Efficiently pricing barrier options in a Markov-switching framework

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Abstract

An efficient Monte-Carlo simulation for the pricing of barrier options in a Markov-switching model is presented. Compared to a brute-force approach, relying on the simulation of discretized trajectories, the presented algorithm simulates the underlying stock-price process only at state changes and at maturity. Given these pieces of information, option prices are evaluated using the probability of Brownian bridges not to fall below some threshold level. It is illustrated how two methods of variance reduction, control variates and antithetic variates, further improve the algorithm. In a small case study, the algorithm is applied to the pricing of options with the EuroStoxx 50 as underlying.

Key words: Markov switching; barrier option; Monte Carlo; Brownian bridge; variance reduction.

1. Introduction

The cyclical behavior of economic time series is extensively treated in the literature, see, e.g., [2] and [13]. Economic crises (e.g. the oil crisis in 1979–80, the burst of the dotcom bubble in 2000, and the financial crisis in 2008–09) and times of a prospering economy are observed. These structural changes of the economy are translated into changes of the return distribution of stocks in the respective environment. One possibility for the modelling of such phenomena, called Markov-switching models, was proposed by [15]. In this framework, the economy is modelled using different states; each state represents a specific economic situation. A Markov-chain, in discrete or continuous time, models the transition between the states.

Recent empirical literature has found strong evidence for different regimes in various economic time series. Empirical support for regime-switching in interest rates was found by, e.g., [4]; strong support for a two-state Markov-switching model in economic growth rates was detected by [23]; evidence for a switching behaviour in stock market returns was provided by [22, 18]. Markov-switching models are often used to extend the classical Black-Scholes framework, allowing for non-constant model parameters of the drift and volatility, respectively. Consequently, the model might (as the aforementioned studies emphasise) describe empirical time series better than the geometric Brownian motion assumed in the Black-Scholes model. Yet, such a generalized market model is analytically less tractable, i.e., closed-form solutions - especially for path dependent payoffs like barrier options - are difficult to obtain.

The aim of this paper is to price barrier options in a Markov-switching extension of the Black-Scholes model. To achieve a fast and unbiased Monte-Carlo simulation, the asset-value process is only simulated at state changes of the Markov chain. Then, barrier option prices are evaluated conditional on this information. Closest to this idea, known as *Conditional Monte-Carlo*, see, e.g., [6], is the algorithm introduced in [19]. The authors of this paper use related techniques for the pricing of barrier options in a jump-diffusion environment. Besides adapting the algorithm to the present situation, variance reduction techniques are used to further improve the algorithm's efficiency.

The paper is organized as follows: Section 2 introduces the Markov-switching model and barrier options. In Section 3, an efficient pricing algorithm is introduced; antithetic and control variates are further proposed to reduce the standard error of the algorithm. In Section 4, this algorithm

is compared to a brute-force Monte-Carlo simulation working on a discrete grid. Section 5 compares these prices to Black-Scholes barrier option prices using the EuroStoxx 50 as underlying.

2. Model description

A well written introduction to Markov-switching models is to be found in [16]. In the present framework, the asset-price process $S = \{S_t\}_{t \geq 0}$, on the filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$, is assumed to follow the stochastic differential equation (sde)

$$dS_t = S_t(\mu_{Z_t}dt + \sigma_{Z_t}dW_t), \quad S_0 > 0, \tag{1}$$

where $Z = \{Z_t\}_{t \geq 0}$ is a time-homogenous Markov chain and $W = \{W_t\}_{t \geq 0}$ is an independent Brownian motion. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is generated by the pair (W, Z), i.e. $\mathcal{F}_t = \sigma\{W_s, Z_s : 0 \leq s \leq t\}$. The price of the risk-free asset $B = \{B_t\}_{t \geq 0}$ is given by $B_t = \exp(rt)$. On a high level, the model might be seen as a special case of the framework presented in [11].

In a two-state world, i.e. $Z_t \in \{1, 2\}$ for all $t \geq 0$, stock returns are described by two sets of parameters: (μ_1, σ_1) and (μ_2, σ_2), respectively. These states might be interpreted as being a crisis and a regular state. One advantage of this simple extension of the Black-Scholes model (where $Z_t \in \{1\}$ for all $t \geq 0$) is the fact that even a two-state model can capture empirically observed properties of asset returns like volatility clustering or heavy tails, see [24]. The time between two state changes of Z is an exponential random variable, i.e. a random variable with cdf $F(x) = 1 - \exp(-\lambda x)$, whose intensity parameter λ depends on the current state. In a two-state model, a 2×2 matrix Q contains on the diagonal the negative of the intensities, i.e. $Q(i, i) = -\lambda_{Z_i}$, $i = 1, 2$. The off-diagonal entries are $Q(i, j) = \lambda_{Z_i}$, $i \neq j$. The model is fully determined if an initial state (or, more general, an initial distribution $\pi^T := (\mathbb{Q}(Z_0 = 1), \mathbb{Q}(Z_0 = 2))$ on the states) is defined. The existence of an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ is shown in, e.g., [10, Prop. 3.3 and Cor. 3.4] and [12, Prop. 3.1], implying that the Markov-switching model with independent asset-price process S and Markov chain Z is free of arbitrage. However, due to the additional uncertainty of having different regimes, the equivalent martingale measure $\mathbb{Q}$ might not be unique. Hence, Arrow-Debreu securities (related to the cost of switching) are used in [14] to complete the market. Another common approach to deal with this

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issue is to simply calibrate $\mathbb{Q}$ to plain vanilla options. This is done in Section 5 as part of a small case study. Note that the pricing algorithm of the next section can be applied irrespectively of the philosophy how $\mathbb{Q}$ is chosen.

The payoff of a barrier option depends on whether or not the stock price S reaches a specified level B during the life of the option. We exemplarily consider a down-and-out call option (DOC). At maturity, this option pays $(S_T - H)^+$, the same amount as a European call option with strike H. However, the option immediately becomes worthless whenever the stock-price process hits the barrier B. The time t value of the DOC option is calculated by

$$V_{DOC}(t) = e^{-r(T-t)} \, \mathbb{E}_{\mathbb{Q}} \left[(S_T - H)^+ 1_{\{\tau > T\}} | \mathcal{F}_t \right], \tag{2}$$

where $1_{\{.\}}$ is the indicator function and τ the first-passage time defined as

$$\tau := \inf\{t > 0 : S_t \le B\}. \tag{3}$$

In the Black-Scholes model, the price $V_{DOC}^{BS}(t)$ is known in closed form. The corresponding formula was obtained by [20]; it is recalled below for the reader's convenience:

$$V_{DOC}^{BS}(t) = S_t \left(\Phi(d_1) - \left(\frac{B}{S_t} \right)^{\frac{2r}{\sigma^2}+1} \Phi(d_2) \right)$$

$$- He^{-r(T-t)} \left(\Phi\left(d_1 - \sigma\sqrt{T-t}\right) - \left(\frac{B}{S_t} \right)^{\frac{2r}{\sigma^2}-1} \Phi\left(d_2 - \sigma\sqrt{T-t}\right) \right), \tag{4}$$

$$d_1 = \frac{\ln\left(\frac{S_t}{H}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \; d_2 = \frac{\ln\left(\frac{B^2}{S_t H}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

with r being the risk-free interest rate and $\Phi(\cdot)$ denoting the cumulative distribution function of a standard-normal distribution. Unfortunately, a similar formula is not known for a Markov-switching extension, so that the valuation has to be done numerically. To overcome this problem, an efficient Monte-Carlo simulation is introduced in the next section.

3. Efficient Monte-Carlo simulation

Basic algorithm

One challenge in the numerical valuation of barrier options (via a Monte-Carlo simulation) is a possible discretization error. The option might knock

out continuously, but, in a typical simulation, the asset process is simulated on a discrete grid. In [21], it is shown that even $5\,000$ time intervals can lead to unstable barrier option prices. The error that occurs by discretization is calculated in, e.g., [7]. However, it is possible to completely avoid a discretization error by using a Brownian bridge technique. With start- (S_{t_i}) and endpoint $(S_{t_{i+1}})$ of a geometric Brownian motion given, the probability of the Brownian bridge (connecting these points) crossing the threshold level B in $[t_i, t_{i+1}]$ can be calculated. The probabilities are given by, see, e.g., [19], for $\min\{S_{t_i}, S_{t_{i+1}}\} > B$:

$$
\begin{aligned}
\wp_i :=& \mathbb{P}\Big(\inf_{t_i \leq s \leq t_{i+1}} \{S_s\} \leq B \Big| S_{t_i} = s_{t_i},\ S_{t_{i+1}} = s_{t_{i+1}} \Big) \\
=& \exp\left(-\frac{2\ln\left(\frac{s_{t_i}}{B}\right)\ln\left(\frac{s_{t_{i+1}}}{B}\right)}{(t_{i+1} - t_i)\sigma_{Z_{t_i}}^2} \right),
\end{aligned}
\tag{5}
$$

Z remaining constant over $[t_i, t_{i+1}]$.

Equation (5) is useful for the present framework, since between any two state changes the underlying is a standard geometric Brownian motion with the respective parameters of the current state. Returning to the model of Section 2, Algorithm 1 provides a numerical procedure for valuing barrier options in a Markov-switching model (without loss of generality (w.l.o.g.) having $M = 2$ states, which is easily extended to $M > 2$ states). Using the properties of a Markov chain, this algorithm simulates the time to the next state change $\ell_i := t_{i+1} - t_i$ by a (conditional on the current state) exponentially distributed random variable $\{\ell_i | Z_{t_i} = s\} \sim \mathrm{Exp}(-Q(s,s))$. Then, the underlying is simulated at t_{i+1}. The (conditional) probability of touching the barrier level B in the interval $[t_i, t_{i+1}]$ can be calculated from Equation (5) and is abbreviated by $\wp_i$. The option valuation is described in Algorithm 1.

Algorithm 1 (Barrier option pricing in a Markov-switching model)
This algorithm estimates the value of a knock-out barrier option, i.e. the expectation in Equation (2), in a two-state Markov-switching model by means of a Monte-Carlo simulation. It is w.l.o.g. assumed that $t = 0$.

  (1) *Repeat Steps (2)-(7) K times, then proceed with Step (8).*

  (2) *Define the initial state Z_0 by drawing a Bernoulli$(\mathbb{Q}(Z_0 = 1))$-distributed random variable. If the initial state is known, simply set Z_0 accordingly.*

(3) *Simulate the vector of state changes $0 < t_1 < t_2 < \ldots < t_{M+1} = T$ by drawing independent exponentially distributed random variables $\ell_i = t_{i+1} - t_i$, where $\ell_i \sim Exp(-Q(Z_{t_i}, Z_{t_i}))$ with $Q(Z_{t_i}, Z_{t_i})$ being the respective element of the intensity matrix Q. This is repeated until $\min(t_i, T) = T = t_{M+1}$. The number of state changes within $[0, T]$ is a random variable denoted M, its realization within the current run is denoted m.*

(4) *Draw a vector of independent standard normal random variables $x_0, x_1, \ldots, x_m$, independent of $\{\ell_i\}_{i=0,1,\ldots,m}$.*

(5) *Simulate asset values S_t^k, indexed by the current simulation run $k \in \{1, 2, \ldots, K\}$, at state changes $\{t_i\}_{i=1,\ldots,m}$ and maturity $T = t_{m+1}$ by $\ln S_0^k = \ln(S_0)$,*

$$\ln S_{t_{i+1}}^k = \ln S_{t_i}^k + \left(\left(r - \frac{\sigma_{Z_{t_i}}^2}{2} \right) \ell_i + x_i \sigma_{Z_{t_i}} \sqrt{\ell_i} \right), \quad i \in \{0, \ldots, m\}.$$
(6)

(6) *Check if $S_{t_i}^k > B$ for all t_i. If so, continue with Step (7). Otherwise, set $PS(k) = 0$ and skip Step (7). The option has knocked out in the latter case.*

(7) *Calculate the probability that the barrier option is not knocked out until T in run k. For this, use Equation (5) and $PS(k) = \prod_{i=0}^m (1 - \wp_i)$, $k \in \{1, 2, \ldots, K\}$. Then return to Step (2).*

(8) *Estimate $V_{DOC}(0)$ by*

$$\hat{V}_{DOC}(0) \quad := \quad \frac{e^{-rT}}{K} \sum_{k=1}^K PS(k) \max \left(S_T^k - H, 0 \right).$$

It is shown in Theorem 3.1 that Algorithm 1 returns unbiased option prices and converges almost surely (a.s.) to the price of a barrier option in a Markov-switching model.

Theorem 3.1 (Unbiasedness and a.s. convergence)
Algorithm 1 generates unbiased option prices and converges almost surely to the price of a barrier option in a Markov-switching model, i.e.

$$\frac{e^{-rT}}{K}\sum_{k=1}^{K} PS(k)\max\left(S_T^k - H, 0\right) \xrightarrow{(K\to\infty)} V_{DOC}(0) \ (a.s.).$$

Proof
Using the tower property of conditional expectation, the barrier option price $V_{DOC}(0)$ in Equation (4) can be expressed as the expectation of a conditional expectation. The inner conditional expectation is taken with respect to: a) the number of state changes M, b) the location of the state changes $0 < t_1 < \ldots < t_M < T$, c) the current state on this random grid $Z_{t_0}, \ldots, Z_{t_M}$, and d) the initial stock price as well as log-returns of the asset process on this random grid, i.e. $x_{t_{i+1}} = \ln(S_{t_{i+1}}/S_{t_i})$, $i \in \{0, \ldots, M\}$. Combined:

$$\mathcal{F}^* = \sigma\{M, 0 < t_1 < \ldots < t_M < T; Z_{t_0}, \ldots, Z_{t_M}; S_0, x_{t_1}, \ldots, x_{t_M}, x_T\}.$$

Integrating out the random variables in $\mathcal{F}^*$ and taking conditional expectations yields

$$V_{DOC}(0) = e^{-rT}\mathbb{E}\left[\mathbb{E}\left[1_{\{\tau>T\}}(S_T - H)^+|\mathcal{F}^*\right]\right]$$

$$= e^{-rT}\sum_{m=0}^{\infty}\mathbb{Q}(M=m)\left(\int\limits_{\substack{(t_1,\ldots,t_m)\\\in(0,T)^m}}\int\limits_{\substack{(x_{t_1},\ldots,x_{t_m},x_T)\\\in(-\infty,\infty)^{m+1}}}\mathbb{E}[1_{\{\tau>T\}}(S_T-H)^+|\mathcal{F}^*]\right.$$

$$\left.\prod_{j=0}^{m}\varphi(x_{t_{j+1}}; r(t_{j+1}-t_j), \sigma_{Z_{t_j}}^2(t_{j+1}-t_j))\ dx_{t_{j+1}}\ d\mathcal{G}_m(t_1,\ldots,t_m)\right),$$

where $\varphi(x; \mu, \sigma^2)$ denotes the probability density function of a normal distribution with mean μ and variance σ^2 and $\mathcal{G}_m$ is the (conditional) distribution of the location of state changes (given that $M = m$, i.e. the state has changed $M = m$ times). The inner conditional expectation can be computed explicitly, i.e.

$$\mathbb{E}[1_{\{\tau>T\}}(S_T - H)^+|\mathcal{F}^*] = 1_{\{I\neq0\}}\ (S_T^k - H)\ \prod_{i=0}^{I}(1 - \wp_i)$$

$$= PS(k)\ \max(S_T^k - H, 0),$$

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where $I := \min\{i \in \{0, \ldots, m+1\} : S_{t_i} \leq B\}$, $\min(\emptyset) := 0$ denotes the index of the first barrier crossing during one of the state changes $\{t_0, \ldots, t_m, t_{m+1} = T\}$. The probability of defaulting within the interval (t_i, t_{i+1}) is given by $\wp_i$. In Steps (2)-(7) of Algorithm 1, in each simulation run $k \in \{1, \ldots, K\}$, the value

$$Z_k = PS(k) \ \max(S_T^k - H, 0)$$

is generated. This is achieved by simulating all random quantities defining $\mathcal{F}^*$. Then, the pricing formula is evaluated (without discretization bias) conditional on this information. Note that specific values for $\mathbb{Q}(M = m)$ and $\mathcal{G}_m$ are not required, as long as one can simulate the required random variables without bias (which is simple). All simulation runs being independent, the generated realizations Z_k are independent and identically distributed (i.i.d.).

Additionally, using the above results, it holds that

$$\mathbb{E}[e^{-rT}Z_k] = \mathbb{E}\left[\mathbb{E}\left[e^{-rT} \ PS(k) \ (S_T^k - H, 0)^+ | \mathcal{F}^*\right]\right] = V_{DOC}(0), \ \forall k,$$
$$Var(Z_k) \leq \mathbb{E}[Z_k^2] \leq \mathbb{E}[\max(S_T^k, H)^2] \leq H^2 + \mathbb{E}[(S_T^k)^2] < \infty.$$

The stated convergence is then implied by the Law of Large Numbers. □

Antithetic variates

One possibility to decrease the variance of $\hat{V}_{DOC}(0)$ in Algorithm 1 is to use antithetic variates. Step (5) is then replaced by Step ($\tilde{5}$). The basic idea when using this technique is to exploit the symmetry of the standard normal law around the y-axis. For each realization x_i, the antithetic variable $-x_i$ is considered. Thus, for each path (corrsponding to $h = 0$ in Equation (7)), an artificial shadow path (corrsponding to $h = 1$ in Equation (7)) of the current stock price simulation is generated. The option payout of the path and its shadow path are averaged in Step (8) and considered as one sample. For additional background on this technique, see, e.g., [17, 6].

($\tilde{5}$) *Simulate asset values S_t^k, indexed by the current simulation run $k \in \{1, 2, \ldots, K\}$, at state changes $\{t_i\}_{i=1,\ldots,m}$ and maturity $T = t_{m+1}$ by $\ln S_0^k = \ln(S_0)$, $h \in \{0, 1\}$, $i \in \{0, \ldots, m\}$,*

$$\ln S_{t_{i+1}}^{k,h} = \ln S_{t_i}^{k,h} + \left(\left(r - \frac{\sigma_{Z_{t_i}}^2}{2}\right)\ell_i + (-1)^h x_i \sigma_{Z_{t_i}}\sqrt{\ell_i}\right). \tag{7}$$

Control variates

A second approach to decrease the variance of $\hat{V}_{DOC}(0)$ is the use of call option prices as control variates, see, e.g., [6]. Call options in a Markov-switching framework can be calculated efficiently using the fast Fourier pricing method as proposed in [9]. The characteristic function of a Markov-switching model is given by, see, e.g., [8, 12], for a more general setting see [11], where $\exp(\cdot)$ denotes the matrix exponential function and $\langle\cdot\rangle$ the scalar product as introduced, e.g., in [3],

$$\phi_t(u) := \mathbb{E}\left[e^{iuS_t}\right]$$

$$= \left\langle \exp\left(Qt + \begin{pmatrix} iu\left(r - \frac{1}{2}\sigma_1^2\right) - \frac{1}{2}\sigma_1^2 u^2 & 0 \\ 0 & iu\left(r - \frac{1}{2}\sigma_2^2\right) - \frac{1}{2}\sigma_2^2 u^2 \end{pmatrix} t\right)\pi, 1 \right\rangle.$$

Using this characteristic function, the price of a call option with strike H and maturity T is recovered as

$$\hat{V}_{Call}(0) \;=\; \frac{e^{-\alpha \ln(H)}}{\pi} \int_0^\infty e^{-iv\ln(H)} \frac{e^{-rT}\phi_T(v - (1+\alpha)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} dv, \quad (8)$$

with α being a constant from the interval $[1, 2]$. Step (8) of Algorithm 1 can now be modified to Step $(\tilde{8})$:

$(\tilde{8})$. *Estimate $V_{DOC}(0)$ by*

$$\hat{V}_{DOC}(0) \;:=\; \frac{1}{K}\sum_{k=1}^K \left(e^{-rT}PS(k)\max(S_T^k - H, 0) + \right.$$

$$\left. + \left(\hat{V}_{Call}(0) - e^{-rT}\max(S_T^k - H, 0)\right)\right).$$

4. Advantages of the Brownian bridge algorithm

Comparison to a simulation using a grid

In this section, Algorithm 1 is compared to an estimation of the barrier option price using a simulation of asset-value trajectories on a discrete grid with mesh $\Delta = T/\#steps$. To implement this, Step (3) of Algorithm 1 must be modified to $t_i = iT/\#steps$, with $i \in \{0, 1, \ldots, \#steps\}$. In Step (7), $PS(k)$ is set to one (if no default is observed on the grid in Step (6)), as default is only considered on the discrete grid.

Table 1 compares the two algorithms regarding estimated option value and computation time. A daily grid, i.e. $\#steps = 252$ trading days per year and $T = 1$ is chosen. The implementation is done using MATLAB$^{\copyright}$, version 2009a. The simulation is run on a 3.67 GHz computer with $100\,000$ simulation runs.

To illustrate the results, six scenarios $\theta_j = (-Q(1,1), -Q(2,2), \sigma_1, \sigma_2, B, H)$, $j \in \{1, \ldots, 6\}$, are specified. The stock value is supposed to start at $S_0 = 1$ with initial state $Z_0 = 1$ in each case. The first three scenarios differ by the barrier level. In θ_1, the barrier is far from the current level. In θ_2 the barrier is closer to the current level, and in θ_3 the barrier is only slightly below the current level:

$$\theta_1 = (0.8, 0.6, 0.15, 0.25, 0.6, 0.6),$$
$$\theta_2 = (0.8, 0.6, 0.15, 0.25, 0.8, 0.8),$$
$$\theta_3 = (0.8, 0.6, 0.15, 0.25, 0.9, 0.9).$$

The remaining scenarios are set to examine the effect of different frequencies of state changes. In θ_4, the regime changes only rarely. In θ_5, it changes several times a year, and in θ_6 very frequently:

$$\theta_4 = (0.2, 0.1, 0.1, 0.25, 0.8, 0.8),$$
$$\theta_5 = (1.0, 0.6, 0.1, 0.25, 0.8, 0.8),$$
$$\theta_6 = (3.0, 2.0, 0.1, 0.25, 0.8, 0.8).$$

Table 1: Comparison of the two Monte-Carlo techniques based on $100\,000$ simulation runs and six scenarios. Asymptotic 95% confidence interval and computation time (in seconds) are given.

	Brownian bridge algorithm			Monte-Carlo on a grid		
	$\hat{V}_{DOC}(0)$	time	95% confid. int.	$\hat{V}_{DOC}(0)$	time	95% confid. int.
θ_1	0.4177	0.57s	$\hat{V}_{DOC}(0) \pm 0.0011$	0.4179	6.05s	$\hat{V}_{DOC}(0) \pm 0.0011$
θ_2	0.2217	0.55s	$\hat{V}_{DOC}(0) \pm 0.0011$	0.2239	5.67s	$\hat{V}_{DOC}(0) \pm 0.0011$
θ_3	0.1186	0.53s	$\hat{V}_{DOC}(0) \pm 0.0009$	0.1229	4.50s	$\hat{V}_{DOC}(0) \pm 0.0010$
θ_4	0.2232	0.52s	$\hat{V}_{DOC}(0) \pm 0.0007$	0.2240	6.04s	$\hat{V}_{DOC}(0) \pm 0.0008$
θ_5	0.2233	0.54s	$\hat{V}_{DOC}(0) \pm 0.0010$	0.2238	5.85s	$\hat{V}_{DOC}(0) \pm 0.0010$
θ_6	0.2225	0.60s	$\hat{V}_{DOC}(0) \pm 0.0011$	0.2233	5.72s	$\hat{V}_{DOC}(0) \pm 0.0011$

Table 1 shows that the Brownian bridge algorithm is, for the chosen examples, about eight to ten times faster. Moreover, Algorithm 1 returns unbiased option prices, whereas the discrete grid of the brute-force algorithm does not allow for continuous knock-out. Thus, the brute-force simulation underestimates the knock-out probability, the consequence is too large option values. This effect is especially apparent in scenario θ_3, a scenario with high knock-out probability. The standard errors of the two approaches are about the same. Considering different state change frequencies, i.e. (θ_4, θ_5, θ_6), it appears as if the standard error is slightly increasing in the frequency of state changes. However, an analysis of several other parameter sets shows that the average number of state changes does not significantly affect the width of the confidence intervals.

Comparison of the variance reduction techniques

Table 2 compares the basic Algorithm 1 to: a) antithetic variates, b) control variates, and c) antithetic and control variates. The criterion used is the width of the 95% asymptotic confidence interval of the option price.

Table 2: Comparison of the different variance reduction techniques based on 100 000 simulation runs. Asymptotic 95% confidence intervals are given.

	Algorithm 1	Antithetic var.	Control var.	Both ext.
θ_1	0.4177 ±5.1e-04	0.4176 ±1.6e-04	0.4177 ±7.6e-06	0.4177 ±5.6e-06
θ_2	0.2217 ±7.2e-04	0.2221 ±2.9e-04	0.2223 ±9.2e-05	0.2220 ±7.2e-05
θ_3	0.1186 ±9.3e-04	0.1188 ±4.3e-04	0.1186 ±2.3e-04	0.1187 ±1.7e-04
θ_4	0.2232 ±7.4e-04	0.2233 ±1.4e-04	0.2233 ±4.2e-05	0.2233 ±3.6e-05
θ_5	0.2233 ±9.8e-04	0.2228 ±2.6e-04	0.2224 ±7.5e-05	0.2226 ±6.1e-05
θ_6	0.2225 ±1.2e-03	0.2221 ±3.2e-04	0.2216 ±1.0e-04	0.2225 ±8.3e-05

It can be observed that especially the use of control variates leads to a significant reduction of the total variance, and, hence, to smaller confidence intervals. The computation time is about the same for all algorithms.

5. Case study: EuroStoxx 50

This section applies the Brownian bridge algorithm to the pricing of barrier options with the EuroStoxx 50 as underlying. The procedure is based on reference [1]. For the calibration of the two-state Markov-chain model, 42 plain-vanilla call-option prices (with homogenous maturity 06-17-2011 but

different strike levels) are used. These are obtained from Reuters on 05-07-2010. The riskless interest rate is approximated by the 1-year Euribor, so $r = 1.25\%$.

The calibration involves minimizing the squared errors between observed call option prices and model prices over the parameter space. This non-convex optimization requires robustness checks to guarantee a global optimum. The calibrated Markov-switching parameters are as follows:

$$Q = \begin{pmatrix} -1.0251 & +0.6039 \\ +1.0251 & -0.6039 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 25.18\% \\ 14.35\% \end{pmatrix}, \pi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The diagonal entries of the transition matrix Q determine the frequency of state changes. If the current state is $Z_0 = 1$ ($Z_0 = 2$), the yearly ($\triangle t = 1$) probability of at least one state change is $1 - \exp(Q(1,1)\triangle t) \approx 64.1\%$ ($1 - \exp(Q(2,2)\triangle t) \approx 45.3\%$), respectively. In the Black-Scholes market, the corresponding volatility parameter is calibrated as $\sigma = 19.29\%$. The respective Black-Scholes prices are computed using the standard Black-Scholes call option formula, see [5]. The resulting average calibration error in the Black-Scholes model is about three times the error in a Markov-switching model. It is now possible to compare option prices in the calibrated Markov-switching model to option prices in the Black-Scholes model, see Figure 1. For this, choose $T = 1$, $S_0 = 2583.93$, different barrier levels B, and $H = B$. For the Markov-switching model, Algorithm 1 is used to compute option prices.

Figure 1 exhibits the resulting option prices for different barrier levels, given in percent of the initial stock price S_0. Observe that barrier option prices in a Markov-switching model are slightly higher compared to their Black-Scholes counterpart. Option prices in both models are close to each other for low barrier levels. In contrast, for high barrier levels, differences of more than 10% occur. This can be explained as follows: If default risk is negligible, the barrier option prices are close to $S_0 - He^{-rT}$, independent of the underlying stock price model. For high barrier levels, however, the regime switching model returns higher option prices than the Black-Scholes model (with the calibrated parameters of the present example).

6. Conclusion

An efficient algorithm for the pricing of barrier options in a Markov-switching framework was presented. This algorithm only simulates the under-
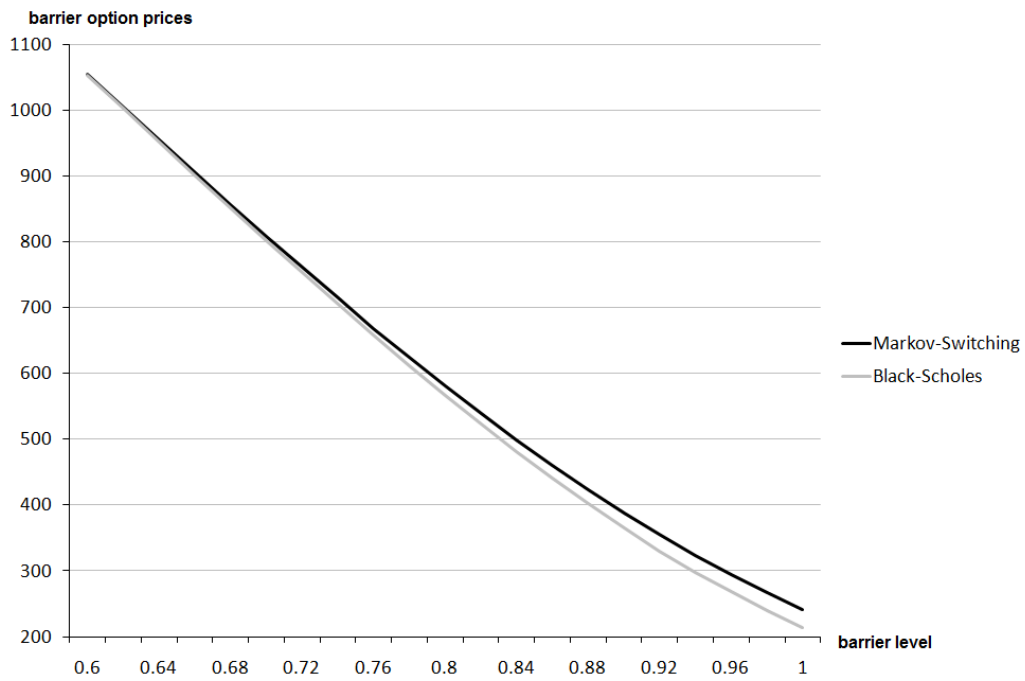
12

Figure 1: Barrier option prices in a Markov-switching framework compared to prices obtained in a Black-Scholes model. The parameters are calibrated to plain-vanilla calls on the EuroStoxx 50, as explained in Section 5.

lying at state changes and maturity, Brownian bridges connect these points. Antithetic and control variates are introduced to further reduce the standard error of the estimation. The presented method leads to unbiased option prices and is significantly faster than the standard simulation on a discrete grid. A case study, using the EuroStoxx 50 as underlying, shows that prices in a Markov-switching model differ from Black-Scholes option prices, even though both models have been calibrated to the same set of plain-vanilla options.

Another field of application for the presented methodology might be the derivation of default probability and bond prices in a structural-default model when the firm-value process follows a Markov-switching diffusion.

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