Duality Methods in Robust Utility Maximization

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Outline

Robust utility functional

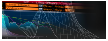
$L^0 = L^0(\Omega, \mathscr{F}, \mathsf{P})$ is the space of real-valued random variables equipped with the topology of convergence in probability (random variables that coincide P-a.s. are identified).
A functional

$$\xi \rightsquigarrow \inf_{\mathsf{Q}\in\mathscr{Q}} \big[\mathsf{E}_{\mathsf{Q}} U(\xi) + \gamma(\mathsf{Q})\big], \quad \xi \in L^0.$$

is called a robust utility functional. If $\gamma \equiv 0$ on Q, we say that it is a coherent robust utility functional. If $\mathsf{Q} = \{\mathsf{P}\}$, then we say that it is a standard utility functional.

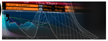
$\inf_{Q\in\mathscr{Q}}\left[E_Q U(\xi)+\gamma(Q)\right]$

- a random variable ξ is interpreted as the terminal wealth of an investor;
- $U\colon \mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ is an increasing concave function (utility function);
- the expectation is assumed to be equal $-\infty$ if it is not defined;
- $\mathscr{Q}$ is a nonempty convex subset of the set of all probability measures on $(\Omega,\mathscr{F})$ that are absolutely continuous wrt P;
- a penalty function $\gamma\colon\mathscr{Q}\to\mathbb{R}_+$ satisfies the following properties: γ is convex, $\inf_{Q\in\mathscr{Q}}\gamma(Q)\geqslant 0$, the set $\{dQ/dP\colon Q\in\mathscr{Q},\ \gamma(Q)\leqslant c\}$ is closed in $L^1(P)$ and uniformly integrable wrt P for any $c\geqslant 0$.



References I

Standard utility maximization: Merton (1969, 1971), Samuelson (1969), Pliska (1986), Karatzas, Lehoczky & Shreve (1987), Cox & Huang (1989, 1991), He & Pearson (1991a, 1991 b), Karatzas, Lehoczky, Shreve & Xu (1991), Karatzas & Žitković (1996), Kramkov & Schachermayer (1999, 2003), Cvitanić, Schachermayer & Wang (2001), Cvitanić & Wang (2001), Schachermayer (2001, 2003), Goll & Rüschendorf (2001), Deelstra, Pham & Touzi (2001), Bellini & Frittelli (2002), Owen (2002), Yan (2002), Karatzas & Žitković (2003), Bouchard & Pham(2004), Bouchard, Touzi & Zeghal (2004), Hugonnier & Kramkov (2004), Hugonnier, Kramkov & Schachermayer (2005), Pratelli (2095), Žitković (2005), Biagini & Frittelli (2005, 2008), Kramkov & Sîrbu (2006, 2006), Oertel & Owen (2007), Biagini (2008), Owen & Žitković (2009), . . .



References II

Coherent robust utility maximization: Talay & Zheng (2002), Quenez (2004), Schied (2004, 2005), Burgert & Rüschendorf (2005), Schied & Wu (2005), Gundel (2005), Müller (2005), Föllmer & Gundel (2006), Morozov (2010), . . .

Robust utility maximization: Schied (2007), Wittmüss (2008), . . .

Outline

Assumptions

Assumption (on the utility function)

$U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is an increasing concave function, $U(x) = -\infty$ for $x < 0$ and $U(x) \in \mathbb{R}$ for $x > 0$.

Assumption (on the set of terminal wealths)

$\mathscr{A}$ is a convex subset of L^0_+ containing a random variable $\xi_0 \geqslant \varkappa > 0$.

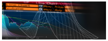
Assumptions on the penalty function γ were formulated above.

Minimax theorem

Theorem (1)

Let the above assumptions be satisfied. Then

$$\sup_{\xi\in\mathscr{A}}\inf_{\mathsf{Q}\in\mathscr{Q}}\big[\mathsf{E}_{\mathsf{Q}}U(\xi)+\gamma(\mathsf{Q})\big]=\min_{\mathsf{Q}\in\mathscr{Q}}\sup_{\xi\in\mathscr{A}}\big[\mathsf{E}_{\mathsf{Q}}U(\xi)+\gamma(\mathsf{Q})\big].$$



Conjugate function

Put

$$V(y) = \sup_{x>0} [U(x) - xy], \quad y \in \mathbb{R}.$$

$V\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function, $\{V < +\infty\} \subseteq \mathbb{R}_+$, V is decreasing,

$$\lim_{y \uparrow +\infty} \frac{V(y)}{y} = 0,$$

and

$$\inf_{y \geqslant 0} [V(y) + xy] = \begin{cases} U(x), & \text{if } x \neq 0, \\ \lim_{x \downarrow 0} U(x), & \text{if } x = 0. \end{cases}$$

Polar set

Let $\mathscr{C}_+ = (\mathscr{A} - L^0_+) \cap L^\infty_+$, and let $\overline{\mathscr{C}}_+$ be the closure of $\mathscr{C}_+$ in L^0. Of course, $\mathscr{A} \subseteq \overline{\mathscr{C}}_+$. It is useful to note that our optimization problem has the same value if $\mathscr{A}$ is replaced by $\mathscr{C}_+$, or by $\overline{\mathscr{C}}_+$. Define a 'polar' set $\mathscr{D}$ by

$$\mathscr{D} = \bigl\{\eta \in L^0_+ \colon \mathsf{E}_\mathsf{P}\eta\xi \leqslant 1 \text{ for all } \xi \in \mathscr{A}\bigr\}.$$

Then $\mathscr{D} \subseteq L^1_+$. In fact,

$$\mathscr{D} = \bigl\{\eta \in L^0_+ \colon \mathsf{E}_\mathsf{P}\eta\xi \leqslant 1 \text{ for all } \xi \in \overline{\mathscr{C}}_+\bigr\},$$
$$\overline{\mathscr{C}}_+ = \bigl\{\xi \in L^0_+ \colon \mathsf{E}_\mathsf{P}\eta\xi \leqslant 1 \text{ for all } \eta \in \mathscr{D}\bigr\}.$$

Value functions of primal and dual problems

For $x > 0$ and $y \geqslant 0$ put

$$\mathscr{A}(x) = x\mathscr{A}, \quad \mathscr{D}(y) = y\mathscr{D}.$$

Define value functions by

$$u(x) = \sup_{\xi \in \mathscr{A}(x)} \inf_{\mathsf{Q} \in \mathscr{Q}} \big[\mathsf{E}_{\mathsf{Q}} U(\xi) + \gamma(\mathsf{Q}) \big], \quad x > 0, \tag{1}$$

$$v(y) = \inf_{\eta \in \mathscr{D}(y),\, \mathsf{Q} \in \mathscr{Q}} \left[\mathsf{E}_{\mathsf{Q}} V\left(\frac{\eta}{d\mathsf{Q}/d\mathsf{P}} \right) + \gamma(\mathsf{Q}) \right], \quad y \geqslant 0. \tag{2}$$

Static case : Main theorem

Theorem (2)

Let the above assumptions be satisfied. Then
(i) *The function* $u(x)$, $x > 0$, *takes values in* $\mathbb{R} \cup \{+\infty\}$, *is increasing and concave.*
(ii) *The function* $v(y)$, $y \geqslant 0$, *takes values in* $\mathbb{R} \cup \{+\infty\}$, *is convex, lower semicontinuous and decreasing. The infimum in* (2) *is attained.*
(iii) *If* $v(y) = +\infty$ *for all* $y \geqslant 0$, *then* $u(x) = +\infty$ *for all* $x > 0$. *If* $v(y) < +\infty$ *for some* $y \geqslant 0$, *then* $u(x) \in \mathbb{R}$ *for* $x > 0$.

Static case : Main theorem

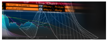
Theorem (2, continued)

(iv) *u and v are connected by*

$$u(x) = \min_{y \geqslant 0}[v(y) + xy], \quad x > 0, \tag{3}$$

and

$$v(y) = \sup_{x > 0}[u(x) - xy], \quad y \geqslant 0.$$



Static case : Main theorem

Theorem (2, continued)

(v) *Fix* $x > 0$*. If the minimum in* (3) *is attained at* y^* *and the minimum in* (2) *for* $y = y^*$ *is attained at the pair* $(\eta^*, \mathrm{Q}^*) \in \mathscr{D}(y) \times \mathscr{Q}$*, then*

$$u(x) = \sup_{\xi \in \mathscr{A}(x)} \left[\mathrm{E}_{\mathrm{Q}^*} U(\xi) + \gamma(\mathrm{Q}^*) \right].$$

Conversely, if the previous relation is satisfied for $\mathrm{Q}^* \in \mathscr{Q}$*, then there exist* $y^* \in \mathbb{R}_+$ *and* $\eta^* \in \mathscr{D}(y^*)$ *such that the minimum in* (3) *is attained at* y^**, and the minimum in* (2) *for* $y = y^*$ *is attained at the pair* (η^*, Q^*)*.*

Static case : Main theorem

Theorem (2, continued)

(vi) *Fix* $x > 0$. *Let* $u(x) < \infty$, *the minimum in* (3) *attained at* y^*, *and the minimum in* (2) *for* $y = y^*$ *attained at the pair* $(\eta^*, \mathrm{Q}^*) \in \mathscr{D}(y) \times \mathscr{Q}$. *If the problem* (1) *has a solution, i.e. there exists a r.v.* $\xi^* \in \mathscr{A}(x)$ *such that* $u(x) = \inf_{\mathrm{Q} \in \mathscr{Q}} \left[\mathrm{E}_{\mathrm{Q}} U(\xi^*) + \gamma(\mathrm{Q}) \right]$, *then*

$$\mathrm{E}_{\mathrm{P}} \xi^* \eta^* = xy^*$$

and P*-a.s.*

$$\frac{\eta^*}{d\mathrm{Q}^*/d\mathrm{P}} \in \partial U(\xi^*) \text{ on } \left\{ \frac{d\mathrm{Q}^*}{d\mathrm{P}} > 0 \right\} \text{ and } \xi^* \eta^* = 0 \text{ on } \left\{ \frac{d\mathrm{Q}^*}{d\mathrm{P}} = 0 \right\}.$$

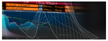
Static case : Main theorem

Theorem (2, continued)

Conversely, if these conditions are satisfied, then $u(x) = \mathrm{E}_{\mathrm{Q}^*} U(\xi^*) + \gamma(\mathrm{Q}^*)$.

(vii) *If* $v \not\equiv +\infty$, *then*

$$\lim_{y \uparrow +\infty} \frac{v(y)}{y} = 0. \tag{4}$$



Static case : Main theorem

Theorem (2, continued)

(viii) *Assume that the function U satisfies the Inada condition at* 0, *i.e.*

$$\lim_{x \downarrow 0} U'_{-}(x) = +\infty.$$

Assume also that $u(x) < \infty$ and(only if U is bounded) that there is at least one measure $\mathrm{Q} \in \arg\min \gamma$ and a positive r.v. ξ_1 such that $\mathrm{Q}(\xi < \xi_1) > 0$ for any $\xi \in \overline{\mathscr{C}}_+$. Then u satisfies the Inada condition at 0 as well.

Static case : Main theorem

Theorem (2, continued)

(ix) *Assume that U is differentiable on* $(0, +\infty)$ *and satisfies the Inada condition at* 0, *and* γ *is strictly convex on* $\mathcal{Q}$. *If* $v(y) < +\infty$ *for a given* $y > 0$, *and the minimum in* (2) *is attained at pairs* (η_1, Q_1) *and* (η_2, Q_2), *then* $Q_1 = Q_2$ *and* $\eta_1 = \eta_2$ Q_1*-a.s. Then if u is finite, it is differentiable on* $(0, +\infty)$. *Only if U is bounded, assume additionally that, for a* $Q \in \arg\min \gamma$, *there is a positive r.v.* ξ_1 *such that* $Q(\xi < \xi_1) > 0$ *for any* $\xi \in \overline{\mathscr{C}}_+$. *Then v is strictly convex on* $\{v < \infty\}$.

Outline

Setting of the problem

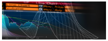
Assume that $\mathscr{D}_0$ is a convex subset of $\mathscr{D}$. Recall that the value function of the dual problem is defined by

$$v(y) = \inf_{\eta \in \mathscr{D},\, Q \in \mathscr{Q}} \left[E_Q V\left(\frac{y\eta}{dQ/dP} \right) + \gamma(Q) \right], \quad y \geqslant 0.$$

Here we consider the problem when the function v does not change if $\mathscr{D}$ is replaced by $\mathscr{D}_0$.

Given a probability measure $Q \ll P$, put

$$v_Q(y) = \inf_{\eta \in \mathscr{D}} \left[E_Q V\left(\frac{y\eta}{dQ/dP} \right) \right],$$
$$\widetilde{v}_Q(y) = \inf_{\eta \in \mathscr{D}_0} \left[E_Q V\left(\frac{y\eta}{dQ/dP} \right) \right], \quad y \geqslant 0.$$



So we consider the question when v_Q and $\widetilde{v}_Q$ coincide.

Superreplication prices

Let $f \in L^0_+$. If $\mathscr{A}(x)$ is interpreted as the set of terminal wealths corresponding to the initial wealth x, a superreplication price of f is usually defined by

$$\mathbb{C}^*(f) = \inf\{x\colon \text{there exists } \xi \in \mathscr{A}(x) \text{ such that } \xi \geqslant f\}.$$

However, here $\mathscr{A}$ does not satisfy any closedness assumption. Instead, we shall call the superreplication price of f the following amount:

$$\overline{\mathbb{C}}^*(f) = \inf\{x\colon \text{there exists } \xi \in \overline{\mathscr{C}}_+ \text{ such that } x\xi \geqslant f\}.$$

From now on, E stands for expectation wrt P.

Connection to superreplication prices

Theorem (3)

Let the above assumptions be satisfied and $\mathscr{D}_0 \subseteq \mathscr{D}$ *a convex nonempty set. Introduce the following conditions:*

(i) *for any* $\eta \in \mathscr{D}$ *there is* $\widetilde{\eta} \in \mathscr{D}_0$ *such that* $\eta \leqslant \widetilde{\eta}$;

(ii) $v_{\mathsf{Q}}(y) = \widetilde{v}_{\mathsf{Q}}(y)$ *for all* $\mathsf{Q} \ll \mathsf{P}$ *and* $y \geqslant 0$ *for any u.f.* U;

(iii) $v_{\mathsf{Q}}(y) = \widetilde{v}_{\mathsf{Q}}(y)$ *for all* $\mathsf{Q} \ll \mathsf{P}$ *and* $y \geqslant 0$ *for some strictly increasing utility function* U;

(iv) *for any* $f \in L^0_+$,

$$\overline{\mathbb{C}}^*(f) = \sup_{g \in \mathscr{D}_0} \mathsf{E} fg.$$

Then (i)⇒(ii)⇒(iii)⇒(iv). *If the closure* $\overline{\mathscr{D}_0}$ *of* $\mathscr{D}_0$ *in* L^0 *satisfies* $\overline{\mathscr{D}_0} \subseteq \mathscr{D}_0 - L^0_+$, *then all the conditions are equivalent.*

Connection to superreplication prices

Corollary

Let a convex nonempty set $\mathscr{D}_0 \subseteq \mathscr{D}$ *satisfy* (iv). *Then* $\overline{\mathscr{D}}_0$ *satisfies* (i)–(iv).

Outline

Assumptions

Let a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathrm{P})$ be given, $\mathscr{F} = \sigma\big(\bigcup_{t\geqslant 0} \mathscr{F}_t\big)$ and $\mathscr{F}_0$ contains only sets of P-measure 0 or 1. We denote by $\mathbb{D}$ the family of adapted real-valued càdlàg processes $X = (X_t)_{t\geqslant 0}$, $\mathbb{D}_+ = \{X \in \mathbb{D} \colon X \geqslant 0\}$, $\mathbb{D}_{++} = \{X \in \mathbb{D} \colon \mathrm{P}(\inf_t X_t > 0) = 1\}$. If $X \in \mathbb{D}$ and P-a.s. a finite limit $\lim_{t\to\infty} X_t$ exists, it will be denoted by X_∞.

Now we introduce assumptions on a set $\mathscr{X}$ of stochastic processes, whose elements are interpreted as wealth processes corresponding to all possible strategies of an investor with the unit initial wealth.If the investor has the initial wealth $x > 0$, then the family of wealth processes corresponding to all his strategies is $\mathscr{X}(x) = x\mathscr{X}$.

Assumptions

Assumption (on a family of wealth processes)

A set $\mathscr{X} \subseteq \mathbb{D}_+$ is convex, $X_0 = 1$ for any $X \in \mathscr{X}$, $1 \in \mathscr{X}$ and P*-a.s. a finite limit* $\lim_{t\to\infty} X_t$ *exists for any $X \in \mathscr{X}$.*

Now let us consider the robust utility maximization problem with $\mathscr{A} = \{X_\infty \colon X \in \mathscr{X}\}$. We are interested in the question: What are the conditions under which one can take the set $\mathscr{D}_0 = \{Y_\infty \colon Y \in \mathscr{Y}\}$ instead of $\mathscr{D}$ in the definition of the function v? Here $\mathscr{Y}$ is the class of supermartingale densities defined on the next slide.

Supermartingale densities

Definition

A nonnegative process Y with $Y_0 = 1$ is called a supermartingale density for $\mathscr{X}$ if, for any $X \in \mathscr{X}$, the product $XY = (X_t Y_t)_{t \geqslant 0}$ is a P-supermartingale. The class of all supermartingale densities is denoted by $\mathscr{Y}$.

Since $1 \in \mathscr{X}$, any $Y \in \mathscr{Y}$ is a P-supermartingale. If R is a probability measure, $\mathsf{R} \ll \mathsf{P}$, and any $X \in \mathscr{X}$ is a supermartingale (in particular, a local martingale) under R, then the density process $\left(d\mathsf{R}|_{\mathscr{F}_t} / d\mathsf{P}|_{\mathscr{F}_t} \right)_{t \geqslant 0}$ is a supermartingale density. Let us call such a measure R a supermartingale measure.

Forked families

Definition

A family $\mathscr{X} \subseteq \mathbb{D}_+$ is called forked if for any $X^i \in \mathscr{X} \cap \mathbb{D}_{++}$, $i = 1, 2$, any $s \geqslant 0$ and any $A \in \mathscr{F}_s$ the process

$$X_t = X_t^1 1_{\{t<s\}} + X_s^1 \left(1_A \frac{X_t^1}{X_s^1} + 1_{\Omega \setminus A} \frac{X_t^2}{X_s^2} \right) 1_{\{t \geqslant s\}}$$

belongs to $\mathscr{X}$.

There is a similar notion of fork-convexity introduced by Žitković (2002). A fork-convex family is forked and convex, the converse, in general, is not true.

Supermartingale densities : Necessary and sufficient conditions

The smallest forked family containing a family $\mathscr{X} \subseteq \mathbb{D}_+$ is called a forked hull of $\mathscr{X}$ and is denoted by fork($\mathscr{X}$).

Theorem (4)

Let the above assumptions be satisfied and $\mathscr{D} \neq \{0\}$. Then the set $\mathscr{D}_0 = \{Y_\infty \colon Y \in \mathscr{Y}\}$ is nonempty and satisfies conditions (i)–(iv) *of Theorem* (3) *if and only if*

$$\{X_\infty \colon X \in \mathrm{fork}(\mathscr{X})\} \subseteq \overline{\mathscr{C}}_+.$$

Rokhlin (2010)

Corollary

Let $\mathscr{W} \subseteq \mathbb{D}_+$ *be a forked and convex family of stochastic processes,* $1 \in \mathscr{W}$ *and* $X_0 = 1$ *for any* $X \in \mathscr{W}$*. The following statements are equivalent*:

(i) *the set*

$$\{X_t \colon X \in \mathscr{W},\ t \in \mathbb{R}_+\}$$

is bounded in probability;

(ii) *there exists a supermartingale density* Y *for the family* $\mathscr{W}$ *such that* $\mathsf{P}(Y_\infty > 0) = 1$*.*

Supermartingale measures : Necessary and sufficient conditions

Theorem (5)

Let the above assumptions be satisfied and $\mathscr{D} \neq \{0\}$*. Then the set* $\mathscr{D}_0 = \{dR/dP \colon R$ *is a supermartingale measure*$\}$ *is nonempty and satisfies condition* (iv) *of Theorem* (3) *if and only if the following two conditions are satisfied*:

$$\{X_\infty \colon X \in \mathrm{fork}(\mathscr{X})\} \subseteq \overline{\mathscr{C}}_+;$$

$$(\lambda_n) \subseteq \mathbb{R}_+,\ (\xi_n) \subseteq \overline{\mathscr{C}}_+,\ f_n = \lambda_n \xi_n - (\lambda_n - 1) \xrightarrow{P} f \geqslant 0,$$
$$(f_n) \text{ is uniformly bounded from below} \quad \Longrightarrow \quad f \in \overline{\mathscr{C}}_+.$$

Outline

A model with arbitrage

Let $B = (B_t)_{0 \leqslant t < \infty}$ be a standard Brownian motion. For $\alpha \in (0,1)$ put

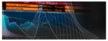
$$Z_t = \alpha \exp(B_t - t/2) + 1 - \alpha, \quad S_t = 1/Z_t,$$

$$\mathscr{X} = \left\{ X = (X_t)_{t \geqslant 0} \in \mathbb{D}_+ : X_t = 1 + \int\limits_0^t H_u \, dS_u, \; H \in L(S) \right\}.$$

This model admits arbitrage: put $H_t^* \equiv 1$, then the wealth process satisfies

$$X_t^* = 1 + \int_0^t H_u^* \, dS_u = 1 + (S_t - 1) = S_t \to 1/(1-\alpha) \quad \text{a.s.},$$

i.e. $X^* \in \mathscr{X}$ and $X_\infty^* = 1/(1-\alpha) > 1$ a.s.



Utility maximization makes sense and H^* is the optimal strategy

Take an arbitrary $X \in \mathscr{X}$. It is easy to check using Itô's formula that XZ is a local martingale and, hence, a supermartingale (in particular, Z is a supermartingale density). Therefore, by Jensen's inequality,

$$\begin{aligned} \mathsf{E}[U(X_\infty)] &= \mathsf{E}\left[U\left(\frac{X_\infty Z_\infty}{1-\alpha}\right)\right] \leqslant U\left(\frac{\mathsf{E}(X_\infty Z_\infty)}{1-\alpha}\right) \leqslant U\Big(\frac{1}{1-\alpha}\Big) \\ &= \mathsf{E}[U(X^*_\infty)]. \end{aligned}$$

A model with an arbitrary number of assets

Let $(\mathscr{F}_n)_{n=0,1,\dots,N}$ be an increasing family of sub-σ-algebras on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, $\mathscr{F}_0 = \{\varnothing, \Omega\}$, $\mathscr{F}_n = \mathscr{F}$.
Let $\mathscr{S} = (S^i)_{i \in I}$ be an arbitrary family of adapted processes $S^i = (S^i_n)_{n=0,1,\dots,N}$ on this space. S^i_n is interpreted as a discounted price of asset i at time n.

All possible strategies

All possible strategies of an investor is the set $\mathscr{L}$ with elements $H = (H^i)_{i \in I}$, where, for every i, $H^i = (H^i_n)_{n=1,\dots,N}$ is a predictable (H^i_n is $\mathscr{F}_{n-1}$-measurable for $n = 1, \dots, N$) random sequence; moreover, there exists a finite subset $J = J(H) \subseteq I$ such that $H^i \equiv 0$ if $i \notin J$. A r.v. H^i_n is interpreted as the amount of the ith asset in an investor's portfolio at time n.

Wealth processes

Given $H \in \mathscr{L}$, define the process $H \circ S = (H \circ S_n)_{n=0,1,\dots,N}$ by

$$H \circ S_n = \sum_{k=1}^{n} \sum_{i \in J(H)} H_k^i \Delta S_k^i, \quad \Delta S_k^i = S_k^i - S_{k-1}^i.$$

Then $1 + H \circ S$ is the wealth process of an investor with an initial wealth 1 and a strategy H.

Admissible strategies

Let now $\mathscr{H}$ be a subset of $\mathscr{L}$ satisfying the following properties:

- $\mathscr{H}$ is convex.
- $0 \in \mathscr{H}$.
- $H + K \in \mathscr{H}$ if $H, K \in \mathscr{H}$ and $HK = 0$.
- if $H \in \mathscr{H}$, $n = 1, \dots, N-1$ and ξ is an $\mathscr{F}_n$-measurable nonnegative r.v., then $H 1_{[0,n]} \in \mathscr{H}$ and $H\xi 1_{[n+1,N]} \in \mathscr{H}$.

Put

$$\mathscr{X} := \{1 + H \circ S \colon H \in \mathscr{H}\} \cap \mathbb{D}_+.$$

Then $\mathscr{X}$ is a forked family.