A note on first-passage times of continuously time-changed Brownian motion[1]

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Abstract

The probability of a Brownian motion with drift to remain between two constant barriers (for some period of time) is known explicitly. In mathematical finance, this and related results are required, e.g., for the pricing of single- and double-barrier options in a Black-Scholes framework. One popular possibility to generalize the Black-Scholes model is to introduce a stochastic time-scale. This equips the modelled returns with desirable stylized facts such as volatility clusters and jumps. For continuous time transformations, independent of the Brownian motion, we show that analytical results for the double-barrier problem can be obtained via the Laplace transform of the time-change. The result is a very efficient power series representation for the resulting exit probabilities. We discuss possible specifications of the time change based on integrated intensities of shot-noise type and of basic-affine process type.

Keywords: Double-barrier problem, first-exit time, first-passage time, time change, time-changed Brownian motion, Fourier pricing, barrier option.

1 Introduction

One- and two-sided exit problems for stochastic processes are classical problems with various applications in mathematical finance, see e.g. Darling and Siegert [1953], Black and Cox [1976], Kunitomo and Ikeda [1992], Geman and Yor [1996], Bertoin [1998], Lin [1999],

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Kyprianou [2000], Pelsser [2000], Rogers [2000], Sepp [2004], and Hurd [2009]. For instance, first-passage time problems appear when barrier options are to be priced or when default probabilities in structural models are to be computed. For most stochastic processes, however, a closed-form solution to the various problem specifications is unknown; an exception is the case of Brownian motion with drift, which is sufficient to price single- and double-barrier options in the seminal Black-Scholes model. This model, however, is often criticized for its simplicity. Consequently, various extensions have been proposed. One popular way of generalizing the Black-Scholes model is to replace the calendar time by some suitable increasing stochastic process[2]. Related techniques can also be applied to credit risk models, see, e.g., Kammer [2007], Hurd [2009]. In the present model, we show how several results can be generalized to the situation of a continuous[3] time shift. Required is the Laplace-transform of the time-change, which is known for several popular specifications. We explicitly discuss models where the time-change is constructed as an integrated intensity. This includes, e.g., models where the intensity is a basic-affine process or a shot-noise process. All results are verified using Monte-Carlo techniques. Also note that we present an alternative approach for some single barrier problems that are currently solved via Fourier inversion.

2 Notation and problem formulation

Throughout we work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supporting all required stochastic processes. We consider a Brownian motion $B = \{B_t\}_{t\geq 0}$ with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and initial value $B_0 = 0$, satisfying the stochastic differential equation (sde) $dB_t = \mu dt + \sigma dW_t$, where $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion. Assume that there are two constant barriers $b < 0 < a$ and define

$$T_{ab} := \inf\{t \geq 0 \;:\; B_t \notin (b,a)\}\,, \tag{1}$$

as well as $T_{ab}^{+} := \{T_{ab} \mid T_{ab} = T_{a,-\infty}\}$ and $T_{ab}^{-} := \{T_{ab} \mid T_{ab} = T_{\infty,b}\}$. If the lower barrier b is hit first, the first-exit time is $T_{ab} = T_{ab}^{-}$; if the upper barrier a is hit first, the first-exit time is $T_{ab} = T_{ab}^{+}$.

[2]Clark [1973] gave the following motivation: *"the different evolutions of price series on different days is due to the fact that information is available to traders at a varying rate. On days when no new information is available, trading is slow, and the price process evolves slowly. On days when new information violates old expectations, trading is brisk, and the price process evolves much faster."*. Geman et al. [2000] show that a time change represents a measure of activity in the economy.

[3]To stress the limitations of this technique, we discuss why subordination with jumps must be treated differently.

Lemma 1 (Double exit problem for a Brownian motion with drift)
Consider a Brownian motion B_t with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$. Then

$$\mathbb{P}(T_{ab}^{+} \le T) = \frac{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - 1}{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - \exp\left\{-\frac{2\mu a}{\sigma^2}\right\}} + \exp\left(\frac{\mu a}{\sigma^2}\right) K_T^{\infty}(b), \tag{2}$$

$$\mathbb{P}(T_{ab}^{-} \le T) = \frac{1 - \exp\left\{-\frac{2\mu a}{\sigma^2}\right\}}{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - \exp\left\{-\frac{2\mu a}{\sigma^2}\right\}} - \exp\left(\frac{\mu b}{\sigma^2}\right) K_T^{\infty}(a), \tag{3}$$

$$\mathbb{P}(T_{ab} \le T) = 1 - \Big(\exp\left(\frac{\mu b}{\sigma^2}\right) K_T^{\infty}(a) - \exp\left(\frac{\mu a}{\sigma^2}\right) K_T^{\infty}(b)\Big), \tag{4}$$

where

$$K_T^N(k) := \frac{\sigma^2 \pi}{(a-b)^2} \sum_{n=1}^{N} \frac{n(-1)^{n+1}}{\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a-b)^2}} \ \exp\left(-\left(\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a-b)^2}\right) T\right) \ \sin\left(\frac{n\pi k}{a-b}\right).$$

Proof
Denote by $f_{ab}^{+}(t)$, $f_{ab}^{-}(t)$, and $f_{ab}(t)$ the corresponding first-passage time densities. Their Laplace transforms $\hat{f}_{ab}(\lambda) := \int_0^\infty \exp(-\lambda t) f_{ab}(t) dt$ are derived in Darling and Siegert [1953] as

$$\hat{f}_{ab}^{+}(\lambda) = \exp\left(\frac{\mu a}{\sigma^2}\right) \frac{\sinh\left(\frac{\sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2} b\right)}{\sinh\left(\frac{\sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}(b-a)\right)}, \quad \hat{f}_{ab}^{-}(\lambda) = -\exp\left(\frac{\mu b}{\sigma^2}\right) \frac{\sinh\left(\frac{\sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2} a\right)}{\sinh\left(\frac{\sqrt{\mu^2 + 2\sigma^2\lambda}}{\sigma^2}(b-a)\right)}.$$

From Laplace inversion tables, e.g. Oberhettinger and Badii [1973], p. 295, one obtains

$$\mathcal{L}^{-1}\left[f_{ab}^{+}\right](t) = -\frac{\sigma^2 \pi}{(a-b)^2} \sum_{n=1}^{\infty} n(-1)^{n+1} \exp\left(-\left(\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a-b)^2}\right) t\right) \sin\left(\frac{n\pi b}{a-b}\right) \exp\left(\frac{\mu a}{\sigma^2}\right).$$

Finally, $\mathbb{P}(T_{ab}^{+} \le T) = \int_0^T f_{ab}^{+}(t) dt$ can be obtained by integration. To obtain the given representation in Lemma 1, the identity $\sinh(kx) = \frac{2}{\pi} \sinh(k\pi) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 + k^2} \sin(nx)$ (see, e.g., Rottmann [2008], p. 126) has to be used. By setting $x = \pi b/(a-b)$, $k = \pm(a-b)\mu/(\pi\sigma^2)$, and using that $\sinh(-y) = -\sinh(y)$ for all $y \in \mathbb{R}$, we find

$$\begin{aligned} &\exp\left(\frac{\mu a}{\sigma^2}\right) \frac{\pi\sigma^2}{(a-b)^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a-b)^2}} \sin\left(\frac{n\pi b}{a-b}\right) \\ &= \exp\left(\frac{\mu a}{\sigma^2}\right) \frac{\sinh\left(\frac{b\mu}{\sigma^2}\right)}{\sinh\left(\frac{(a-b)\mu}{\sigma^2}\right)} = \frac{1 - \exp\left(-\frac{2\mu b}{\sigma^2}\right)}{\exp\left(-\frac{2\mu b}{\sigma^2}\right) - \exp\left(-\frac{2\mu a}{\sigma^2}\right)}. \end{aligned}$$

By symmetry, $\mathbb{P}(T_{ab}^{-} \le T)$ (respectively $\hat{f}_{ab}^{-}$) can be obtained from $\mathbb{P}(T_{ab}^{+} \le T)$ (respectively $\hat{f}_{ab}^{+}$) by replacing $a \mapsto -b$, $b \mapsto -a$, and $\mu \mapsto -\mu$. The expression for $\mathbb{P}(T_{ab} \le T) =$

$\mathbb{P}(T^+_{ab} \leq T) + \mathbb{P}(T^-_{ab} \leq T)$ *is given in, e.g., Darling and Siegert [1953], p. 633*[4] *and in Dominé [1996], p. 173.* □

Using Lemma 1, we now investigate the situation of a time-changed Brownian motion. To do so, let $\Lambda = \{\Lambda_t\}_{t\geq 0}$ be a (pathwise) continuous and increasing stochastic process with $\lim_{t\nearrow\infty} \Lambda_t = \infty$ $\mathbb{P}$-a.s. and $\Lambda_0 = 0$. This stochastic time-scale is used to time-change B, i.e. we consider the process $S_t := B_{\Lambda_t}$, for $t \geq 0$. The idea to approach exit problems for the process S is conceptually simple. Conditioned on Λ_T, we are back in a situation that was already solved. When we integrate out Λ_T to obtain unconditional probabilities, we observe that the required quantities can be interpreted as functions of the Laplace transform of Λ_T, for which we exploit the specific structure of (1). Fortunately, this Laplace transform is known for most popular specifications of Λ, see the examples presented below.

Theorem 2 (Double exit problem for a time-changed Brownian motion)
Consider a time-changed Brownian motion $S_t := B_{\Lambda_t}$ with a (pathwise) ***continuous*** *time-change Λ, independent of B. Denote the Laplace transform of Λ_T by $\vartheta_T(u) := \mathbb{E}[\exp(-u\Lambda_T)]$, $u \geq 0$. Then*

$$\mathbb{P}(T^+_{ab} \leq T) = \frac{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - 1}{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - \exp\left\{-\frac{2\mu a}{\sigma^2}\right\}} + \exp\left(\frac{\mu a}{\sigma^2}\right) K^{\infty}_{\Lambda_T}(b), \tag{5}$$

$$\mathbb{P}(T^-_{ab} \leq T) = \frac{1 - \exp\left\{-\frac{2\mu a}{\sigma^2}\right\}}{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - \exp\left\{-\frac{2\mu a}{\sigma^2}\right\}} - \exp\left(\frac{\mu b}{\sigma^2}\right) K^{\infty}_{\Lambda_T}(a), \tag{6}$$

$$\mathbb{P}(T_{ab} \leq T) = 1 - \left(\exp\left(\frac{\mu b}{\sigma^2}\right) K^{\infty}_{\Lambda_T}(a) - \exp\left(\frac{\mu a}{\sigma^2}\right) K^{\infty}_{\Lambda_T}(b)\right), \tag{7}$$

where

$$K^N_{\Lambda_T}(k) := \frac{\sigma^2\pi}{(a-b)^2} \sum_{n=1}^{N} \frac{n(-1)^{n+1}}{\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2\pi^2}{2(a-b)^2}} \;\; \vartheta_T\left(\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a-b)^2}\right) \;\; \sin\left(\frac{n\pi k}{a-b}\right).$$

Proof
The first-passage times of Brownian motion ($S_t = B_t$, i.e. $\Lambda_t \equiv t$) are given in Lemma

[4] Note that the expression in Darling and Siegert [1953], p. 633, contains two typos: π^2 has to be replaced by π and $(-1)^n$ by $(-1)^{n+1}$.

1. Then,

$$\mathbb{P}(T^+_{ab} \le T) = \mathbb{E}\Big[\mathbb{P}(T^+_{ab} \le \tilde{T} \mid \tilde{T} = \Lambda_T)\Big] = \frac{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - 1}{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - \exp\left\{-\frac{2\mu a}{\sigma^2}\right\}}$$

$$+ \frac{\exp\left(\frac{\mu a}{\sigma^2}\right)\sigma^2\pi}{(a-b)^2} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a-b)^2}} \mathbb{E}\left[\exp\left(-\left(\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a-b)^2}\right)\Lambda_T\right)\right] \sin\left(\frac{n\pi b}{a-b}\right)$$

$$= \frac{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - 1}{\exp\left\{-\frac{2\mu b}{\sigma^2}\right\} - \exp\left\{-\frac{2\mu a}{\sigma^2}\right\}} + \exp\left(\frac{\mu a}{\sigma^2}\right)\frac{\sigma^2\pi}{(a-b)^2} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}}$$

$$\cdot\, \vartheta_T\left(\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}\right) \sin\left(\frac{\pi n b}{a-b}\right).$$

Note that the first equality holds for continuous time changes only (see Remark 5). The expressions for $\mathbb{P}(T^-_{ab} \le T)$ and $\mathbb{P}(T_{ab} \le T)$ are obtained analogously. □

Example 3 (CIR process)
A popular possibility to include a continuous stochastic time change is an integrated CIR process *as introduced in, e.g., Duffie et al. [2000]. The CIR process is given via the sde*

$$d\lambda_t = \theta(\nu - \lambda_t)dt + \gamma\sqrt{\lambda_t}d\tilde{W}_t, \quad \lambda_0 > 0, \tag{8}$$

where θ, ν, and γ are non-negative constants, $\{\tilde{W}_t\}_{t\ge 0}$ a one-dimensional Brownian motion. The Feller condition, see Feller [1951], $2\theta\nu > \gamma^2$ guarantees that the process is almost surely positive. The Laplace transform of the integrated process $\Lambda_T := \int_0^T \lambda_s ds$ is given by, see e.g. Cox et al. [1985], Dufresne [2001]

$$\vartheta_T^{CIR}(u) := \mathbb{E}\Big[\exp\Big(-u\int_0^T \lambda_s ds\Big)\Big] \tag{9}$$

$$= \left[\frac{\exp(\theta T/2)}{\cosh(\varrho T/2) + \frac{\theta}{\varrho}\sinh(\varrho T/2)}\right]^{\frac{2\theta\nu}{\gamma^2}} \exp\left[-\frac{u\lambda_0}{\varrho}\frac{2\sinh(\varrho T/2)}{\cosh(\varrho T/2) + \frac{\theta}{\varrho}\sinh(\varrho T/2)}\right], \tag{10}$$

where $\varrho = \sqrt{\theta^2 + 2u\gamma^2}$. Furthermore

$$\mathbb{E}[\Lambda_T] = \frac{\lambda_0}{\theta} - \frac{\nu}{\theta} + \nu T + \exp(-\theta T)\Big(-\frac{\lambda_0}{\theta} + \frac{\nu}{\theta}\Big). \tag{11}$$

The case $\mu = -1/2$, $\sigma = 1$ is the special case of a Heston-type stochastic volatility model treated in Lipton [2001], p. 492ff. As a generalization of the CIR process, one can use basic affine processes, see, e.g., Duffie et al. [2000]. Those processes allow for an additional jump component of the intensity process. Note that the integrated intensity remains continuous in t.

Example 4 (Shot-noise process)
Consider a shot-noise process[5] *as in Cox and Isham [1980], Dassios and Jang [2003]*

$$\lambda_t = \lambda_0 \exp(-\delta t) + \sum_{s_i \le t} M_i \exp\big(-\delta(t-s_i)\big), \tag{12}$$

where $\lambda_0 > 0$ *is the initial intensity,* $\delta > 0$ *the exponential decay rate,* $\{s_i\}_{i=1}^{\infty}$ *are the jump times of a time-homogeneous Poisson process with intensity* $\rho > 0$*, and* $M_i \sim Exp(\zeta)$ *are the jump sizes. The Laplace transform of the integrated process* $\Lambda_T := \int_0^T \lambda_s ds$ *is given by*[6]

$$\begin{aligned}\vartheta_T^{sn}(u) &:= \mathbb{E}\Big[\exp\Big(-u\int_0^T \lambda_s ds\Big)|\lambda_0\Big]\\ &= \exp\left(-\frac{u\lambda_0}{\delta}\big(1-\exp(-\delta T)\big) - \rho T + \frac{\rho}{1+\frac{u}{\delta\zeta}}\Big(T + \frac{1}{\delta}\ln\Big(1+\frac{u}{\delta\zeta}\big(1-\exp(-\delta T)\big)\Big)\Big)\right).\end{aligned}$$

Furthermore

$$\mathbb{E}[\Lambda_T] = \left(\frac{\lambda_0}{\delta} - \frac{\rho}{\delta^2\zeta}\right)\big(1-\exp(-\delta T)\big) + \frac{\rho T}{\delta\zeta}. \tag{13}$$

Remark 5 (Restriction to continuous time-transformations)
Note that Theorem 2 does not hold for discontinuous time-transformations, e.g. for Lévy subordinators. If there is a jump in the time transformation Λ*, we do not observe the time-changed Brownian motion on the whole interval* $[0, \Lambda_T]$*. Thus, the true barrier hitting probabilities in the case of a discontinuous time transformation are always lower than the expressions given in Theorem 2. Hurd [2009] introduces a (slightly modified)* **first-passage time of second kind** *to account for this issue.*

3 Implementation

To implement the probabilities given in Lemma 1 and Theorem 2, the infinite sums $K_{\Lambda_T}^{\infty}$ have to be approximated by finite sums $K_{\Lambda_T}^{N}$. Given a predefined precision $\epsilon > 0$, Lemma 6 gives lower bounds for the number of required summands N. These are conveniently small.

Lemma 6 (Truncating $K_{\Lambda_T}^{\infty}$***)***
Define the (absolute) computation error of $\mathbb{P}(T_{ab} \le T)$ *by*

$$\epsilon := \left|\mathbb{P}(T_{ab}^S \le T) - \Big(1 - \Big(\exp\Big(\frac{\mu b}{\sigma^2}\Big)K_{\Lambda_T}^N(a) - \exp\Big(\frac{\mu a}{\sigma^2}\Big)K_{\Lambda_T}^N(b)\Big)\Big)\right|. \tag{14}$$

To obtain a given precision ϵ*, a lower bound for the summation index* $N \in \mathbb{N}$ *is*

[5] Note that the presented model is not the most general shot-noise process. It is without difficulty possible to, e.g., change the jump size distribution, see Dassios and Jang [2003].

[6] This Laplace transform can be obtained from Dassios and Jang [2003] by introducing exponential jump sizes $\hat{g}(u) = 1/(1 + u/\zeta)$ in Equation (2.10), p. 79.

a) in the case of Brownian motion

$$N \geq \sqrt{\max\left(1, -\frac{2(a-b)^2}{\pi^2\sigma^2 T}\left[\ln\left(\frac{\pi^3\sigma^2 T\epsilon}{4(a-b)^2}\right) - \frac{a\mu}{\sigma^2}\right]\right)}. \tag{15}$$

b) in the case of an integrated CIR process with parameters $(\theta, \nu, \gamma, \lambda_0)$

$$N \geq \max\left(1, \frac{(a-b)\sqrt{2\theta^2+4\gamma^2}}{\pi\sigma\lambda_0(1-\exp(-\sqrt{\theta^2+2\gamma^2}T))} \cdot \left(\frac{\mu a}{\sigma^2} + \frac{\theta^2\nu T}{\gamma^2} - \ln\left(\frac{\epsilon\pi^2\sigma\lambda_0(1-\exp(-\sqrt{\theta^2+2\gamma^2}T))}{4\sqrt{2}(a-b)\sqrt{\theta^2+2\gamma^2}}\right)\right)\right). \tag{16}$$

c) in the case of an integrated shot-noise process with parameters $(\delta, \zeta, \rho, \lambda_0)$

$$N \geq \sqrt{\max\left(1, -\frac{2\delta(a-b)^2}{\pi^2\sigma^2\lambda_0\big(1-\exp(-\delta T)\big)}\left[\ln\left(\frac{\pi^3\sigma^2\lambda_0\big(1-\exp(-\delta T)\big)\epsilon}{4\delta(a-b)^2}\right) - \frac{a\mu}{\sigma^2}\right]\right)}.$$

The same lower bounds hold for the (absolute) computation error of $\mathbb{P}(T^+_{ab} \leq T)$ and $\mathbb{P}(T^-_{ab} \leq T)$.

Proof
If $\vartheta_T(u) \leq J\exp(-Mu)$, where $J > 0$, $M > 0$ are constants, we get from Equation (14)

$$\begin{aligned}
\epsilon &= \left|\exp\left(\frac{\mu b}{\sigma^2}\right)\big(K^\infty_{\Lambda_T}(a) - K^N_{\Lambda_T}(a)\big) - \exp\left(\frac{\mu a}{\sigma^2}\right)\big(K^\infty_{\Lambda_T}(b) - K^N_{\Lambda_T}(b)\big)\right| \\
&\overset{(*)}{\leq} \left|\left(\exp\left(\frac{\mu b}{\sigma^2}\right) + \exp\left(\frac{\mu a}{\sigma^2}\right)\right)\frac{\sigma^2\pi}{(a-b)^2}\sum_{n=N+1}^{\infty}\frac{n}{\frac{\sigma^2 n^2\pi^2}{2(a-b)^2}}\,\vartheta_T\left(\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2\pi^2}{2(a-b)^2}\right)\right| \\
&\leq \left|\exp\left(\frac{\mu a}{\sigma^2}\right)\frac{4}{\pi}\sum_{n=N+1}^{\infty}\frac{1}{n}\,\vartheta_T\left(\frac{\sigma^2 n^2\pi^2}{2(a-b)^2}\right)\right| \\
&\leq \left|J\exp\left(\frac{\mu a}{\sigma^2}\right)\frac{4}{\pi}\int_N^\infty \frac{1}{n}\,\exp\left(-M\frac{\pi^2\sigma^2}{2(a-b)^2}n^2\right)dn\right| \\
&\leq \left|J\exp\left(\frac{\mu a}{\sigma^2}\right)\frac{4(a-b)^2}{\pi^3\sigma^2 M}\exp\left(-M\frac{\pi^2\sigma^2}{2(a-b)^2}N^2\right)\right|.
\end{aligned}$$

Thus, we get

$$N \geq \sqrt{\max\left(1, -\frac{2(a-b)^2}{\pi^2\sigma^2 M}\left[\ln\left(\frac{\pi^3\sigma^2 M\epsilon}{4(a-b)^2 J}\right) - \frac{a\mu}{\sigma^2}\right]\right)}. \tag{17}$$

In the Gaussian case ($\vartheta_T(u) = \exp(-uT)$), we set $M = T$, $J = 1$.

Since one can show by a lengthy calculation that $g(u) := \frac{\rho}{1+\frac{u}{\delta\zeta}}\Big(T + \frac{1}{\delta}\ln\big(1 + \frac{u}{\delta\zeta}\big(1 - \exp(-\delta T)\big)\big)\Big)$ *is decreasing for $u \in [0,\infty)$ with $g(0) = \rho T$, we get*

$$\vartheta_T^{sn}(u) \le \exp\Big(-\frac{\lambda_0}{\delta}\big(1 - \exp(-\delta T)\big)u\Big)$$

and thus $M = \lambda_0\big(1 - \exp(-\delta T)\big)/\delta$, $J = 1$ if the time change is an integrated shot-noise process.

If $\vartheta_T(u) \le J^* \exp(-M^*\sqrt{u})$, where $J^* > 0$, $M^* > 0$ are constants, we get analogously

$$\begin{aligned}\epsilon &\overset{(*)}{\le} \left| J^* \exp\Big(\frac{\mu a}{\sigma^2}\Big)\frac{4}{\pi}\int_N^\infty \frac{1}{n}\exp\left(-M^*\frac{\pi\sigma}{\sqrt{2}(a-b)}n\right)dn\right| \\ &\le \left| J^* \exp\Big(\frac{\mu a}{\sigma^2}\Big)\frac{4\sqrt{2}(a-b)}{\pi^2\sigma M^*}\exp\left(-M^*\frac{\pi\sigma}{\sqrt{2}(a-b)}N\right)\right|.\end{aligned}$$

Then,

$$N \ge \max\left(1, -\frac{\sqrt{2}(a-b)}{\pi\sigma M^*}\left[\ln\left(\frac{\pi^2\sigma M^*\epsilon}{4\sqrt{2}(a-b)J^*}\right) - \frac{a\mu}{\sigma^2}\right]\right). \tag{18}$$

If the time-change is an integrated CIR process, using that $\cosh(\varrho T/2) + \theta/\varrho\sinh(\varrho T/2) \ge \cosh(\varrho T/2) \ge 1$, $\exp(\varrho T/2) \ge \exp(-\varrho T/2)$, *and $\theta/\varrho \le 1$, we get:*

$$\begin{aligned}\vartheta_T^{CIR}(u) &= \left[\frac{\exp(\theta T/2)}{\cosh(\varrho T/2) + \frac{\theta}{\varrho}\sinh(\varrho T/2)}\right]^{\frac{2\theta\nu}{\gamma^2}} \exp\left[-u\frac{\lambda_0}{\varrho}\frac{2\sinh(\varrho T/2)}{\cosh(\varrho T/2) + \frac{\theta}{\varrho}\sinh(\varrho T/2)}\right] \\ &\le \exp\left(\frac{\theta^2\nu T}{\gamma^2}\right)\exp\left(-u\frac{\lambda_0}{\sqrt{\theta^2 + 2u\gamma^2}}\big(1 - \exp\big(-\sqrt{\theta^2 + 2u\gamma^2}T\big)\big)\right) \\ &\le \exp\left(\frac{\theta^2\nu T}{\gamma^2}\right)\exp\left(-\sqrt{u}\frac{\lambda_0}{\sqrt{\theta^2 + 2\gamma^2}}\big(1 - \exp\big(-\sqrt{\theta^2 + 2\gamma^2}T\big)\big)\right)\end{aligned}$$

where the last inequality holds for $u \ge 1$. Thus $M^* = \lambda_0\big(1 - \exp(-\sqrt{\theta^2 + 2\gamma^2}T)\big)/\sqrt{\theta^2 + 2\gamma^2}$, $J^* = \exp\big(\theta^2\nu T/\gamma^2\big)$.

The estimation $\overset{(*)}{\le}$ is an upper bound for the (absolute) computation error of $\mathbb{P}(T_{ab}^- \le T)$ and $\mathbb{P}(T_{ab}^+ \le T)$. Thus, the given lower bounds for N hold for those expressions, too. $\square$

4 Numerical examples

The parameters in this section are chosen such that $\mathbb{E}[\Lambda_T] = T$, i.e. on average our time change Λ_T increases as the calendar time. If we set $a = -b = 0.2$ and want to evaluate $\mathbb{P}(T_{ab} \leq T)$ for a precision ϵ = 1e-08, we obtain for reasonable parameters $(\mu, \sigma) = (0.1, 0.2)$, $(\theta, \nu, \gamma, \lambda_0) = (0.12, 1.00, 0.05, 1.00)$, and $(\delta, \zeta, \rho, \lambda_0) = (0.80, 0.80, 0.64, 1.00)$ lower bounds of only $N \leq 5$ for the Brownian motion and the shot-noise type time change. If the time change is an integrated CIR process, our error bound yields $N \approx 25$. Note, however, that the error bounds in Lemma 6 are rather coarse and the actual precision is supposedly much higher.

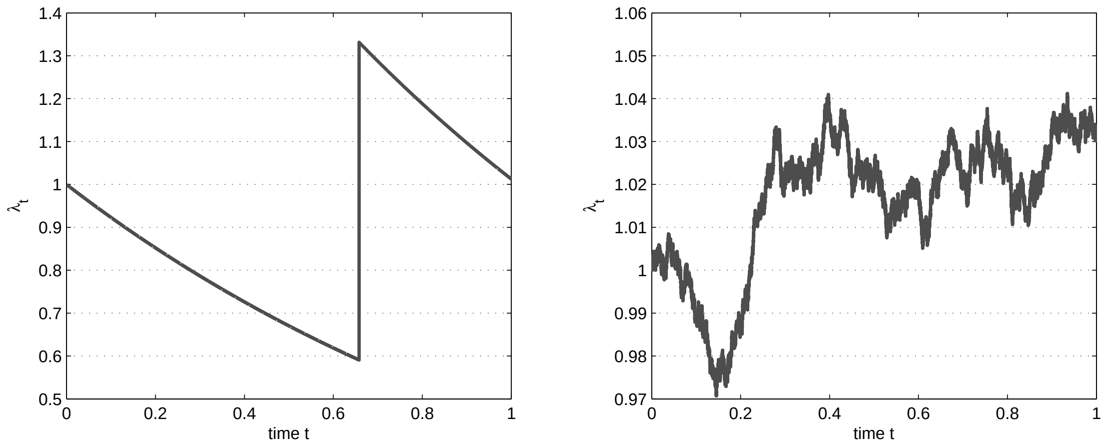

Figure 1 Sample paths of a shot noise process (left) and a CIR process (right). Whenever $\lambda_t > 1$, we approach faster than calendar time.

First-passage time probabilities for single barrier problems can be obtained by letting $a \to \infty$ (or $b \to -\infty$, respectively). If we choose a such that $\mathbb{P}(T^+_{ab} \leq T) \approx 0$, then $\mathbb{P}(T^-_{ab} \leq T) \approx \mathbb{P}(T_{\infty,b} < T)$. In the case of Brownian motion, it is a classical result (used in credit risk, see Black and Cox [1976]) that

$$\mathbb{P}(T_{\infty,b} \leq T) = \Phi\left(\frac{b - \mu T}{\sigma\sqrt{T}}\right) + \exp\left(\frac{2b\mu}{\sigma^2}\right)\Phi\left(\frac{b + \mu T}{\sigma\sqrt{T}}\right). \tag{19}$$

For time-changed Brownian motion, Hurd [2009] derived the following equation using Fourier inversion:

$$\begin{aligned}\mathbb{P}(T_{\infty,b} \leq T) &= \frac{1 + \exp(2\mu b/\sigma^2)}{2} \\ &+ \frac{1}{\pi}\mathcal{R}\left[\int_0^\infty \frac{\exp(2\mu b/\sigma^2)\cdot\exp(ixb) - \exp(-ixb)}{ix}\ \vartheta_T\big(-ix\mu T + x^2\sigma^2 T/2\big)\ dx\right],\end{aligned} \tag{20}$$

where $\mathcal{R}[x+iy] = x$ denotes the real part of the complex numer $x+iy$. Due to the fast convergence of our power series expansion, we get very fast and accurate results that provide a meaningful alternative to the Fourier inversion algorithm by Hurd [2009]. The power series expansion has got two advantages: 1) error bounds can easily be computed and 2) the computation is extremely fast due to the exponentially decaying error term. Numerical results are presented in Table 1. To obtain $\mathbb{P}(T_{ab}^{+} \leq T) < 1\text{e-}16$, we set $a = 3.0$.

Table 2 compares the power series technique to a Monte-Carlo simulation on a discrete grid. Due to the binary character (default vs. no default) of the first-exit time probabilities, one needs a surprisingly fine grid to minimize the discretization bias. Our results indicate that even $k = 1000$ discretization intervals per unit of time are not enough.

5 Conclusion

We showed how first-exit time problems with two constant barriers are solved analytically if the underlying process is a continuously time-changed Brownian motion. The resulting infinite power series converges very fast, for most parameter constellations $N \leq 10$ summands are enough to obtain a precision of $\epsilon = 1\text{e-}08$. Furthermore, we applied the result to single-barrier first-exit problems by setting the barriers appropriately. This provides a reasonable alternative to the Fourier inversion technique presented in Hurd [2009]. Due to the discretization bias, one should avoid simulating first-exit times on a grid.

Brownian motion in calendar time	True	Power series ($a = 3.0,\ N = 40$)		Fourier inversion	
	$\mathbb{P}(T_{\infty,b} \leq 1)$	$\mathbb{P}(T_{3,b} \leq 1)$	*error*	$\mathbb{P}(T_{\infty,b} \leq 1)$	*error*
$b = -0.80$	7.81e-06	7.81e-06	<1e-08	7.81e-06	2.8e-16
$b = -0.20$	0.1803	0.1803	<1e-08	0.1803	1.7e-15
$b = -0.10$	0.4619	0.4619	<1e-08	0.4619	8.9e-16
$b = -0.05$	0.6929	0.6929	<1e-08	0.6929	1.0e-15

Brownian motion with integrated CIR process as time change	True	Power series ($a = 3.0,\ N = 40$)		Fourier inversion	
	$\mathbb{P}(T_{\infty,b} \leq 1)$	$\mathbb{P}(T_{3,b} \leq 1)$	*error*	$\mathbb{P}(T_{\infty,b} \leq 1)$	*error*
$b = -0.80$	7.66e-06	7.66e-06	<1e-08	7.66e-06	4.8e-15
$b = -0.20$	0.1768	0.1768	<1e-08	0.1768	1.3e-15
$b = -0.10$	0.4574	0.4574	<1e-08	0.4574	6.6e-15
$b = -0.05$	0.6899	0.6899	<1e-08	0.6899	1.1e-15

Brownian motion with integrated shot-noise process as time change	True	Power series ($a = 3.0,\ N = 40$)		Fourier inversion	
	$\mathbb{P}(T_{\infty,b} \leq 1)$	$\mathbb{P}(T_{3,b} \leq 1)$	*error*	$\mathbb{P}(T_{\infty,b} \leq 1)$	*error*
$b = -0.80$	0.0002	0.0002	<1e-08	0.0002	2.5e-08
$b = -0.20$	0.1641	0.1641	<1e-08	0.1641	1.1e-05
$b = -0.10$	0.4431	0.4431	<1e-08	0.4431	2.4e-05
$b = -0.05$	0.6806	0.6806	<1e-08	0.6806	2.5e-05

Table 1 Comparison of the Power series and the Fourier inversion technique for $\mathbb{P}(T_{\infty,b} \leq 1)$ of a Brownian motion ($\mu = 0.1$, $\sigma = 0.2$) in calendar time (above), time-changed by an integrated CIR process (middle; $(\theta, \nu, \gamma, \lambda_0) = (0.12, 1.00, 0.05, 1.00)$), and time-changed by an integrated shot-noise process (below, $(\delta, \zeta, \rho, \lambda_0) = (0.80, 0.80, 0.64, 1.00)$). The Fourier integral in Equation (20) was evaluated using a Trapezoid rule with discretization $\triangle = 0.01$ on the interval $[0, 100]$. The absolute error of both approaches is given.

Brownian motion in calendar time	Monte Carlo $k = 100$		Monte Carlo $k = 1000$		Power series $N = 10$	
	$\mathbb{P}(T_{ab} \leq 1)$	*error*	$\mathbb{P}(T_{ab} \leq 1)$	*error*	$\mathbb{P}(T_{ab} \leq 1)$	*error*
$(b, a) = (-0.80, 0.80)$	0.0003	9.2e-05	0.0004	7.2e-05	0.0004	< 1e-08
$(b, a) = (-0.20, 1.00)$	0.1601	0.0202	0.1743	0.0058	0.1803	< 1e-08
$(b, a) = (-0.20, 0.20)$	0.6181	0.0469	0.6499	0.0001	0.6650	< 1e-08
$(b, a) = (-0.05, 0.20)$	0.9369	0.0289	0.9575	0.0083	0.9658	< 1e-08

Brownian motion with integrated CIR process as time change	Monte Carlo $k = 100$		Monte Carlo $k = 1000$		Power series $N = 40$	
	$\mathbb{P}(T_{ab} \leq 1)$	*error*	$\mathbb{P}(T_{ab} \leq 1)$	*error*	$\mathbb{P}(T_{ab} \leq 1)$	*error*
$(b, a) = (-0.80, 0.80)$	0.0004	0.0001	0.0004	0.0001	0.0004	< 1e-08
$(b, a) = (-0.20, 1.00)$	0.1601	0.0202	0.1734	0.0069	0.1803	< 1e-08
$(b, a) = (-0.20, 0.20)$	0.6183	0.0464	0.6502	0.0145	0.6647	< 1e-08
$(b, a) = (-0.05, 0.20)$	0.9365	0.0302	0.9574	0.0083	0.9657	< 1e-08

Brownian motion with integrated shot-noise process as time change	Monte Carlo $k = 100$		Monte Carlo $k = 1000$		Power series $N = 10$	
	$\mathbb{P}(T_{ab} \leq 1)$	*error*	$\mathbb{P}(T_{ab} \leq 1)$	*error*	$\mathbb{P}(T_{ab} \leq 1)$	*error*
$(b, a) = (-0.80, 0.80)$	0.0079	0.0007	0.0084	0.0002	0.0086	<1e-08
$(b, a) = (-0.20, 1.00)$	0.1471	0.0195	0.1601	0.0065	0.1666	<1e-08
$(b, a) = (-0.20, 0.20)$	0.5550	0.0413	0.5834	0.0129	0.5963	<1e-08
$(b, a) = (-0.05, 0.20)$	0.8975	0.0362	0.9231	0.0106	0.9337	<1e-08

Table 2 Comparison of the Power series technique ($N = 10$ and $N = 40$) to a naïve Monte-Carlo simulation (1.000.000 simulation runs) on a discrete grid with mesh $\triangle = 1/k$ according to the absolute error. The barrier hitting probability $\mathbb{P}(T_{ab} \leq 1)$ of a Brownian motion ($\mu = 0.1$, $\sigma = 0.2$) in calendar time (above), time-changed by an integrated CIR process (middle; $(\theta, \nu, \gamma, \lambda_0) = (0.12, 1.00, 0.05, 1.00)$), and time-changed by an integrated shot-noise process (below, $(\delta, \zeta, \rho, \lambda_0) = (0.80, 0.80, 0.64, 1.00)$) is presented. Note that a simulation on a discrete grid underestimates exit probabilities, the bias is larger as one might expect, even for $\triangle = 0.001$.

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