Securitization of Senior Life Settlements: Managing Interest Rate Risk with a *Planned Duration Class*

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Using duration to measure interest rate risk for securities such as MBS, callable bonds and securities backed by life insurance policies is problematic because for these securities the timing of cash flows is uncertain. In order to measure duration for securities with embedded options like callable bonds and MBS, cash flow timing must be assumed or modeled. In the case of senior life settlements, duration is only a useful summary of interest rate risk if the estimated life of the insured is accurate. It is precisely because the life of the insured is uncertain that the duration of a pool of senior life settlement contracts will not offer a meaningful summary of interest rate risk. In this paper we illustrate how a pool of senior life settlement contracts can be funded with a capital structure that is composed of two classes of securities; one which has a duration that is insulated from variations in the life of the insured around the estimated life expectancy and the other with a duration which is highly sensitive to variations in the life of the insured around the estimated life expectancy. We name the security class with a stable duration the *Planned Duration Class* (PDC) while the class with the unstable duration is called the *support duration class*.

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Senior life settlements are life insurance contracts that have been purchased from senior citizens by investors interested in the value offered by the death benefit. In return for assuming the stream of insurance premiums necessary to keep the policy active, the purchaser becomes the beneficiary of the policy. Pools of senior life settlements have been funded on the capital markets with issues of fixed income securities that are backed by the future death benefits of the life insurance policies[1]. The vehicle that funds the pool of life settlement contracts must issue securities to fund the purchase price of the contracts and the expected premiums or have access to a line of credit or other source of liquidity to fund the future premiums.

In this paper we illustrate how a pool of senior life settlement contracts can be funded with a capital structure that is composed of two classes of securities whose duration is altered from the duration of the underlying pool of insurance contracts. One class is constructed so that its duration is insulated from variations in the mortality rates of the insured. We accomplish this by calculating the first derivative of the Macaulay duration with respect to a change in the time the pool of insurance contracts is outstanding above or below the expected life (LE). Fluctuations in the time the pool is outstanding is a measurement of longevity risk; the risk that the insured lives beyond or short of some expected value. By fixing this derivative at zero, we are able to find the yield/LE combinations for which duration is a stable measure of interest rate risk across premium and death benefit combinations. The resulting matrix is the basis for our design of various classes of securities with different exposures to interest rate risk.

By default the creation of a class of securities that has a stable duration necessitates the creation of an accompanying security class that has an exceptionally unstable duration. The key to the success of this capital structure is that sufficient interest rate risk can be leveraged onto the unstable class and that this levered class can be funded at a yield that does not erode the savings garnered by financing the portion of the pool of life settlement contracts with the stable duration class.

The actual lifespan of the insured whose policies compose a life settlement portfolio will vary from what is projected because of inaccurate estimates of life expectancies and negative and positive shocks to the mortality tables such as epidemics, medical costs and the approval of new pharmaceuticals and treatments. Variations in actual life spans around expected life throw off the accuracy of duration as a measurement of interest rate risk making it difficult for investors to manage their portfolios within chosen ranges of interest rate risk and immunization strategies. Longevity risk is the chance that an insured lives beyond the expected mortality date. As the market for mortgage-backed securities has proven, there is value in the ability to distill and securitize various dimensions of risk. The most obvious case is the market for principal and interest only strips derived from mortgage pass-through securities. In this case the risk that prepayments are slower than expected is allocated to the principal only class and the risk that prepayment rate are faster than expected are allocated to the interest only class.

Fund and asset/liability managers frequently use duration as a metric to summarize the interest rate risk of portfolios of fixed income securities. Portfolios can be managed to target duration ranges, and interest rate risk of a firm can be moderated by managing the duration gap between assets and liabilities. Managers will rebalance assets and or liabilities when targeted durations become misaligned. The duration of a security is the weighted average

[1] *The Lifetrade Fund* (2006) and the *Senior Life Settlement Asset-Backed Securitization Bond* (*SLS ASB* 2006) are examples of funds that purchase life settlement contracts.

time that the value of the security is returned to the investor. This duration measure can be modified so that it is price elasticity with respect to yield. As a price elasticity, duration measures how sensitive the price of the security is relative to changes in the yield of the security. When duration is modified to measure interest rate risk, it is referred to as modified duration.

Any measure of interest rate risk for a pool of senior life settlement contracts must be based on the expected mortality rate of the pool of insured. Deviations from the expected mortality rate will change the timing and magnitude of cash flows to and from the pool of contracts. When the actual mortality rate is slower than the expected rate, the value of the premiums that must be paid to keep the policies in force increases and the receipt of fixed death benefits are pushed further into the future.

A clear difference between the complexities of measuring interest rate risk of securities with embedded options and senior life settlements is that people do not exercise an option to die. Our objective is to design a capital structure which is less costly than one in which all classes of securities issued to fund a pool of senior life settlements share on a pro rata basis all premium payments and all death benefits. We do this by creating one class of securities that has a stable duration, e.g. predictable interest rate risk that is independent of fluctuations in mortality rates. This class of securities is the unlevered class or *Planned Duration Class* (*PDC*). The natural result of creating an unlevered class is the creation of a class of securities that is levered with respect to longevity risk. We call this class the levered class or *Support Duration Class* (*SDC*). In this tranching process the longevity risk embedded in the pool of life settlement contracts is stripped out and reallocated to one class of securities and away from another class of securities. The success of this type of tranching is the ability to place the levered class at a yield that does not offset the savings generated by the unlevered class relative to a single security structure. This tranching of risk is at the core of the secondary mortgage market. It is standard for issues of collateralized mortgage obligations (*CMOs*) to have certain classes of securities that are insulated from prepayment risk, and the insulating classes or leveraged classes that finance the prepayment risk that has been shifted away from the insulated class. In a *CMO* transaction the class that is insulated from prepayment risk is called the *planned amortization class* or *PAC* and the class that is leveraged with respect to prepayment risk is called the *support class*. The efficiency and liquidity of the market for *CMOs* depends on the ability of investment bankers to place the riskier support classes at yields that make the overall CMO transaction valuable to the issuer.

An investor who buys an interest in a pool of life settlement contracts is exposed to the following risks: 1) credit risk 2) interest rate risk, and most significantly 3) longevity or mortality risk. It is the entanglement of the interest rate risk and longevity risk essentially makes the sorting, ranking and selection of life settlement investments based on duration inaccurate. Yet there is value in being able to fit life settlement investments into specified duration ranges. Our solution to this problem will give investors this ability to select and compare life settlement contracts based on duration.

In 2004 the first securities backed by a pool of senior life settlement contracts was structured and issued. It was a $63 Million *class A* senior life settlement-securitization backed by $195 million in face value of life insurance policies, issued by Tarrytown Second, LLC. There have been several other public securitizations of pools of senior life settlements, and many private deals. It has been estimated by Conning & Co. that in the next ten years, the life settlements market may grow to over $125 billion. If the market for asset-backed securities collateralized by senior life settlements is to grow to its full potential, the measurements of risks specific to this class of securities must be developed and refined.

We find the conditions for which the Macaulay duration of a life settlement contract is not affected by changes in the insured's life above or below that of his life expectancy. Once we calculate the combinations of discount rates and life expectancies for which duration remains a stable measure of interest rate risk for a life settlement contract, we use this information to structure a class of senior life settlement backed securities whose interest rate risk can be summarized using the standard measure of duration.

Of course by deleveraging one class of securities, we are left with a highly leveraged class of securities. For this smaller but more volatile class with respect to deviations of the actual life of a pool of insured from the life expectancy, duration becomes even more unreliable as a measure of interest rate risk. Just as the most volatile classes of collateralized mortgage obligations with respect to prepayment risk must be placed with investors who specialize in estimating and assuming prepayment risk, our solution to the duration problem for senior life settlements relies on the existence of group of investors who would be willing to take positions in securities that are leveraged with respect to longevity risk. It is important to keep in mind that an unexpected increase in longevity perhaps due to a new pharmaceutical that is approved will reduce the value of a pool of life settlements as the pool of insured lives longer and pushes out the date that death benefits are received. . The risk cuts both ways, an extreme heat wave or flu epidemic would reduce the average life of a life settlement pool increasing the yield on the life settlement contracts.

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[3] J.M. Keynes (1936) introduced the concept of a bond's price elasticity with respect to interest rates. J.R. Hicks' (1938) research focused on that same elasticity concept. It was Macaulay (1938) that expanded Keynes's and Hick's work by developing the duration risk metric, which has since taken on his name, "*Macaulay duration"*. While the Macaulay duration is calculated for parallel shits in a flat yield curve, other variations of duration have been developed to account for non-parallel shifts and for securities with embedded options. Macaulay and modified duration (which is simply the Macaulay duration divided by $1+y$), are widely used metrics by portfolio and asset/liability managers.

It is not only shocks to the mortality tables that lead to changes in the value of a life settlement contract, simple errors in estimating someone's life span will also lead to deviations in actual value from expected value. For the leveraged class interest rate risk becomes quite unpredictable as even slight differences in actual life from life expectancy create large changes in the value of the life settlement contract. It should be noted however, that in order to achieve an investment-grade rating, a securitized pool of senior life settlements must have a group of settlers that is diversified across diseases.

In the following section of the paper we derive the changes in the duration of a pool of senior life settlement contracts with respect to changes in the average life of the insured whose contracts compose the life settlement pool. We show that for a set of combinations of discount rates and life expectancies, duration is a stable consistent measure of interest rate risk for life settlement contracts and securities backed by senior life settlements. When the life of the pool of insured people deviates from the expected life, duration still offers an accurate measure of interest rate risk We demonstrate our results with numerical examples.

Macaulay Duration and Its Sensitivity to Shifts in Life Expectancy Tables

The value of an individual senior life settlement is equal to the present value of the premia *p* paid periodically during the life of the senior life settler, plus the discounted value of the benefit *B* received at death of the settler.

$$P = -p[1/(1+y)^1 + 1/(1+y)^2 + \ldots..+1/(1+y)^n] + B/(1+y)^n \qquad (1)$$

Where *p* stands for the insurance "premium" paid each year- first by the original owner of the policy and then by the life settlement company, *B* is the death benefit at the time of death of the life settler, *y* is the discount rate and *P* is the present value of the life settlement contract.

The general formula for the Macaulay Duration, *D*, of a fixed income security is

$$D = \sum_{1=i}^{t} (i)CF_i/(1+y)^i/P$$

Where *i* is the time at which the cash flow is paid, *y* is the discount rate, *P* is the price of the security at the time the duration is computed, and CF_i represents the cash flow at time *i*. *t* is the time when the settler dies, and is unknown. The Macaulay duration can be rearranged so that it measures the percentage change in a security's price over the percentage change in yield, the price elasticity of the security. [3]

The cash flows to and from a life settlement contract are the yearly premia *p* and the death benefit *B* received at the time of the insured's death, which we denote *t*.

Equation (2) is the calculation of the Macaulay duration *D* for a life settlement contract. The present value of each cash flow is multiplied by the time at which it is paid, *i*, where *i* runs from year *1* to year *t*, *t* being the time of death so the premia cease and the death benefit is received. The sum of the present values, multiplied by the time at which they are received, is divided by the present value of the life settlement contract, *P*.

$$D = [\sum_{i=1}^{t} p(i)/(1+y)^{i} - tB/(1+y)^{t}]/P \qquad (2)$$

Changes in Duration for shifts in life expectancy

Investors in mortgage-backed securities are exposed to the risk that the prepayment rate on the underlying pool of mortgages will be above or below the rate they use to price the mortgage-backed security. Prepayments affect the timing and magnitude of cash flows generated by mortgage-backed securities. For example when prepayments accelerate, principal is prepaid faster so that the total interest paid declines, a change in magnitude, and the principal is collected faster, a change in timing. For MBS the cash flows generated by a pool of mortgages are always positive. Investors in pools of life settlement contracts are exposed to longevity risk—the risk that the pool of insured lives longer than the expectation upon which the pool of contracts were priced. Unlike a pool of mortgages which generate positive cash flows composed of interest and principal payments, the underlying pool of life settlements contracts generates a stream of negative cash flows, the insurance premiums and a stream of positive cash flows, the death benefits. In the case of a pool of senior life settlements, the timing, magnitude and direction of cash flows are affected by the longevity of underlying insured. The change in the direction of the cash flows from negative to positive for a pool of senior life settlements is the reason that under certain conditions, a deviation from life expectancy of the settlers does not affect the Macaulay duration of the pool.

In *Appendix A* we calculate the first derivative of the Macaulay duration with respect to the change in time *t* above or below the expected time of death, and in *Appendix B* we find the conditions for which the derivative is equal to zero. Fixing this derivative at zero is the constraint we use to locate the yields of the life settlement contracts at which deviations from life expectancy have no impact on the duration measurement. This calculation is done for various yield/life expectancy combinations. As a matter of simplifying the derivative calculation in *Appendix A*, we set $a = (1)/(1+y)$ and equation (2) then becomes:

$$D = [(\sum_{i=1}^{t} pi(a)^{i}) - Bt(a)^{t}]/P \qquad (2)$$

Conditions for a reliable Senior Life Settlement's Macaulay Duration

In *Appendix B* we develop the conditions for which the Macaulay duration of a pool of senior life settlements is not affected by changes in the life of the insured above or below life expectancies computed at the time the insurance contracts were sold. We use equation (13) from *Appendix A*, to calculate the conditions for which $D' = 0$. This is expressed in equation (14).

$t^* = [1/\ln(1+y)] + [p(1+y)/y(-p-By)]$	**eq. 14**

Equation (14) shows the constraint between a settler's life expectancy (we call it t^* in the equation) the premium *p*, the death benefit *B* and the yield *y* of the corresponding policy, in order to obtain a life settlement with a duration that is not affected by deviations around LE.

Carving out the *Planned Duration Class* and the *Support Duration Class*

Exhibit 1 shows a typical block of senior life settlements that could be securitized. It has a total principal value of $82 million, a weighted average α of 4.4% (we define *α* as the ratio of premium *p* to death benefit *B*, *α = p/B*), and a weighted average *LE* of 4.538. It is standard for the industry of senior life settlements to quote life insurance policies in terms of the premium to death benefit ratio, i.e. a 4% policy may corresponds to a premium of $4000 and a death benefit of $100,000.

Exhibit 1

Principal Amount	Annual Premium	LE
10,000,000	500,000.00	4
15,000,000	700,000.00	5
5,000,000	200,000.00	2
2,000,000	50,000.00	7
1,000,000	75,000.00	5
3,000,000	150,000.00	6
6,000,000	200,000.00	3
8,000,000	350,000.00	4
1,000,000	100,000.00	5
4,000,000	200,000.00	8
10,000,000	450,000.00	3
7,000,000	250,000.00	2
10,000,000	400,000.00	5

We next find the premia and death benefits that satisfy equation (14) for different *yield/LE* combinations. We have assumed an upward yield curve so that yields do increase with life expectancy in eq. (14). We assume a yield of 5.25% for a *LE* of 2, a yield of 5.50% for a *LE* of 3 and continue to increase the yield by 25bps for each *LE* increase by one additional year, as shown in Exhibit 2.

When we plug a *5.25%* yield, a *LE* of *2*, and a death benefit *B* of *$1,100,000* in eq. (14), and solve for premium *p*, we obtain *$400,500* as shown in Exhibit 2. We solve for premia, using several other combination of yields, *LE*s and death benefits, all represented in Exhibit 2.

Exhibit 2

LE	2	3	4	5
y	5.25%	5.50%	5.75%	6%
P^*	400,500	274,000	190,000	145,000
B^*	1,100,000	1,100000	1,100000	1,100000

Exhibit 3

	2	3	4	5
∑p	450,000.00	650,000.00	850,000.00	1,275,000.00
∑B	12,000,000	16,000,000	18,000,000	27,000,000

Exhibit 3 summarizes exhibit 1 across *LE*s: it adds all premia generated by life settlements with a LE of 2 and all corresponding death benefits. There are only two life settlements with *LE* of 2, and the corresponding premia are $200,000 and $250,000, a total of $450,000, shown in exhibit 3 in the second column. The corresponding death benefits are $5 million and $7 million, a total of $12 million. In each column we add the premia and death benefits for each *LEs* going from 2 to 5.

Exhibit 4: Planned Duration Class

PDC	2	3	4	5
$\sum p/p^*$	1.12	2.37	4.47	8.79
$\sum B/B^*$	11	15	16	25

Exhibit 4 shows the ratio of the optimal premium p^* (from Exhibit 2) over the total premia across *LEs* ranging from 2 to 5, computed in Exhibit 3, and the corresponding ratio of the optimal B^* (computed in Exhibit 2) over the total death benefits computed in Exhibit 3. If we look at the *LE* of 5, we can create 8.79 *PDCs* with an annual premium of $145,000 each, and a corresponding death benefit of $1,100,000 (from Exhibit 2). More specifically, the *planned duration class*, *PDC*, for the *LE* of 5 and a 6% yield, can be structured by stripping 8.79 times the amount of $145,000 in premia (from a total of $1,275,000 in Exhibit 3), with a corresponding 8.79 times $1,100,000 in death benefits (from a total of $27,000,000 in Exhibit 3). The *support duration class*, *CDC*, is then allocated the difference between the sum of the total premia under the *LE* of 5, minus those allocated to the *PDC*, and the sum of the death benefits under *LE* of 5 minus those allocated to the *CDC*.

We show the total amount of premia p and corresponding death benefits B for the *planned duration classes* created under different *LEs* in Exhibit 5 and the corresponding premia and death benefits for the *support duration classes* in Exhibit 6. The corresponding α is shown in the last row of Exhibits 5 and 6. It is interesting to observe that α is high for the *planned duration class*, and goes to zero for the *support duration class*. We can actually generalize the relationship between α and *LE* for a *planned duration class* in Exhibit 7. We observe that for a *PDC* with a high *LE*, a lower α is needed, whilst for a *PDC* with low *LE*, a higher α must be constructed by stripping different premia.

The resulting *support duration class*, *SDC*, is a zero-coupon bond that pays death benefits only (no negative premium), at actual death of the insured, which qualifies as *maturity* of the zero-coupon bond. The *SDC's* Macaulay duration is then equal to the *LE* (the maturity) if deviation around *LE* is zero.

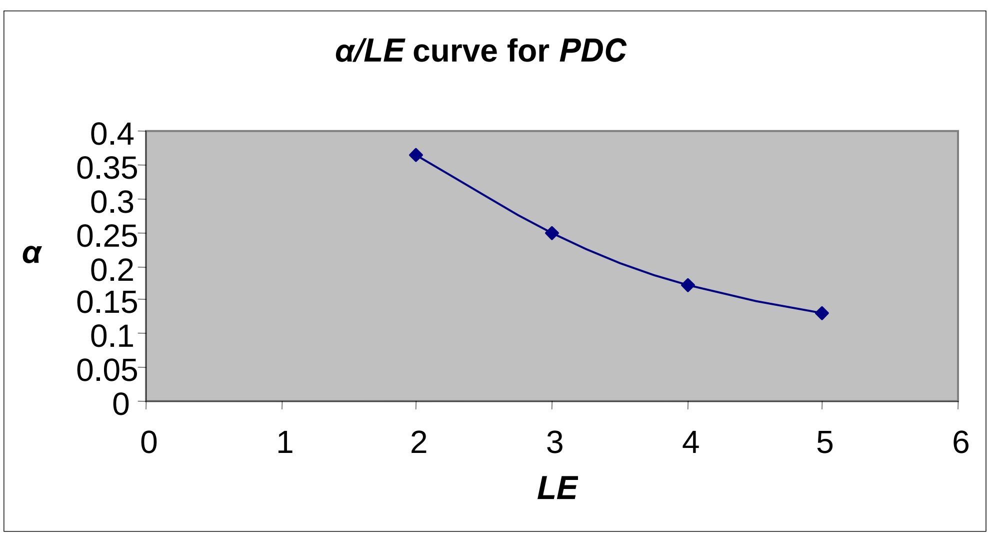
Exhibit 5: ***Planned Duration Class***

LE	2	3	4	5
y	0.0525	0.055	0.0575	0.06
(PDC)p	450000	650000	850000	1275000
(PDC)B	1235955.056	2609489.051	4921052.632	9672413.793
(PDC)α	0.364090909	0.249090909	0.172727273	0.131818182

Exhibit 6: ***Support Duration Class***

LE	2	3	4	5
(SDC)p	0.00	0.00	0.00	0.00
(SDC)B	10,764,045	13,390,511	13,078,947	17,327,586
(SDC)α	0.0000000	0.0000000	0.0000000	0.0000000

Exhibit 7



For simplicity, we only analyzed *LEs* ranging from 2 to 5 in Exhibit 1, which corresponded to a total face value of $73 million. This means that we created a *planned duration class* in the amount of 25% of \$73 million and a *support duration class* in the amount of 75% of the \$73 million face value.

Conclusion

The derivation of the *yield/LE* combinations for which the duration of a pool of life settlement contracts does not change as the age of the life settlers deviates from the life expectancy, opens up the possibility of structuring two classes of securities to finance a pool of senior life settlement contracts; one class would be designed to have a stable duration measure across a spectrum of pool longevity and the other class would pick up the slack by being designed to have a duration that is very sensitive to small changes in pool longevity. In fact the design of the second class is imposed by the design of the first. When financing a fixed pool of assets such as senior life settlements with various classes of asset-backed securities, the deleveraging of one class with respect to a risk dimension, in this case longevity risk, must be accompanied by a leveraging of another class. The first class would be structured to address the needs of those investors looking for investments with fairly certain durations, we call this class the *Planned Duration Class (PDC)*. This is done by carving out cash flows from the pool of life settlement contracts that satisfy equation (14). Cash flows generated by the life settlement pool but not allocated to the *Planned Duration Class* would be directed to the *Support Duration Class* (S*DC*). This can be achieved by combining seasoned with unseasoned portfolios of senior life settlement contracts. In exchange for assuming the longevity risk of the pool, investors in the S*DC class* would be offered a higher yield.

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APPENDIX A

The next step is to isolate from eq. (2) the term, that we call *f(t)*:

$$f(t) = \sum_{i=1}^{t} pi(a)^{i}$$

In the next section we find an expression for *f(t)*.

Expression of *f(t)*

Recall that for every real number *b* and natural number *k<n*, the following formula holds:

$$\sum_{i=k}^{t} b^i = (b^{n+1} - b^k)/(b-1) \qquad (3)$$

We can rewrite *f(t)* as

$$f(t) = \sum_{i=1}^{t} pi(a)^i = p\sum_{i=1}^{t} i(a)^i = \qquad (4)$$
$$p(a+a^2+a^3+\ldots a^t+ \qquad (5)$$
$$a^2+a^3+\ldots a^t+ \qquad (6)$$
$$a^3+\ldots a^t+ \qquad (7)$$
$$\ldots + \qquad (8)$$
$$a^t) \qquad (9)$$

By applying formula (3) to every single line of the previous segments (fragments (5), (6), etc.) we get that

$$f(t) = \sum_{i=1}^{t} pi(a)^i = p\sum_{i=1}^{t} i(a)^i =$$

$$p[(a^{t+1}-a^1)/(a-1) + (a^{t+1}-a^2)/(a-1) + (a^{t+1}-a^3)/(a-1) + \ldots + (a^{t+1}-a^t)/(a-1)]$$

The above expression can be simplified into

$$f(t) = \sum_{i=1}^{t} pi(a)i = p\sum_{i=1}^{t} i(a)^i =$$

$$p[ta^{t+1} - (a + a^2 + \ldots + a^t)]/(a-1)$$

Applying formula (3) again we get that

$$f(t) = \sum_{i=1}^{t} pi(a)^i = p\sum_{i=1}^{t} i(a)^i = p[ta^{t+1} - (a^{t+1}-a)/(a-1)]/(a-1)$$

$$[p/(a-1)][(ta^{t+2} - ta^{t+1} - a^{t+1}+a]/(a-1) = [p/(a-1)^2][ta^{t+2} -(t+1)a^{t+1} +a]$$

In summary, we proved that

$$f(t) = [p/(a-1)^2][ta^{t+2} - (t+1)a^{t+1} +a]$$

where *a=1/(1+y)* and *p*, *y*, are constant with respect to *t*.

Expression for D

We can add now the new expression of *f(t)* to get the final expression for *D*:

$$D = [(\sum_{i=1}^{t} pi(a)^i) - Bt(a)^t]/P =$$

$$[f(t) - Bt(a)^t]/P = \{(p/(a-1)^2[ta^{t+2} - (t+1)a^{t+1} + a]) - Bt(a)^t\}/P =$$

$$(1/P)\{ta^t[a^2p/(a-1)^2 - ap/(a-1)^2 - B] - a^t[ap/(a-1)^2] + ap/(a-1)^2\} =$$

$$(1/P)\{ta^t[ap/(a-1) - B] - a^t[ap/(a-1)^2] + ap/(a-1)^2\}$$

In summary, we get that

$$D = (1/P)\{ta^t[ap/(a-1) - B] - a^t[ap/(a-1)^2] + ap/(a-1)^2\} \qquad (10)$$

For the purpose of having an expression of *D* using the original parameters, we can now replace *a* by *1/(1+y)* in *eq. (10)* to obtain that

$$D = (1/P)\{ta^t[ap/(a-1) - B] - a^t[ap/(a-1)^2] + ap/(a-1)^2\} =$$

$$(1/P)\{t[1/(1+y)^t][[p/(1+y)]/[(1/(1+y)-1]-B]-(1/(1+y))^t[p/(1+y)]/[(1/(1+y))-1]^2 +$$
$$[(p/(1+y)]/[(1/(1+y)) - 1]^2$$

To conclude, we have the following close expression for the function *D*:

$$D = \{t[p(-y)^{-1} - B]/(1+y)^t - p/[y^2(1+y)^{t-1}] + p(1+y)/y^2\}/P \qquad (11)$$

The Derivative of Duration With Respect to Time

Note that the expression of *D* in eq.(10) is of the form

$$D = \{ta^t[C(a-1) - B] - a^tC + C\}/P =$$

$$(1/P)\{ta^t[C(a-1) - B] - a^tC + C\} \qquad (12)$$

Where *a*, *B*, *P* are constants and *C* is a constant defined as

$C = ap/(a-1)^2$

Now we take the derivative of duration as expressed in equation (12) with respect to t. The solution is given is equation 13.

$D' = (1/P)\{a^t[C(a-1) – B] + ta^t[C(a-1) – B] [\ln(a)] – a^tC[\ln(a)]\}$

Where *C* and *a* are the constants previously specified.

We simplify the last expression to arrive at equation (13).

$D' = (a^t/P)\{[C(a-1) – B] + t[C(a-1) – B] [\ln(a)] – C[\ln(a)]\} =$

$$\boxed{\mathbf{D' = (a^t/P)\{[t[C(a-1) – B] \ln(a)] + [C(a-1) – B – C[\ln(a)]]\} \qquad (13)}}$$

APPENDIX B

Note that *(a^t/P)* can never be equal to *0* because *a* is always positive (1/1+y). This means that the derivative of duration with respect to time (t) equals zero (*D' = 0)* when

$t(C(a-1) – B)(\ln(a)) + C(a-1) – B – C(\ln(a)) = 0$

This is a simple linear equation that we solve for t.

$t = [(-(C(a-1) – B) + C(\ln(a))]/[(C(a-1) – B)(\ln(a))] =$

$[-1/\ln(a)] + C/[C(a-1) – B]$

Now we replace C with *$(ap)/(a-1)^2$* to arrive at *D' = 0* when

$t = [-1/\ln(a)] + [ap/(a-1)^2]/\{[ap/(a-1)]– B]\}$

This previous equation can be simplified to :

$t = [-1/\ln(a)] + [ap/(a-1)^2]/\{[ap(a-1)– B(a-1)^2]/(a-1)^2\} =$

$[-1/\ln(a)] + ap/[ap(a-1) – B(a-1)^2]$

Finally, we replace *a* by its value of *1/(1+y)* to show that *D'=0* when

$$t^{*} = [1/\ln(1+y)] + [p(1+y)/y(-p-By)] \qquad \text{eq. 14}$$