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Anscombe's Theorem
Stopped Random Walks
Applications

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Background

Standard testing procedures:

Fixed sample + analysis

Two (polemic) problems:

♢ an unnecessarily large sample;
a smaller one would have saved lives.

♢ the sample too small;
no significant conclusion.

How can we escape from this terrible dilemma ?

Solution

Sequential procedure

♣ Random sample size ;

♣ Sample until $\min\{n : \cdots\}$, i.e.,
typically, some stopping time.

Problems in the i.i.d. setting:

♠ LLN ?

♠ CLT ?

♠ LIL ?

♠ Moments ?

Example 1

$X, X_1, X_2, \ldots$ i.i.d. coin-tossing r.v's, viz.,

$$P(X = 1) = P(X = -1) = 1/2, \qquad S_n = \sum_{k=1}^{n} X_k,$$

$$N = \min\{n : S_n = 1\}.$$

Obviously:

$$E\,S_n = 0 \quad \text{for all} \quad n.$$

A natural guess:

$$E\,S_N = E\,N \cdot E\,X = \cdots\cdots = E\,N \cdot 0 = 0. \tag{1}$$

However, $S_N = 1$ a.s., $\Longrightarrow$

$$E\,S_N = 1 \neq 0 \qquad :-($$

"Problem": $E\,N = \infty$.

But ... could (1) be true "sometimes"?

Example 2

The same, but

$$N(n) = \text{the index of } S_k \text{ at the } n\text{th visit to } 0, \quad n \geq 1.$$

Well-known:

$P(S_n = 0 \text{ i.o.}) = 1, \quad N(n) \stackrel{a.s.}{\to} \infty, \quad$ CLT holds.

A natural guess:

$$\frac{S_{N(n)}}{\sqrt{N(n)}} \stackrel{d}{\to} N(0,1) \quad \text{as} \quad n \to \infty. \tag{2}$$

However,

$$\frac{S_{N(n)}}{\sqrt{N(n)}} = 0 \quad \text{for all} \quad n \qquad : -($$

But ... could (2) be true "sometimes"?

Conclusion so far

Something more is needed

for “the obvious” to be true.

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A.s. convergence

- $Y_1, Y_2, \ldots$ random variables,

$$Y_n \stackrel{a.s.}{\to} Y \quad \text{as} \quad n \to \infty,$$

- $\{\tau(t),\, t \geq 0\}$ positive, integer valued r.v's,

$$\tau(t) \stackrel{a.s.}{\to} \infty \quad \text{as} \quad t \to \infty.$$

Then

$$Y_{\tau(t)} \stackrel{a.s.}{\to} Y \quad \text{as} \quad t \to \infty.$$

Proof: The union of two null sets is a null set.

In particular

- $X, X_1, X_2, \ldots$ i.i.d. $\quad E\,X = \mu, \quad S_n = \sum_{k=1}^{n} X_k.$
- $\{\tau(t),\ t \geq 0\}$ positive, integer valued r.v's,

$$\tau(t) \stackrel{a.s.}{\to} \infty \quad \text{as} \quad t \to \infty.$$

Then

$$\frac{S_{\tau(t)}}{\tau(t)} \stackrel{a.s.}{\to} \mu \quad \text{and} \quad \frac{X_{\tau(t)}}{\tau(t)} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad t \to \infty.$$

Anscombe's theorem

- $Y_1, Y_2, \ldots$ random variables,

$$Y_n \xrightarrow{d} Y \quad \text{as} \quad n \to \infty,$$

- $\{\tau(t),\ t \geq 0\}$ positive, integer valued r.v's,

$\{b(t) > 0,\ t \geq 0\}, \quad b(t) \nearrow \infty \quad \text{as} \quad t \to \infty,$

$$\frac{\tau(t)}{b(t)} \xrightarrow{p} 1 \quad \text{as} \quad t \to \infty. \tag{3}$$

- Given $\varepsilon > 0$ and $\eta > 0$, there exist $\delta > 0$ and n_0, such that, for all $n > n_0$,

$$P\Big(\max_{\{k:|k-n|<n\delta\}} |Y_k - Y_n| > \varepsilon\Big) < \eta. \tag{4}$$

Then

$$Y_{\tau(t)} \xrightarrow{d} Y \quad \text{as} \quad t \to \infty.$$

The Anscombe condition

- Given $\varepsilon > 0$ and $\eta > 0$, there exist $\delta > 0$ and n_0, such that, for all $n > n_0$,

$$P\Big(\max_{\{k:|k-n|<n\delta\}} |Y_k - Y_n| > \varepsilon\Big) < \eta.$$

Uniform continuity in probability

Convergence in distribution ...... CLT ....

$Y_n \quad \longleftrightarrow \quad S_n/\sqrt{n} \qquad \cdots \longrightarrow \cdots$

$$P\Big(\max_{\{k:|k-n|<n\delta\}} \Big|\frac{S_k}{\sqrt{k}} - \frac{S_n}{\sqrt{n}}\Big| > \varepsilon\Big) < \eta.$$

$$\approx \qquad P\Big(\max_{\{k:|k-n|<n\delta\}} |S_k - S_n| > \varepsilon\sqrt{n}\Big) < \eta.$$

$\approx$ Kolomgorov's inequality.

Rényi's theorem with a direct proof

$X, X_1, X_2, \ldots$ i.i.d. $\quad E\,X = 0, \quad \operatorname{Var} X = \sigma^2 < \infty,$

$S_n = \sum_{k=1}^n X_k, \ n \geq 1.$

$\{\tau(t),\ t \geq 0\}$ positive, integer valued r.v's, such that

$$\frac{\tau(t)}{t} \xrightarrow{p} \theta \quad (0 < \theta < \infty) \quad \text{as} \quad t \to \infty. \tag{5}$$

Then

$$\begin{cases} \dfrac{S_{\tau(t)}}{\sigma\sqrt{\tau(t)}} \xrightarrow{d} N(0,1), \\ \\ \dfrac{S_{\tau(t)}}{\sigma\sqrt{\theta t}} \xrightarrow{d} N(0,1), \end{cases} \quad \text{as} \quad t \to \infty.$$

A weighted Rényi

$X_1, X_2, \ldots$ i.i.d. mean 0 $\sigma^2 < \infty$,
$\gamma > 0, \qquad S_n = \sum_{k=1}^n k^\gamma X_k,\ n \geq 1.$
$\{\tau(t),\ t \geq 0\}$ positive, integer valued r.v's, such that

$$\frac{\tau(t)}{t^\beta} \xrightarrow{p} \theta \quad (0 < \theta < \infty) \quad \text{as} \quad t \to \infty, \tag{6}$$

for some $\beta > 0$. Then

$$\frac{S_{\tau(t)}}{(\tau(t))^{\gamma+(1/2)}} \xrightarrow{d} N\Big(0, \frac{\sigma^2}{2\gamma+1}\Big),$$

$$\frac{S_{\tau(t)}}{t^{\beta(2\gamma+1)/2}} \xrightarrow{d} N\Big(0, \frac{\sigma^2\theta^{2\gamma+1}}{2\gamma+1}\Big) \quad \text{as} \quad t \to \infty.$$

Proof: The same, although a bit more elaborate.

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Renewal theory

$X_1, X_2, \dots$ i.i.d. $\quad S_n = \sum_{k=1}^{n} X_k,\ n \geq 1, \quad S_0 = 0.$

$\{S_n,\ n \geq 0\}$ is a random walk

If $X_1, X_2, \dots$ all ≥ 0, then

$\{S_n,\ n \geq 0\}$ is a renewal process.

Set $\quad N(t) = \max\{n : S_n \leq t\}, \quad t \geq 0.$

$\{N(t),\ t \geq 0\}$ is the (renewal) counting process.

Limit theorems via inversion:

$$\{S_n \leq t\} = \{N(t) > n\}$$

Examples: Light bulbs, queueing, insurance risk ...

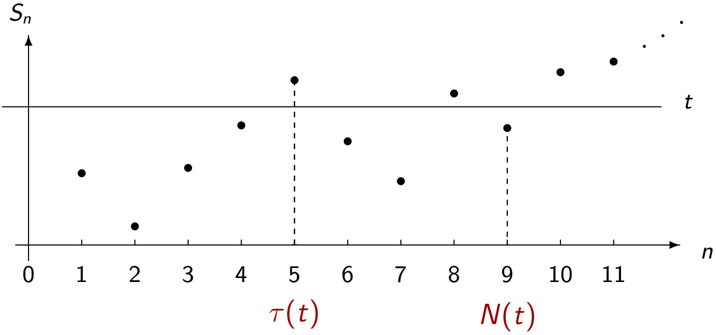
Renewal theory for random walks

$$N(t) = \max\{n : S_n \leq t\}, \quad t \geq 0.$$

Inversion no longer true

Better: $\tau(t) = \min\{n : S_n > t\}, \quad t \geq 0;$

The *first passage time* process.



Remarks

♡ For practical purposes, more reasonable to "take action"

at first occurrence of some strange event

rather than

at last occurrence.

Besides ... how do we now that "this" was the last occurrence?

♡ Mathematically:

- First passage times are stopping times;
- Counting variables are not.

$\{S_{\tau(t)},\, t \geq 0\}$ is a Stopped Random Walk.

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Sandwich lemma

$$t < S_{\tau(t)} \leq t + X_{\tau(t)} = t + X^+_{\tau(t)}.$$

Proof:

$$S_{\tau(t)-1} \leq t < S_{\tau(t)},$$

and

$$X_{\tau(t)} > 0.$$

The strong law

Theorem

$$\frac{\tau(t)}{t} \stackrel{a.s.}{\to} \frac{1}{\mu} \quad \text{as} \quad t \to \infty.$$

Proof:

$$\tau(t) \stackrel{a.s.}{\to} \infty \quad \text{as} \quad t \to \infty \qquad \Longrightarrow$$

$$\frac{S_{\tau(t)}}{\tau(t)} \stackrel{a.s.}{\to} \mu \quad \text{and} \quad \frac{X_{\tau(t)}}{\tau(t)} \stackrel{a.s.}{\to} 0 \quad \text{as} \quad t \to \infty.$$

$+$ sandwich lemma.

Central limit theorem

Theorem If, in addition, $\operatorname{Var} X = \sigma^2 < \infty$, then

$$\frac{\tau(t) - t/\mu}{\sqrt{\frac{\sigma^2 t}{\mu^3}}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty.$$

Proof: CLT + Anscombe (Rényi's version) $\Longrightarrow$

$$\frac{S_{\tau(t)} - \mu\tau(t)}{\sqrt{\sigma^2 \tau(t)}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty.$$

$+$ sandwich lemma $+$ SLLN $\Longrightarrow$

$$\frac{t - \mu\tau(t)}{\sqrt{\sigma^2 \frac{t}{\mu}}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty.$$

Summary so far

We need:

- SLLN, CLT, etc ;
- Transition; Random SLLN, Anscombe,etc ;
- Sandwich inequality .

This constitutes

The S R W – method.

Additional results

- ♯ Finiteness of moments;
- ♯ Marcinkiewicz–Zygmund type moment inequalities;
- ♯ Marcinkiewicz–Zygmund laws;
- ♯ LIL results;
- ♯ Stable analogs;
- ♯ Weak invariance principles, viz., Anscombe–Donsker;
- ♯ Strong invariance principles;
- ♯ Analogs for curved barriers, typically
  $\tau(t) = \min\{n : S_n > tn^{\alpha}\}, \qquad 0 < \alpha < 1;$
- ♯ Results for random processes with i.i.d. increments.

Renewal theory with a trend

Now instead,

$X_k = Y_k + k^\gamma \mu, \quad k \geq 1, \gamma \in \mathbb{R}, \mu > 0,$
where $Y_1, Y_2, \ldots$ are i.i.d. with mean 0.

Also, $T_n = \sum_{k=1}^n Y_k, \qquad S_n = \sum_{k=1}^n X_k, \; n \geq 1,$

$$\tau(t) = \min\{n : S_n > t\}, \quad t \geq 0.$$

Note: $\gamma = 0 \longrightarrow$ "Renewal theory for random walks".

Note: Only $\gamma \in (0, 1]$ is of interest.

Results

LLN

$$\frac{\tau(t)}{t^{1/(\gamma+1)}} \overset{a.s.}{\to} \left(\frac{\gamma+1}{\mu}\right)^{1/(\gamma+1)} \quad \text{as} \quad t \to \infty.$$

Proof: "The same".

CLT

Now $\gamma \in (0, 1/2)$, $\operatorname{Var} Y = \sigma^2 < \infty$. Then

$$\frac{\tau(t) - \left(\frac{(\gamma+1)t}{\mu}\right)^{1/(\gamma+1)}}{t^{(1-2\gamma)/(2(\gamma+1))}} \overset{d}{\to} N\Big(0,\ \sigma^2 \cdot \frac{(\gamma+1)^{(1-2\gamma)/(\gamma+1)}}{\mu^{3/(\gamma+1)}}\Big).$$

Proof: "The same" + delta method.

Stopped two-dimensional random walks

$\{(U_n^{(1)}, U_n^{(2)}), n \geq 1\}$, a two-dimensional random walk

i.i.d. increments $(X_k^{(1)}, X_k^{(2)})$, $k \geq 1$,

$\mu_2 = E\,X^{(2)} > 0$ and $\mu_1 = E\,X^{(1)} \in \mathbb{R}$.

$$\tau(t) = \min\{n : U_n^{(2)} > t\}, \quad t \geq 0\,.$$

The process of interest:

$$\{U_{\tau(t)}^{(1)}, \quad t \geq 0\}.$$

Note 1 No assumption about independence between components!

Note 2 "Everything" so far applies to

$$\{\tau(t),\, t \geq 0\} \quad \text{and} \quad \{U_{\tau(t)}^{(2)},\, t \geq 0\}.$$

Chromatography

This is how it started...

- A sample of molecules is injected onto a column;
- The molecules oscillate between a

  mobile phase and a stationary phase;
- This separates the compounds;
- Problem: Determine the elution time.

Multiple paths

- Velocity v in the mobile phase;
- $L =$ the length of the column;
- $\{(X_k^{(1)}, X_k^{(2)}),\ k \geq 1\}$ are times in the mobile and stationary phases, respectively;
- $U_n^{(1)} = \sum_{k=1}^n (X_k^{(1)} + X_k^{(2)}) \quad = \quad$ time ;
- $U_n^{(2)} = \sum_{k=1}^n v \cdot X_k^{(1)} \quad = \quad$ distance.
- Finally: With $\tau(L) = \min\{n : U_n^{(2)} > L\}$,
- $\Longrightarrow \quad U_{\tau(L)}^{(2)} \approx L$ (renewal theory);
- and $U_{\tau(L)}^{(1)} =$ the desired information.

The alternating renewal process

More generally — mobile times/stationary times ....

- Light bulbs, etcetera, allowing for repair times:

  $\{(X_k^{(1)}, X_k^{(2)}),\ k \geq 1\}$ are active/repair times.

  $U_{\tau(t)}^{(1)} =$ the "good" time in $(0, t]$.

- Queueing theory: $\{(X_k^{(1)}, X_k^{(2)}),\ k \geq 1\}$ are busy/idle times.

  $U_{\tau(t)}^{(1)} =$ the busy time in $(0, t]$.

We stop one component ...

... and check the other one

Theorem (LLN)

$$\frac{U^{(1)}_{\tau(t)}}{t} \stackrel{a.s.}{\to} \frac{\mu_1}{\mu_2} \quad \text{as} \quad t \to \infty.$$

Proof:

$$\frac{U^{(1)}_{\tau(t)}}{t} = \frac{U^{(1)}_{\tau(t)}}{\tau(t)} \cdot \frac{\tau(t)}{t} \stackrel{a.s.}{\to} \mu_1 \cdot \frac{1}{\mu_2} \quad \text{as} \quad t \to \infty.$$

Theorem (CLT)

If $\sigma_1^2 = \operatorname{Var} X^{(1)} < \infty, \quad \sigma_2^2 = \operatorname{Var} X^{(2)} < \infty,$ and

$$v^2 = \operatorname{Var}\left(\mu_2 X^{(1)} - \mu_1 X^{(2)}\right) > 0,$$

then

$$\frac{U_{\tau(t)}^{(1)} - \frac{\mu_1}{\mu_2} t}{v \mu_2^{-3/2} \sqrt{t}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t \to \infty.$$

Proof: Rényi's device:

$$S_n = \mu_2 U_n^{(1)} - \mu_1 U_n^{(2)}, \quad n \geq 1$$

is a random walk, with mean 0 and variance nv^2.

Anscombe + sandwich for $U_{\tau(t)}^{(2)}$ + LLN for $\tau(t)$.

Additional, more sophisticated, examples

Queueing theory

- $\{X_k^{(2)},\ k \geq 1\}$ are the interarrival times,
  $X_k^{(1)} = 1$ if customer k makes a purchase, 0 otherwise.
  $U_{\tau(t)}^{(1)} = \#$ purchasing customers in $(0, t]$.

- $\{X_k^{(1)},\ k \geq 1\} =$ amounts of the purchases.
  $U_{\tau(t)}^{(1)} =$ the amount of cash at time t.

Replacement based on age

$\{X_k^{(2)}, k \geq 1\}$ interreplacement times,

$X_k^{(1)} = 1$ if replacement due to failure, 0 due to age.

$U_{\tau(t)}^{(1)} = \#$ replacements due to failure in $(0, t]$.

Cumulative shock models

$\{X_k^{(1)}, k \geq 1\} =$ intershock times,

$X_k^{(2)} =$ the magnitude of the k th shock.

$U_{\tau(t)}^{(1)} =$ the failure time.

Stopped two-dim. random walks with a trend

For $i = 1, 2$, $\{Y_k^{(i)},\ k \geq 1\}$ i.i.d. with mean 0,
$X_k^{(i)} = Y_k^{(i)} + k^{\gamma_i}\mu_i$, with $\mu_1 \in \mathbb{R}$, $\mu_2 > 0$, and $\gamma_i \in [0, 1]$
$\{(U_n^{(1)},\ U_n^{(2)}),\ n \geq 1\}$ as expected.

$$\tau(t) = \min\{n : U_n^{(2)} > t\}, \quad t \geq 0.$$

Object of interest:

$$\{U_{\tau(t)}^{(1)}, \quad t \geq 0\}.$$

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Renewal theory for perturbed random walks

- ♠ $X_1, X_2, \dots$ i.i.d., $E\,X = \mu > 0$, $S_n = \sum_{k=1}^{n} X_k$.
- ♠ In addition: $\{\xi_n,\ n \geq 1\}$, arbitrary r.v's, such that

$$\frac{\xi_n}{n} \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty\,. \tag{7}$$

- ♠ Object in focus: $Z_n = S_n + \xi_n, \quad n \geq 1,$ and

$$\tau(t) = \min\{n : Z_n > t\}, \quad t \geq 0.$$

Remark

More general than nonlinear renewal theory.

Results

"As before" + taking care of noise.

The case $Z_n = n \cdot g(\bar{Y}_n)$

- ♣ $Y_1, Y_2, \ldots$ i.i.d., positive mean θ, variance ν^2;
- ♣ $g > 0$, twice continuously differentiable around θ;
- ♣ $\bar{Y}_n = \frac{1}{n} \sum_{k=1}^{n} Y_k$;
- ♣ $Z_n = n \cdot g(\bar{Y}_n), \quad n \geq 1$;
- ♣ $\tau(t) = \min\{n : Z_n > t\}, \quad t \geq 0.$

This is a special case of a perturbed random walk.

Namely...

Namely

Taylor expansion

$$Z_n = n \cdot g(\theta) + n \cdot g'(\theta)(\bar{Y}_n - \theta) + n \cdot \frac{g''(\rho_n)}{2}(\bar{Y}_n - \theta)^2$$

$$= \text{random walk} + \text{noise}.$$

Further results

- ♡ Perturbed random walks with a trend;
- ♡ Stopped two-dim. perturbed random walks;
- ♡ The same with a trend.

In particular: The case

$$(Z_n^{(1)}, Z_n^{(2)}) = \big(n \cdot g_1(\bar{Y}_n^{(1)}), n \cdot g_2(\bar{Y}_n^{(2,1)}, \bar{Y}_n^{(2,2)})\big). \quad (8)$$

Proofs

The same basic pattern, additional technicalites.

Repeated significance tests

One-parameter exponential families

$$G_\theta(dx) = \exp\{\theta x - \psi(\theta)\}\lambda(dx), \quad \theta \in \Theta\,,$$

- λ is a non-degenerate, σ-finite measure on $\mathbb{R}$;
- Θ is a non-degenerate interval on $\mathbb{R}$;
- ψ is convex;
- θ unknown.

$Y_1, Y_2, \ldots$ i.i.d. random variables $\sim G_\theta$.

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0.$$

The log-likelihood ratio is

$$\begin{aligned} T_n &= \sup_{\theta\in\Theta} \log \prod_{k=1}^{n} \exp\{\theta Y_k - \psi(\theta)\} \\ &= n \cdot \sup_{\theta\in\Theta} \{\theta \bar{Y}_n - \psi(\theta)\} = n \cdot g\big(\bar{Y}_n\big), \end{aligned}$$

where $g(x) = \sup_\theta \big(\theta x - \psi(\theta)\big)$, $x \in \mathbb{R}$,
is the convex (Fenchel) conjugate of ψ.

$\{T_n, n \geq 1\}$ is a perturbed random walk :–)

Sequential test procedure:
Reject H_0 as soon as T_n large $\Longrightarrow$

$$\tau(t) = \min\{n : T_n > t\}, \quad t > 0,$$

which has well-known properties ....

Repeated significance tests

Two-parameter exponential families

More than just an extension from the previous setup, in that two-parameter models may provide relations between marginal one-parameter tests and joint tests.

Special scenario:
The two-dimensional test statistic falls into its (two-dimensional) critical region, whereas none of the (one-dimensional) marginal test statistics fall into theirs.

Thus ...... Something is wrong somewhere ...

but ... where or what?

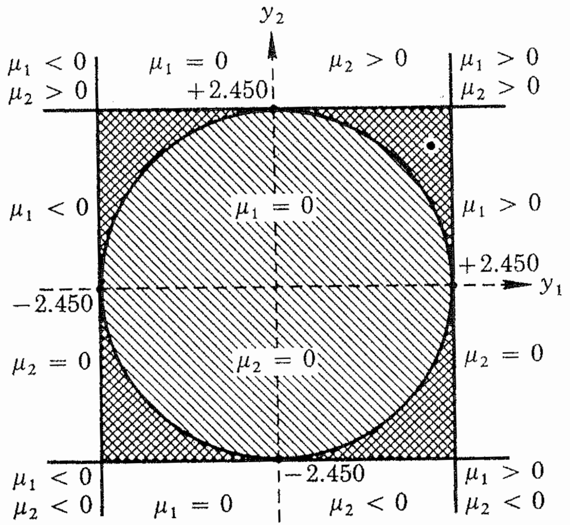
y_2
$\mu_1 < 0$
$\mu_2 > 0$
$\mu_1 = 0$
$+2.450$
$\mu_2 > 0$
$\mu_1 > 0$
$\mu_2 > 0$
$\mu_1 < 0$
$\mu_1 = 0$
$\mu_1 > 0$
$+2.450$
y_1
-2.450
$\mu_2 = 0$
$\mu_2 = 0$
$\mu_2 = 0$
$\mu_1 < 0$
$\mu_2 < 0$
$\mu_1 = 0$
-2.450
$\mu_2 < 0$
$\mu_1 > 0$
$\mu_2 < 0$

Figure 6

Formally — analogously

$$G_{\theta_1,\theta_2}(dy_1, dy_2) = \exp\{\theta_1 y_1 + \theta_2 y_2 - \psi(\theta_1, \theta_2)\}\lambda(dy_1, dy_2),$$

$$H_0 : \theta_1 = \theta_{01}, \quad \theta_2 = \theta_{02}$$
$$H_1 : \theta_1 \neq \theta_{01} \quad \text{or} \quad \theta_2 \neq \theta_{02},$$

where ... normalization.... convex conjugate ...

$$g(y_1, y_2) = \sup_{\theta_1,\theta_2} \left(\theta_1 y_1 + \theta_2 y_2 - \psi(\theta_1, \theta_2)\right).$$

The log-likelihood ratio: $T_n = n \cdot g\left(\bar{Y}_n^{(1)}, \bar{Y}_n^{(2)}\right)$.

$\{T_n,\ n \geq 1\}$ is a perturbed random walk.

Marginals

We may interpret T_n, as the second component of a two-dimensional perturbed random walk.

Example

$(Y_k^{(1)}, Y_k^{(2)})'$, $k \geq 1$, i.i.d. normal, mean $(\theta_1, \theta_2)'$, variances 1. Then ...

$$T_n = \frac{n}{2}\Big(\big(\bar{Y}_n^{(1)}\big)^2 + \big(\bar{Y}_n^{(2)}\big)^2\Big)$$
$$= \frac{1}{2n}((\Sigma_n^{(1)})^2 + (\Sigma_n^{(2)})^2)\,.$$

With "obvious notation"

$$\tau(t) = \min\{n : \|\mathbf{\Sigma}_n\| > \sqrt{2tn}\}, \quad t \geq 0,$$

generalizes the square root boundary problem.

One conclusion

$$g_1(x) \equiv 1 \quad \text{and} \quad g_2(y_1, y_2) = g(y_1, y_2)$$

$$\Longrightarrow \quad \frac{\tau(t)}{t} \stackrel{a.s.}{\to} \frac{2}{\theta_1^2 + \theta_2^2} \quad \text{as} \quad t \to \infty.$$

Strong laws for the marginal tests:

$$\frac{\tau_i(t)}{t} \stackrel{a.s.}{\to} \frac{2}{\theta_i^2} \quad \text{as} \quad t \to \infty, \quad i = 1, 2.$$

Note $$\frac{2}{\theta_i^2} > \frac{2}{\theta_1^2 + \theta_2^2}.$$

Thus, under the alternative, we would, at stopping, encounter a two-dimensional rejection, but, possibly not (yet?) a one-dimensional rejection ...

i.e., something is wrong but no further information.

Tack så mycket !

Vielen Dank !

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