Precise Asymptotics – A General Approach

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The Hsu-Robbins-Erdős-Baum-Katz Theorems

Consider an i.i.d. sequence $X; X_1, X_2, \ldots$ and set $S_n = \sum_{k=1}^{n} X_k, \; n \geq 1$.

- **Hsu and Robbins (1947):** If $EX^2 < \infty$ and $EX = 0$, then **"complete convergence"**, i.e.

$$\sum_{n=1}^{\infty} P(|S_n| > n\varepsilon) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

- **Erdős (1949, 1950):** Conversely, if sum is finite for some $\varepsilon > 0$, then $EX^2 < \infty$ and $EX = 0$, and sum is finite for all $\varepsilon > 0$.
- **Baum and Katz (1965):** Let $0 < p < 2, \; r \geq p$. If $E|X|^r < \infty$ and $EX = 0$ in case $r \geq 1$, then

$$\sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| > \varepsilon n^{1/p}) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

- **Baum and Katz (1965):** Conversely, if

$$\sum_{n=1}^{\infty} n^{r/p-2}\, P(|S_n| > \varepsilon n^{1/p}) < \infty \quad \text{for some} \quad \varepsilon > 0,$$

  then $E|X|^r < \infty$ and, if $r \geq 1$, $EX = 0$. In particular, the sum is finite for all $\varepsilon > 0$.

Remarks. a) $r = 2,\ p = 1$: **Hsu-Robbins (1947), Erdős (1949, 1950)**; $r = p = 1$: **Spitzer (1956)**; $r \geq 1,\ p = 1$: **Katz (1963)**.

b) Note that $\lim\limits_{\varepsilon \searrow 0} \sum_{n=1}^{\infty} n^{r/p-2}\, P(|S_n| > \varepsilon n^{1/p}) = \infty.$

Precise Asymptotics

- **Heyde (1975):** If $EX^2 < \infty$ and $EX = 0$, then one has a **"complete convergence rate"**, i.e.

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| > n\varepsilon) = EX^2 \quad \textbf{(Hsu-Robbins-Erdős)}.$$

- **Chen (1978):** Let $r \geq 2$, $0 < p < 2$. If $EX = 0$, $EX^2 = \sigma^2 > 0$, and $E|X|^r < \infty$, then

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| > \varepsilon n^{1/p}) = \frac{p}{r-p} E|Y|^{\frac{2(r-p)}{2-p}},$$

where Y is normal $(0, \sigma^2)$ **(Baum-Katz)**.

- **Gut and Spătaru (2000):** Let $1 \leq p < r < \alpha$, $\alpha \in (1,2]$. If X in normal domain of attraction of a nondegenerate stable law G_α and $EX = 0$, then

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{\alpha(r-p)}{\alpha-p}} \sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| > \varepsilon n^{1/p}) = \frac{p}{r-p} E|Z|^{\frac{\alpha(r-p)}{\alpha-p}},$$

where Z has distribution G_α.

- **Gut and Spătaru (2000):** Let $1 \leq p < \alpha$, $\alpha \in (1,2]$. If X in domain of attraction of G_α as above and $EX = 0$, then

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| > \varepsilon n^{1/p}) = \frac{\alpha p}{\alpha - p}.$$

In particular, if $\operatorname{Var} X = \sigma^2 < \infty$, then limit equals $2p/(2-p)$.

The General Pattern of Proofs

- **Step I:** Prove the result for the limiting random variable Z, i.e.

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{\alpha(r-p)}{\alpha-p}} \sum_{n=1}^{\infty} n^{r/p-2} P(|Z| > \varepsilon n^{1/p-1/\alpha}) = \frac{p}{r-p} E|Z|^{\frac{\alpha(r-p)}{\alpha-p}}.$$

- **Step II:** Prove that

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{\alpha(r-p)}{\alpha-p}} \sum_{n=1}^{\infty} n^{r/p-2} \left| P(|S_n| > \varepsilon n^{1/p}) - P(|Z| > \varepsilon n^{1/p-1/\alpha}) \right| = 0.$$

Note that, since Z has a continuous distribution,

$$\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} \sup_z \left| P(|S_n|/n^{1/\alpha} > z) - P(|Z| > z) \right| = 0.$$

Step II is typically divided into the following **two substeps**:

Steps II a and II b

- **Step II a:** Choose suitable $N = N(\varepsilon, M) \nearrow \infty$ as $\varepsilon \searrow 0$, $M \nearrow \infty$, and show that

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{\alpha(r-p)}{\alpha-p}} \sum_{n=1}^{N} n^{r/p-2} \left| P(|S_n| > \varepsilon n^{1/p}) - P(|Z| > \varepsilon n^{1/p-1/\alpha}) \right| = 0.$$

Just make use of

$$\Delta_n(\varepsilon) = \left| P(|S_n| > \varepsilon n^{1/p}) - P(|Z| > \varepsilon n^{1/p-1/\alpha}) \right| \leq \Delta_n \,.$$

- **Step II b:** Prove that

$$\lim_{M \nearrow \infty} \lim_{\varepsilon \searrow 0} \varepsilon^{\frac{\alpha(r-p)}{\alpha-p}} \sum_{n>N} n^{r/p-2} \Delta_n(\varepsilon) = 0.$$

For the latter relation one typically makes use of an estimate

$$\max \left\{ P(|S_n|/n^{1/\alpha} > z),\, P(|Z| > z) \right\} \leq C\, z^{-r},$$

which, in turn, may follow from a corresponding r-th moment bound.

General Results on Precise Asymptotics

Assumptions. Let $\{Z_n\}$ be a general sequence of r.v.'s satisfying

(A.1) For some $0 < \alpha \leq 2$, $|Z_n|/n^{1/\alpha} \stackrel{d}{\to} |Z|$ $(n \to \infty)$, where $|Z|$ has a continuous distribution.

(A.2) For some $0 < r < \alpha$, there exists a $C > 0$ such that, for all $n \geq 1$ and $z > 0$,

$$\max\left\{P(|Z_n|/n^{1/\alpha} \geq z),\, P(|Z| \geq z)\right\} \leq C\, z^{-r}.$$

Note that, by Markov's inequality, a sufficient condition for **(A.2)** is

(A.2') For some $0 < r < \alpha$, there exists a $C > 0$ such that, for all $n \geq 1$,

$$\max\left\{E(|Z_n|/n^{1/\alpha})^r,\, E|Z|^r\right\} \leq C.$$

General Results on Precise Asymptotics

Theorem 1 (Gut - St., 2012 a)

Suppose that $E|Z|^{\alpha(r-p)/(\alpha-p)} < \infty$ *for some* $0 < p < r \ (< \alpha)$. *Then, under Assumptions* **(A.1)** *and* **(A.2)**,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{\alpha(r-p)}{\alpha-p}} \sum_{n=1}^{\infty} n^{r/p-2} P(|Z_n| \geq \varepsilon n^{1/p}) = \frac{p}{r-p} E|Z|^{\frac{\alpha(r-p)}{\alpha-p}}.$$

The proof is based on the following two propositions:

Proposition 1 a

Suppose that $E|Z|^{\alpha(r-p)/(\alpha-p)} < \infty$ *for some* $0 < p < r \ (< \alpha)$. *Then, under Assumption* **(A.1)**,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{\alpha(r-p)}{\alpha-p}} \sum_{n=1}^{\infty} n^{r/p-2} P(|Z| \geq \varepsilon n^{1/p-1/\alpha}) = \frac{p}{r-p} E|Z|^{\frac{\alpha(r-p)}{\alpha-p}}.$$

General Results on Precise Asymptotics

Proposition 1 b

Under Assumptions **(A.1)** *and* **(A.2)**, *for all* $0 < p < r \ (< \alpha)$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{\alpha(r-p)}{\alpha-p}} \sum_{n=1}^{\infty} n^{r/p-2} \left| P(|Z_n| \geq \varepsilon n^{1/p}) - P(|Z| \geq \varepsilon n^{1/p-1/\alpha}) \right| = 0.$$

Proof of Proposition 1 a. Direct calculation.

Proof of Proposition 1 b. Proceed as in **Step II** above and choose

$$N = N(\varepsilon, M) = \left[M \, \varepsilon^{-\alpha p/(\alpha-p)} \right]$$

with $0 < \varepsilon \leq 1$ and $M \geq 1$.

Under a somewhat stronger assumption, we also have an extension of **Theorem 1** to the case $\alpha = 2$ and $r \geq 2$.

(A.3) For some $r \geq 2$ there exist $C > 0$ and $q \geq 2$ such that, for all $n \geq 1$ and $z > 0$,

$$\max\left\{P(|Z_n|/n^{1/2} \geq z),\ P(|Z| \geq z)\right\} \leq C\left\{nQ_1(n^{1/2}z) + Q_2(z)\right\},$$

where Q_1 and Q_2 are non-increasing functions such that

$$\int_1^\infty z^{r-1}Q_1(z)\,dz < \infty \quad \text{and} \quad \int_1^\infty z^{q-1}Q_2(z)\,dz < \infty.$$

Theorem 2 (Gut - St., 2012 a)

Under Assumption **(A.1)**, *with* $\alpha = 2$, *and Assumption* **(A.3)**, *with* $q = 2(r-p)/(2-p)$, *we have, for* $0 < p < 2$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n=1}^{\infty} n^{r/p-2} P(|Z_n| \geq \varepsilon n^{1/p}) = \frac{p}{r-p} E|Z|^{\frac{2(r-p)}{2-p}}.$$

Next we turn our attention to the limiting case $r = p$ which is not covered by **Theorems 1 and 2**.

Theorem 3 (Gut - St., 2012 a)

Let $0 < r < \alpha$. *Under Assumptions* **(A.1)** *and* **(A.2)**, *we have*

$$\lim_{\varepsilon \searrow 0} \frac{1}{-\log \varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} P(|Z_n| \geq \varepsilon n^{1/r}) = \frac{\alpha r}{\alpha - r}.$$

The proof is again based on **two propositions** in which (say)

- **Proposition 3 a** yields the above result with $|Z_n|$ being replaced by $n^{1/\alpha}|Z|$;
- **Proposition 3 b** shows that this replacement is negligible. This **second step** is again verified in **two substeps** by dividing the sum at $N = N(\varepsilon) = [\varepsilon^{-\gamma}]$ with $\gamma > \alpha r/(\alpha - r)$.

Some Examples

The **general pattern of proofs**, which is essentially adapted from the i.i.d. case, is also applicable under much weaker assumptions. For the sake of completeness, however, we begin by describing the i.i.d. case once again as a special case:

- **Sums of i.i.d. r.v.'s attracted to stable law (Gut-Spătaru, 2000):**
  - **Steps I and II a** via convergence to (stable or normal) r.v. Z;
  - **Step II b** via Fuk-Nagaev (1971) tail estimate for S_n.
- **Attraction to semistable law (Scheffler, 2003):**
  - **Steps I and II a** via convergence to (semistable) r.v. Z;
  - **Step II b** via large deviation estimate for S_n (Heyde, 1967).

Some Examples

- **Positive association (Mi, 2005):** Consider partial sums $\{S_n\}$ of strictly stationary, positively associated r.v.'s with $2+\delta$ moments.
  - **Steps I and II a** via Newman's (1980) CLT for S_n;
  - **Step II b** via an estimate $E|S_n|^t \leq Cn^{t/2}$ (Mi, 2005).
- **Martingale differences (Gut-St., 2012 a):**
  - **Steps I and II a** via Haeusler's (1984) functional CLT for martingale difference sequences under $2+\delta$ moments;
  - **Step II b** via a Fuk-Nagaev type inequality (Haeusler, 1984).

Further examples:

- **Independent, but non-i.i.d. summands under domination;**
- **Weighted sums (Cheng-Wang, 2006);**

Some Examples and Other Settings

- **Negatively associated summands (Huang-Zhang, 2005);**
- ϱ**-mixing summands (Huang-Jiang-Zhang, 2005);**
- **Linear processes (Tan-Yang, 2008);**

Other settings:

- **Maximal sums**: $\max_{1\le k\le n} |S_k|$ instead of $|S_n|$;
- **Moments**: $E\,|S_n|^p\, I\{|S_n| > \varepsilon n\},\ 0 < p < 2,$ instead of $|S_n|$;
- **Self-normalized sums;**
- **Records and record times;**
- **Renewal counting processes;**
- **. . . ($\approx$ 75 references).**

Klesov's Convergence Rate in Heyde's (1975) Result

Some concluding remarks concerning the special case of partial sums $S_n = \sum_{k=1}^{n} X_k$ $(n \geq 1)$ of i.i.d. r.v.'s :

Theorem (Klesov, 1993/94)

a) *If* X *is normal* $(0, \sigma^2)$ *with* $\sigma^2 > 0$, *then*

$$\varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) - \sigma^2 = -\frac{1}{2}\varepsilon^2 + o(\varepsilon^2) \quad \textit{as} \quad \varepsilon \searrow 0\,.$$

b) *If* $EX = 0$, $EX^2 = \sigma^2 > 0$, *and* $E|X|^3 < \infty$, *then*

$$\varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) - \sigma^2 = o(\varepsilon^{1/2}) \quad \textit{as} \quad \varepsilon \searrow 0\,.$$

Klesov's (1993/94) theorem can be generalized as follows :

Theorem 4 (Gut - St., 2012 b)

Let $r \geq 2,\ 0 < p < 2$, *and let* Y *be* *normal* $(0, \sigma^2)$ *with* $\sigma^2 > 0$.

a) *If* $EX = 0,\ EX^2 = \sigma^2 > 0,$ *and* $E|X|^q < \infty$ *for some* $r < q \leq 3$, *then, as* $\varepsilon \searrow 0$,

$$\varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| \geq \varepsilon n^{1/p}) - \frac{p}{r-p} E|Y|^{\frac{2(r-p)}{2-p}} = o\big(\varepsilon^{\frac{p(r-p)(q-2)}{(2-p)(q-p)}}\big) .$$

b) *If* $EX = 0,\ EX^2 = \sigma^2 > 0,$ *and* $E|X|^q < \infty$, *for some* $q \geq 3$ *with* $q > (2r - 3p)/(2 - p)$, *then, as* $\varepsilon \searrow 0$,

$$\varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| \geq \varepsilon n^{1/p}) - \frac{p}{r-p} E|Y|^{\frac{2(r-p)}{2-p}} = o\big(\varepsilon^{\frac{2p(r-p)}{(2-p)(p+q(2-p))}}\big) .$$

Convergence Rates in Precise Asymptotics

Remarks. a) If $p = 1$, $r = 2$, $q = 3$, the latter rates reduce to **Klesov's (1993/94)** rate $o(\varepsilon^{1/2})$.

b) If $r \geq 2$, $0 < p < 2$, **Theorem 4** yields convergence rates in **Chen's (1978)** precise asymptotic for the **Baum - Katz (1965) theorem** under r (≥ 2) moments.

c) If $1 \leq p < r < \alpha$, $\alpha \in (1, 2]$ and X is in the normal domain of attraction of a non-degenerate stable law G_α, with $EX = 0$, then convergence rates in the **Gut - Spătaru (2000)** precise asymptotics for the **Baum - Katz (1965) theorem** can also be obtained (see **Gut - St., 2012 c**).

References

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Thank you for your interest !