \tag{35}$$

□

The bound K^* can easily be derived for specific conditional payoff functions $g(S_T)$. For double digital options, one obtains $K^* = 2/\sqrt{K} - 2/\sqrt{P}$, for double barrier options $K^* = 2\sqrt{P} - 2K/\sqrt{P}$.

6 Numerical case study

In this section, we give a numerical example comparing the FFT technique (Section 3) to the analytic formulas using the time-change representation of the given models (Section 4). We compare the results of the one-factor stochastic volatility models, i.e. the Heston and Stein–Stein model, with regard to accuracy and computation time. Additional improvements could be obtained if multi-factors models were used.

6.1 One barrier

First, we compare the two approaches to price digital options. In all models, the parameters were chosen such that the average volatility of the annualized stock returns equals 21%. The corresponding pricing formulas are to be found in Lemma 6 (FFT technique) and Theorem 14 (time-change representation). As discussed in Remark 16, we set the upper barrier to $a = \ln(P) = 10\sigma\sqrt{T}$, a value that guarantees that the probability of hitting the upper barrier is negligible, i.e. it is smaller than 1e-16.

Apart from that, the infinite series has to be truncated. Error bounds for this truncation are easy to obtain, see Section 5. In our parameter sets, $N = 90$ terms turned out to be enough to obtain an acceptable relative error.

Table 1 gives the results for different parameter sets in the Black–Scholes, the Heston, and the Stein–Stein model. We aim at obtaining relative pricing errors below 1e-04. If the time-change representation is used, a higher accuracy of 1e-12 comes at almost no additional computational cost. In the stochastic volatility models, the true value is computed using the time-change representation with $a = 20\sigma$ and $N = 200$. The Black–Scholes model is also displayed, since the more convenient closed-form expression presented as Equation (15) allows to compare the results to existing pricing formulas. If we aim at an accuracy of at least 1e-04, in all models and over all the considered parameter sets, we find that the time-change representation is about 30-40 times faster than FFT. This is mainly due to the fact that the Laplace transform of the time change has to be evaluated only $N = 90$ times for the time-change representation, whereas a reasonably small error in the FFT technique requires several thousand evaluations of the characteristic function. This explains why the benefit of the time change representation is even higher if more complex or multi-factor stochastic volatility models are used.

Black–Scholes	analytic expression			FFT			true value
	$\hat{I_K}(S_0;D,T)$	rel. err.	time	$\hat{I_K}(S_0;D,T)$	rel. err.	time	$I_K(S_0;D,T)$
D=0.80	0.6143907366	1e–16	0.21ms	0.6143882493	4e–06	16.1ms	0.6143907366
D=0.85	0.4750496283	1e–16	0.23ms	0.4750486755	2e–06	28.8ms	0.4750496283
D=0.90	0.3184738022	1e–16	0.17ms	0.3184759717	7e–06	16.7ms	0.3184738022
D=0.95	0.1563530857	1e–16	0.16ms	0.1563541729	7e–06	18.6ms	0.1563530857

Heston	analytic expression			FFT			true value
	$\hat{I_K}(S_0;D,T)$	rel. err.	time	$\hat{I_K}(S_0;D,T)$	rel. err.	time	$I_K(S_0;D,T)$
D=0.80	0.6226859185	3e–12	0.32ms	0.6226834064	4e–06	19.2ms	0.6226859185
D=0.85	0.4852440603	2e–12	0.23ms	0.4852429997	2e–06	15.3ms	0.4852440602
D=0.90	0.3274529660	2e–12	0.19ms	0.3274553781	7e–06	15.5ms	0.3274529660
D=0.95	0.1614407693	2e–12	0.19ms	0.1614419955	8e–06	15.3ms	0.1614407693

Stein–Stein	analytic expression			FFT			true value
	$\hat{I_K}(S_0;D,T)$	rel. err.	time	$\hat{I_K}(S_0;D,T)$	rel. err.	time	$I_K(S_0;D,T)$
D=0.80	0.6362552464	1e–08	1.06ms	0.6362530653	4e–06	36.5ms	0.6362552383
D=0.85	0.5063024193	1e–08	1.58ms	0.5063011967	2e–06	33.8ms	0.5063024113
D=0.90	0.3488272033	2e–08	2.15ms	0.3488301222	8e–06	33.2ms	0.3488271958
D=0.95	0.1745901944	3e–08	3.30ms	0.1745917779	8e–06	33.2ms	0.1745901886

Table 1 Prices $\hat{I_K}(S_0;D,T)$ of digital options in the Black–Scholes model (top, $\sigma = 21\%$), in the Heston model (middle, $\lambda_0 = 0.0441$, $\theta = 0.005$, $\nu = 0.0441$, $\gamma = 0.10$, $\rho = 0$), in the Stein–Stein model (below, $\lambda_0 = 0.21$, $\xi = 0.002$, $\varkappa = 0.70$, $k = 0.10$) calculated by the analytic expression (left column, $N = 90$, $P = \exp(10\sigma\sqrt{T})$, see Theorem 14) and by FFT (middle column, see Lemma 6). The true value $I_K(S_0;D,T)$ (right column) was calculated using $N = 200$ and $P = \exp(20\sigma\sqrt{T})$ in the analytic expression. The remaining parameters are chosen as $S_0 = 1$, $K = D$, $r_t = 0.10$, and $T = 1$. Absolute errors of both approaches are given. The computation time was calculated using Matlab on a 2.0 GHz PC.

6.2 Two barriers

For double digital options, the pricing formulas are presented in Lemma 10 (FFT technique) and in Theorem 14 (time-change representation). The advantage of the FFT technique is the fact that call and digital options with different strikes can be evaluated simultaneously (see, e.g., Carr and Madan [1999]). Table 2 presents the pricing results together with both computation time and relative error in the Black–Scholes model (top), the Heston model (middle), and the Stein–Stein model (below). Again, we aim at an accuracy (in terms of the relative error) of 1e-04. In the two barrier case, the advantage of the time-change representation is more significant than in the single barrier case. Since the upper barrier now does not have to be set to infinity, $N = 20$ terms in the series representation (Theorem 14) are sufficient to obtain a very high accuracy. The results in Table 2 show that the computation time for the time-change representation is now 50-100 times faster than the FFT technique. Although the FFT technique is now also an infinite series of call options (truncated at $N = 20$), its computation time is about the same as in the single barrier case since digital options with different strikes can be computed simultaneously.

Black–Scholes	analytic expression			FFT			true value
	$\hat{I_K}(S_0;D,P,T)$	rel. err.	time	$\hat{I_K}(S_0;D,P,T)$	rel. err.	time	$I_K(S_0;D,P,T)$
$D{=}S_0^2/P{=}0.80$	0.5902066317	1e–16	0.05ms	0.5902061072	9e–07	9.4ms	0.5902066317
$D{=}S_0^2/P{=}0.85$	0.3834252443	1e–16	0.05ms	0.3834234415	5e–06	9.4ms	0.3834252443
$D{=}S_0^2/P{=}0.90$	0.1095096997	1e–16	0.05ms	0.1095104165	6e–06	9.3ms	0.1095096997
$D{=}S_0^2/P{=}0.95$	0.0000999252	1e–16	0.05ms	0.0000979657	2e–02	7.5ms	0.0000999252

Heston	analytic expression			FFT			true value
	$\hat{I_K}(S_0;D,P,T)$	rel. err.	time	$\hat{I_K}(S_0;D,P,T)$	rel. err.	time	$I_K(S_0;D,P,T)$
$D{=}S_0^2/P{=}0.80$	0.5968486249	1e–16	0.22ms	0.5968439794	6e–06	14.5ms	0.5968486249
$D{=}S_0^2/P{=}0.85$	0.3961295282	1e–16	0.18ms	0.3961276952	5e–06	19.8ms	0.3961295282
$D{=}S_0^2/P{=}0.90$	0.1289954367	1e–16	0.22ms	0.1289962389	6e–06	21.1ms	0.1289954367
$D{=}S_0^2/P{=}0.95$	0.0009498677	1e–16	0.25ms	0.0009484332	9e–04	20.0ms	0.0009498677

Stein–Stein	analytic expression			FFT			true value
	$\hat{I_K}(S_0;D,P,T)$	rel. err.	time	$\hat{I_K}(S_0;D,P,T)$	rel. err.	time	$I_K(S_0;D,P,T)$
$D{=}S_0^2/P{=}0.80$	0.6057024792	1e–16	0.88ms	0.6056960521	6e–06	34.6ms	0.6057024792
$D{=}S_0^2/P{=}0.85$	0.4197102727	1e–16	0.97ms	0.4197085100	5e–06	37.1ms	0.4197102727
$D{=}S_0^2/P{=}0.90$	0.1721021525	1e–16	0.94ms	0.1721035072	6e–06	35.4ms	0.1721021525
$D{=}S_0^2/P{=}0.95$	0.0094331898	1e–16	0.84ms	0.0094333509	1e–03	37.3ms	0.0094331898

Table 2 Prices $\hat{I_K}(S_0;D,P,T)$ of double digital options in the Black–Scholes model (top, $\sigma = 21\%$), in the Heston model (middle, $\lambda_0 = 0.0441$, $\theta = 0.005$, $\nu = 0.0441$, $\gamma = 0.10$, $\rho = 0$), in the Stein–Stein model (below, $\lambda_0 = 0.21$, $\xi = 0.002$, $\varkappa = 0.70$, $k = 0.10$) calculated by the analytic expression (left column, $N = 20$, see Theorem 14) and by FFT (middle column, $N = 20$, see Lemma 10). The true value $I_K(S_0;D,P,T)$ (right column) was calculated using $N = 200$ in the analytic expression. The remaining parameters are chosen as $S_0 = 1$, $K = D$, $r_t = 0.10$, and $T = 1$. Absolute errors of both approaches are given. The computation time was calculated using Matlab on a 2.0 GHz PC.

The same holds for double barrier options. The corresponding pricing formulas are given in Lemma 11 (FFT technique) and in Theorem 15 (time-change representation). Table 3 presents the pricing results together with both computation time and absolute error in the Black–Scholes model (top), the Heston model (middle), respectively the Stein–Stein model (below). Aiming at a relative error of less than 1e-04, the analytic expression resulting from the time-change representation turns out to be superior to the FFT technique, this time being about 100 times faster. A higher precision in the time-change representation – which is, for example, important for at the money barriers – comes at almost no additional computational cost.

7 Conclusion

We showed how barrier derivatives can efficiently be priced in stochastic volatility models. Instead of relying on FFT (see, e.g., Carr and Madan [1999], Carr and Lee [2009]), we derive rapidly converging infinite series that can easily be implemented and allow for a straightforward error control. Those series turn out to be faster and more accurate than FFT.

Black–Scholes	analytic expression			FFT			true value
	$\hat{EOC}_K(S_0;\cdot)$	rel. err.	time	$\hat{EOC}_K(S_0;\cdot)$	rel. err.	time	$EOC_K(S_0;\cdot)$
$D{=}S_0^2/P{=}0.80$	0.1625270208	1e–16	0.06ms	0.1625265175	3e–06	5.83ms	0.1625270208
$D{=}S_0^2/P{=}0.85$	0.0867337308	1e–16	0.05ms	0.0867329170	9e–06	5.16ms	0.0867337308
$D{=}S_0^2/P{=}0.90$	0.0168613875	1e–16	0.04ms	0.0168608922	3e–05	4.79ms	0.0168613875
$D{=}S_0^2/P{=}0.95$	0.0000075922	1e–16	0.03ms	0.0000072895	2e–07	4.81ms	0.0000075922

Heston	analytic expression			FFT			true value
	$\hat{EOC}_K(S_0;\cdot)$	rel. err.	time	$\hat{EOC}_K(S_0;\cdot)$	rel. err.	time	$EOC_K(S_0;\cdot)$
$D{=}S_0^2/P{=}0.80$	0.1612642249	1e–16	0.20ms	0.1612637548	6e–08	12.9ms	0.1612642249
$D{=}S_0^2/P{=}0.85$	0.0879562985	1e–16	0.11ms	0.0879555363	9e–07	13.4ms	0.0879562985
$D{=}S_0^2/P{=}0.90$	0.0197738415	1e–16	0.10ms	0.0197733214	2e–05	13.1ms	0.0197738415
$D{=}S_0^2/P{=}0.95$	0.0000721684	1e–16	0.10ms	0.0000719713	3e–03	13.1ms	0.0000721684

Stein–Stein	analytic expression			FFT			true value
	$\hat{EOC}_K(S_0;\cdot)$	rel. err.	time	$\hat{EOC}_K(S_0;\cdot)$	rel. err.	time	$EOC_K(S_0;\cdot)$
$D{=}S_0^2/P{=}0.80$	0.1576325944	1e–16	0.79ms	0.1576321967	5e–06	32.7ms	0.1576325944
$D{=}S_0^2/P{=}0.85$	0.0888429183	1e–16	0.66ms	0.0888423088	9e–06	32.7ms	0.0888429183
$D{=}S_0^2/P{=}0.90$	0.0256262200	1e–16	0.73ms	0.0256257853	6e–06	32.8ms	0.0256262200
$D{=}S_0^2/P{=}0.95$	0.0007147280	1e–16	0.70ms	0.0007148038	2e–02	35.8ms	0.0007147280

Table 3 Prices $\hat{EOC}_K(S_0;D,P,T)$ of double barrier options in the Black–Scholes model (top, $\sigma = 21\%$), in the Heston model (middle, $\lambda_0 = 0.0441$, $\theta = 0.005$, $\nu = 0.0441$, $\gamma = 0.10$, $\rho = 0$), in the Stein–Stein model (below, $\lambda_0 = 0.21$, $\xi = 0.002$, $\varkappa = 0.70$, $k = 0.10$) calculated by the analytic expression (left column, $N = 20$, see Theorem 15) and by FFT (middle column, see Lemma 11). The true value $I_K(S_0;D,P,T)$ (right column) was calculated using $N = 200$ in the analytic expression. The remaining parameters are chosen as $S_0 = 1$, $K = D$, $r_t = 0.10$, and $T = 1$. Absolute errors of both approaches are given. The computation time was calculated using Matlab on a 2.0 GHz PC.

References

G. Bakshi and D. Madan. Spanning and derivative security valuation. *Journal of Financial Economics*, Vol. 55, No. 2:pp. 205–238, 2000.

C. A. Ball and A. Roma. Stochastic volatility option pricing. *Journal of Financial and Quantitative Analysis*, Vol. 29:pp. 589–607, 1994.

O. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck based models and some of their uses in financial economics. *Journal of the Royal Statistical Society: Series B*, Vol. 63, No. 2: pp. 167–241, 2001.

D. Bates. Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options. *Review of Financial Studies*, Vol. 9, No. 1:pp. 69–107, 1996.

F. Black and J. C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, Vol. 31, No. 2:pp. 351–367, 1976.

P. Brockwell, E. Chadraa, and A. Lindner. Continuous-time GARCH processes. *The Annals of Applied Probability*, Vol. 16, No. 2:pp. 790–826, 2006.

P. Carr and J. Crosby. A class of Lévy process models with almost exact calibration to both barrier and vanilla FX options. *Journal of Quantitative Finance*, Vol. 10, No. 10:pp. 1115–1136, 2010.

P. Carr and R. Lee. Put-call symmetry: Extensions and applications. *Mathematical Finance*, Vol. 19, No. 4:pp. 523–560, 2009.

P. Carr and D. B. Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, Vol. 2:pp. 61–73, 1999.

P. Carr, K. Ellis, and V. Gupta. Static heding of exotic derivatives. *Journal of Finance*, Vol. 53, No. 3:pp. 1165–1190, 1998.

P. Carr, H. Geman, D. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, Vol. 13, No. 3:pp. 345–382, 2003.

P. Carr, H. Zhang, and O. Hadjiliadis. Maximum drawdown insurance. *International Journal of Theoretical and Applied Finance*, Vol. 14, No. 8:pp. 1195–1230, 2011.

P. Christoffersen, S. Heston, and K. Jacobs. The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. *Management Science*, Vol. 55:pp. 1914–1932, 2009.

D. Cox and V. Isham. *Monographs on Applied Probability and Statistics: Point Processes*. Chapman & Hall, 1980.

D. Cox and H. Miller. *Theory of Stochastic Processes*. Chapman & Hall, 1965.

J. Cox, J. Ingersoll, and S. Ross. A theory of the term structure of interest rates. *Econometrica*, Vol. 53:pp. 187–201, 1985.

J. da Fonseca, M. Grasselli, and C. Tebaldi. Option pricing when correlations are stochastic: an analytical framework. *Review of Derivatives Research*, Vol. 10:pp. 151–180, 2007.

D. Darling and A. Siegert. The first passage problem for a continuous Markov process. *The Annals of Mathematical Statistics*, Vol. 24, No. 4:pp. 624–639, 1953.

A. Dassios and J.-W. Jang. Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity. *Finance and Stochastics*, Vol. 7:pp. 73–95, 2003.

E. Derman, D. Ergener, and I. Kani. Forever hedged. *Risk*, Vol. 7:pp. 139–145, 1994.

D. Dufresne. The integrated square-root process. *Working paper, University of Montreal*, 2001.

D. Dupont. Hedging barrier options: Current methods and alternatives. *Working paper*, 2002.

B. Eraker, M. Johannes, and N. Polson. The impact of jumps in volatility and returns. *Journal of Finance*, Vol. 58, No. 3:pp. 1269–1300, 2003.

M. Escobar, T. Friederich, L. Seco, and R. Zagst. A general structural approach for credit modeling under stochastic volatility. *Journal of Financial Transformation*, Vol. 32:pp. 123–132, 2011.

W. Feller. Two singular diffusion problems. *The Annals of Mathematics*, Vol. 54, No. 1:pp. 173–182, 1951.

H. Geman and M. Yor. Pricing and hedging double-barrier options: a probabilistic approach. *Mathematical Finance*, Vol. 6, No. 4:pp. 365–378, 1996.

B. Götz. *Valuation of multi-dimensional derivatives in a stochastic covariance framework*. PhD thesis, TU Munich, 2011.

H. He, W. Keirstead, and J. Rebholz. Double lookbacks. *Mathematical Finance*, Vol. 8, No. 3:pp. 201–228, 1998.

S. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Journal of Finance*, Vol. 42:pp. 327–343, 1993.

P. Hieber and M. Scherer. A note on first-passage times of continuously time-changed Brownian motion. *Statistics & Probability Letters*, Vol. 82, No. 1:pp. 165–172, 2012.

J. Hull and A. White. The pricing of options on assets with stochastic volatility. *Journal of Finance*, Vol. 42, No. 2:pp. 281–300, 1987.

T. Hurd. Credit risk modeling using time-changed Brownian motion. *International Journal of Theoretical and Applied Finance*, Vol. 12, No. 8:pp. 1213–1230, 2009.

S. Kammer. *A general first-passage-time model for multivariate credit spreads and a note on barrier option pricing*. PhD thesis, Justus-Liebig Universität Gießen, 2007.

R. Kiesel and M. Lutz. Efficient pricing of constant maturity swap spread options in a stochastic volatility LIBOR market model. *Journal of Computational Finance*, Vol. 14, No. 4:pp. 37–72, 2011.

F. Kilin. Accelerating the calibration of stochastic volatility models. *Journal of Derivatives*, Vol. 18, No. 3:pp. 7–16, 2011.

C. Klüppelberg, A. Lindner, and M. Ross. A continuous-time GARCH process driven by a Lévy process: stationarity and second-order behaviour. *Journal of Applied Probability*, Vol. 41, No. 3:pp. 601–622, 2004.

X. S. Lin. Laplace transform and barrier hitting time distribution. *Actuarial Research Clearing House*, Vol. 1:pp. 165–178, 1999.

A. Lipton. *Mathematical methods for foreign exchange*. World Scientific, 2001.

V. Naik. Option valuation and hedging strategies with jumps in the volatility of asset returns. *Journal of Finance*, Vol. 48, No. 5:pp. 1969–1984, 1993.

A. Pelsser. Pricing double barrier options using Laplace transforms. *Finance and Stochastics*, Vol. 4: pp. 95–104, 2000.

C. Pigorsch and R. Stelzer. A multivariate Ornstein–Uhlenbeck type stochastic volatility model. *Working paper*, 2009.

S. Raible. *Lévy processes in finance: Theory, numerics, and empirical facts.* PhD thesis, Freiburg University, 2000.

E. Reiner and M. Rubinstein. Breaking down the barriers. *Risk 4*, Vol. 8:pp. 28–35, 1991.

S. Rollin, A. Ferreiro-Castilla, and F. Utzet. A new look at the Heston characteristic function. *Working paper*, 2011.

R. Schöbel and J. Zhu. Stochastic volatility with an Ornstein-Uhlenbeck process: An extension. *Review of Finance*, Vol. 3, No. 1:pp. 23–46, 1999.

A. Sepp. Extended CreditGrades model with stochastic volatility and jumps. *Wilmott Magazine*, September:50–62, 2006.

E. Stein and J. Stein. Stock price distributions with stochastic volatility: An analytical approach. *Review of Financial Studies*, Vol. 4:pp. 727–752, 1991.

A Parameters of the Stein–Stein model

The functions $L(u)$, $M(u)$, and $N(u)$ in the characteristic function are defined as

$$
\begin{aligned}
&A := -\frac{\xi}{k^2}, \quad B := \frac{\varkappa\xi}{k^2}, \quad C_u := -\frac{u}{k^2T}, \quad a_u := \sqrt{A^2 - 2C_u}, \quad b_u = -\frac{A}{a_u},\\
&L(u) := -A - a_u \left(\frac{\sinh(a_u k^2 T) + b_u \cosh(a_u k^2 T)}{\cosh(a_u k^2 T) + b_u \sinh(a_u k^2 T)} \right),\\
&M(u) := B \left(\frac{b_u \sinh(a_u k^2 T) + b_u^2 \cosh(a_u k^2 T) + 1 - b_u^2}{\cosh(a_u k^2 T) + b_u \sinh(a_u k^2 T)} - 1 \right),\\
&N(u) := \frac{a_u - A}{2a_u^2} \left(a_u^2 - AB^2 - B^2 a_u \right) k^2 T\\
&\quad + \frac{B^2(A^2 - a_u^2)}{2a_u^3} \left(\frac{(2A + a_u) + (2A - a_u)e^{2a_u k^2 T}}{A + a_u + (a_u - A)e^{2a_u k^2 T}} \right)\\
&\quad + \frac{2AB^2(a_u^2 - A^2)e^{a_u k^2 T}}{a_u^3\left(A + a_u + (a_u - A)e^{2a_u k^2 T}\right)} - \frac{1}{2} \ln\left(\frac{1}{2}\left(\frac{A}{a_u} + 1 \right) + \frac{1}{2}\left(1 - \frac{A}{a_u} \right) e^{2a_u k^2 T} \right).
\end{aligned}
$$

B Single barrier limit

In our series representation the limit $a := \ln(P) \to \infty$ cannot be exchanged with the infinite summation over n as the series representation for $X_{D,\infty}^{g(S_T)}(S_0)$ is not absolutely convergent. To derive the limiting expression, one has to change the series representation. Then, the limiting option price $X_{D,\infty}^{g(S_T)}(S_0)$ is given by Theorem 5. For $a := \ln(P)$, $b := \ln(D)$, and $x := \ln(S_0)$, we obtain

$$
\begin{aligned}
X_{D,\infty}^{g(S_T)}(S_0) &= \lim_{a\to\infty} \frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b}\sum_{n=1}^{\infty} \vartheta_T^c\left(\frac{1}{8}+\frac{n^2\pi^2}{2(a-b)^2}\right)\sin\left(\frac{n\pi(x-b)}{a-b}\right) Z_n^{g(S_T)} \\
&= \lim_{a\to\infty} \mathbb{E}\left[\frac{1}{B_T}\frac{2e^{\frac{x}{2}}}{a-b}\sum_{n=1}^{\infty}\exp\left(-\frac{\Lambda_T}{8}-\frac{n^2\pi^2\Lambda_T}{2(a-b)^2}\right)\sin\left(\frac{n\pi(x-b)}{a-b}\right) Z_n^{g(S_T)}\right] \\
&= \lim_{a\to\infty} \mathbb{E}\Bigg[\frac{1}{B_T}\frac{2}{a-b}\int_b^a \exp\left(-\frac{x-y}{2}-\frac{\Lambda_T}{8}\right) g(e^y) \\
&\qquad \cdot \sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2\Lambda_T}{2(a-b)^2}\right)\sin\left(\frac{n\pi(x-b)}{a-b}\right)\sin\left(\frac{n\pi(y-b)}{a-b}\right) dy\Bigg].
\end{aligned}
$$

If we change the parameterization (see He et al. [1998], Equations (2.3) and (2.4)), we get

$$
\begin{aligned}
&= \lim_{a\to\infty} \mathbb{E}\Bigg[\frac{1}{B_T}\frac{2}{a-b}\int_b^a \frac{a-b}{2}\exp\left(-\frac{x-y}{2}-\frac{\Lambda_T}{8}\right) g(e^y) \\
&\qquad \cdot \sum_{n=-\infty}^{\infty}\left(\varphi\left(\frac{y-x-2n(a-b)}{\sqrt{\Lambda_T}}\right)-\varphi\left(\frac{y+x-2na+(2n-2)b}{\sqrt{\Lambda_T}}\right)\right) dy\Bigg].
\end{aligned}
$$

This series is absolutely convergent, thus we can change limit and summation. In the limit $a \to \infty$ only the "$n = 0$" term remains, i.e.

$$
\begin{aligned}
&= \mathbb{E}\left[\frac{1}{B_T}\int_b^{\infty}\exp\left(-\frac{x-y}{2}-\frac{\Lambda_T}{8}\right) g(e^y)\left(\varphi\left(\frac{y-x}{\sqrt{\Lambda_T}}\right)-\varphi\left(\frac{y+x-2b}{\sqrt{\Lambda_T}}\right)\right) dy\right] \\
&= \mathbb{E}\Bigg[\frac{1}{B_T}\int_b^{\infty} g(e^y)\Bigg(\varphi\left(\frac{y-x+\Lambda_T/2}{\sqrt{\Lambda_T}}\right) \\
&\qquad - \exp\big(-(b-x)\big)\varphi\left(\frac{y+x-2b+\Lambda_T/2}{\sqrt{\Lambda_T}}\right)\Bigg) dy\Bigg] \\
&= \frac{1}{B_T}\left(\mathbb{E}_{\mathbb{Q},S_0}\left[\mathbb{1}_{\{S_T>D_T\}}\, g(S_T)\right]-\frac{S_0}{D}\,\mathbb{E}_{\mathbb{Q},D^2/S_0}\left[\mathbb{1}_{\{S_T>D_T\}}\, g\left(S_T\right)\right]\right).
\end{aligned}
$$