On optimal stopping of autoregressive sequences

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Outline

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AR(1) process

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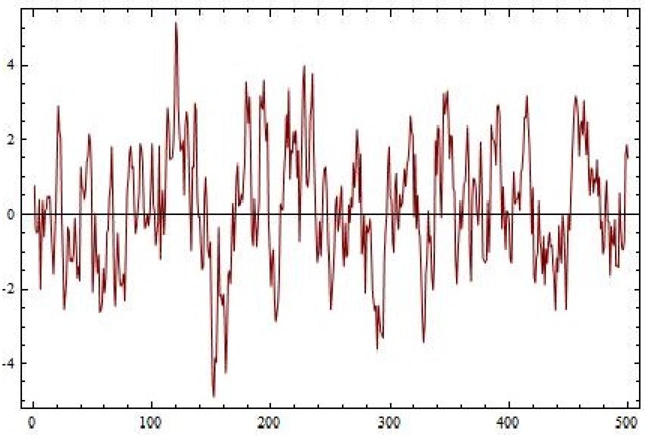
- $\Delta X_n = -(1-\lambda) X_{n-1} \Delta n + \Delta L_n, \qquad L_n = \sum_{i=1}^n Z_i$
- $X_n = \lambda^n X_0 + \sum_{i=0}^{n-1} \lambda^i Z_{n-i}.$

Path of an AR(1) process

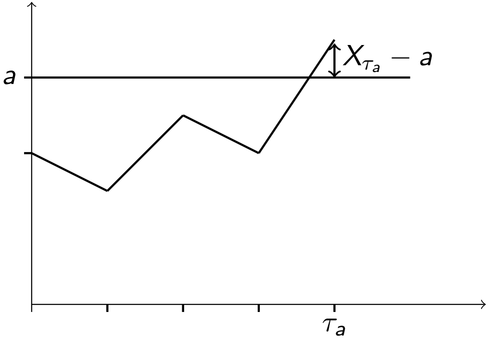


Threshold time, overshoot

In many applications – such as statistical surveillance – it is of interest to find the joint distribution of

$$\tau_a = \inf\{n \in \mathbb{N}_0 : X_n \geq a\} \quad \text{(threshold time)}$$

$$X_{\tau_a} - a \qquad\qquad \text{(overshoot).}$$



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"Solution":

$$v(x) := \sup_\tau E_x(\rho^\tau g(X_\tau)) \quad \text{value function.}$$

$$S := \{x \in E : v(x) \le g(x)\} \quad \text{optimal stopping set.}$$

Optimal stopping time:

$$\tau^* = \inf\{n \in \mathbb{N}_0 : X_n \in S\}.$$

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3. Find the optimal boundary a^*.

Novikov-Shiryaev problem

Let $g(x) = (x^+)^\alpha$, $\alpha > 0$. Consider

$$\sup_\tau E_x(\rho^\tau (X_\tau^+)^\alpha) = \sup_\tau E(\rho^\tau ((\lambda^\tau x + X_\tau)^+)^\alpha).$$

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Proposition

If $(X_n)_{n\in\mathbb{N}_0}$ is a AR(1) process, then there exists a^* such that the optimal stopping set S is given by

$$S = [a^*, \infty)$$

Proof

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Hence $S = \{x : v(x) \leq g(x)\}$ is of the form $[a^*, \infty)$. □

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Motivation

If $(X_n)_{n\in\mathbb{N}_0}$ is a random walk (or an AR(1) process) with $Exp(\beta)$-distributed jumps. Then

$$\tau_a,\ X_{\tau_a} - a \ \text{ are independent}$$

and

$$X_{\tau_a} - a \stackrel{d}{=} Exp(\beta).$$

Distributions of phasetype

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$\eta := \inf\{t \geq 0 : J_t = \Delta\}$

$PH(Q,\alpha) := P^{\eta}_{\hat{\alpha}}$ is called **phasetype distribution with parameters** Q, α.

Properties of phasetype distributions

- density: $f(t) = \alpha e^{Qt} q$
- Laplace transform: $E_{\hat{\alpha}}(e^{s\eta}) = \alpha(-sI - Q)^{-1} q$, $s \in \mathbb{C}$ with $\Re(s) < \delta$.
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Ex: $m = 1,\ Q = (-\lambda)\ \rightarrow\ PH(Q, (1)) := Exp(\lambda)$.

Phasetype distribution and overshoot

Generalization:

If $(X_n)_{n\in\mathbb{N}_0}$ is AR(1) process with innovations $Z_n = S_n - T_n$, $S_n, T_n \geq 0$ independent, $S_n \sim PH(Q,\alpha)$, then

$$E_x(\rho^{\tau_a} g(X_{\tau_a})) = \sum_{i=1}^{m} E_x(\rho^{\tau_a} 1_{K_i}) E(g(a + R^i)),$$

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How to find $E_x(\rho^{\tau_a} 1_{K_i})$?
Answer: Use the stationary distribution, complex martingales and complex analysis.

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For each $a \in \mathbb{R}$ we consider

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For each $a \in \mathbb{R}$ we consider

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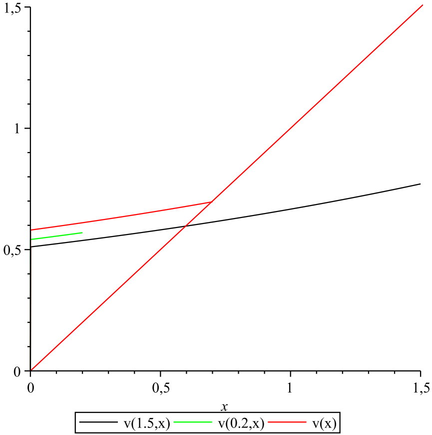
Choose a^* such that

$$v_{a^*}(a^*) = g(a^*).$$

Continuous fit



$a^* \approx 0.7$ is the unique solution to the transcendental equation $g(a^*) = v_{a^*}(a^*)$.

Smooth and continuous fit condition

Principle for continuous time processes

Let g be differentiable, then the following can be expected:

1. If the process X **enters the interior** of the optimal stopping set S immediately after starting on $a^* \in \partial S$, then the value function v is **smooth** at a^*.

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The second principle holds in our situation.

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Summary

- We studied optimal stopping problems driven by autoregressive processes.
- Elementary arguments reduces the problem to finding an optimal threshold time.
- The joint distribution of $(\tau_a, X_{\tau_a} - a)$ can be obtained for phasetype innovations.
- The optimal threshold can be found using the continuous fit principle.